

9. (Statistical Mechanics)

A parallel plate capacitor consists of two square plates, each of area a^2 , separated by the distance d . The first plate, which is located at $z = 0$, is made of conducting metal and is grounded. The second plate, which is located at $z = d$, is made of dielectric and is maintained at a positive potential V with respect to the metal plate. The whole system is at some very high temperature T so that the electrons emitted from the hot metal of the first plate form a dilute gas which is in equilibrium and which fills this capacitor. There is no conductivity of electrons between the gas and the dielectric plate. Assume that the capacitor is so large ($a \gg d$) that the edge effects can be disregarded.

- (a) Write out the system of equations and boundary conditions that determines the potential $\varphi(z)$, and the density of electrons $n(z)$, inside the capacitor as functions of z ($d > z > 0$). Note: since in equilibrium there is no net flux of electrons across $z = 0$, you may assume that $n(z)$ has vanishing gradient there.
- (b) Assuming a weak potential, $\frac{|eV|}{kT} \ll 1$, find $\varphi(z)$ and $n(z)$ inside the capacitor to first order in $\frac{eV}{kT}$.

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This problem is an unusual mix of electromagnetism and statistical mechanics. In part (a), we will focus on the electromagnetism side of the problem. In part (b), we will move on to the statistical mechanics.

Throughout, we will assume the setup is translationally invariant in x and y , since the problem instructs us to ignore any edge effects. We will take the charge of each electron to be $-|e|$.

- (a) The relation between the electric potential $\varphi(\mathbf{r})$ and the charge density $\rho(\mathbf{r})$ is given by the definition of the electric potential and Gauss' law:

$$\mathbf{E} = -\nabla\varphi(\mathbf{r}) \quad \text{and} \quad \nabla \cdot \mathbf{E} = \frac{\rho(\mathbf{r})}{\epsilon_0} \quad (1)$$

Using this information, we can write

$$\begin{aligned} -\nabla^2\varphi(\mathbf{r}) &= \nabla \cdot (-\nabla\varphi(\mathbf{r})) \\ &= \nabla \cdot \mathbf{E} \\ -\nabla^2\varphi(\mathbf{r}) &= \frac{\rho(\mathbf{r})}{\epsilon_0} \end{aligned} \quad (2)$$

Since the setup is translationally invariant in x and y , $\varphi(\mathbf{r})$ depends only on z , so

$$-\nabla^2\varphi(\mathbf{r}) = -\frac{d^2\varphi}{dz^2} \quad (3)$$

If the number density of electrons is $n(z)$, then the charge density inside the capacitor depends only on z and is equal to

$$\rho(\mathbf{r}) = \rho(z) = -|e|n(z) \quad (4)$$

Putting all this together, we get a differential equation relating $\varphi(z)$ to $n(z)$:

$$\begin{aligned} -\frac{d^2\varphi}{dz^2} &= -\frac{|e|n(z)}{\epsilon_0} \\ \frac{d^2\varphi}{dz^2} &= \frac{|e|}{\epsilon_0}n(z) \end{aligned} \quad (5)$$

The problem gives us two boundary conditions for the electric potential:

$$\varphi(0) = 0 \quad \text{since the first plate is grounded} \quad (6)$$

$$\varphi(d) = V \quad (7)$$

It also gives us a third boundary condition for $n(z)$, the fact that $n(z)$ has vanishing gradient at $z = 0$:

$$\left. \frac{dn}{dz} \right|_{z=0} = 0 \quad (8)$$

Summarizing our results, we get the following:

Differential equation: $\frac{d^2\varphi}{dz^2} = \frac{|e|}{\epsilon_0}n(z)$

(9)

Boundary conditions: $\varphi(0) = 0, \quad \varphi(d) = V, \quad \left. \frac{dn}{dz} \right|_{z=0} = 0$

(10)

For a second-order ordinary differential equation like (9), you are probably used to having two boundary conditions. In this problem, we have three boundary conditions; we'll explain more in part (b).

- (b) So far, so good, but we haven't done any statistical mechanics yet. Stat mech doesn't directly tell us that much about the electric potential $\varphi(z)$, but it does have a lot to say about the number density of particles at equilibrium $n(z)$. Since this problem asks us to compare the number density of the particles at equilibrium, the appropriate strategy to use is "chemical potential matching":

Chemical potential matching:

When a statistical mechanics problem asks you to compare the number of noninteracting classical particles of two different types, you should set the chemical potentials of each type of particle equal to one another.

The chemical potential is the free energy needed to add a particle of a certain type, i.e., the partial derivative of F with respect to N : $\mu = \left. \frac{\partial F}{\partial N} \right|_{T,V}$. For that reason, the following statements are all equivalent for a system of noninteracting particles of types 1 and 2, where a particle of one type can become a particle of the other type:

- A. The system is in thermodynamic equilibrium.
- B. The free energy of the system is minimized.
- C. The change in free energy when a particle of type 1 becomes a particle of type 2 is zero.
- D. The free energy needed to subtract a particle of type 1, plus the free energy needed to add a particle of type 2, is zero.
- E. The chemical potential of type 1 particles is equal to the chemical potential of type 2 particles: $-\mu_1 + \mu_2 = 0$, or $\mu_1 = \mu_2$.

To find the chemical potential of a system of noninteracting classical particles of a certain type:

1. Calculate the partition function for that type of particle. For noninteracting, indistinguishable classical particles, the partition function for all the particles Z can be written in terms of the partition function for one particle Z_1 using $Z = Z_1^N / N!$.
2. Calculate the free energy for that type of particle using $F = -kT \ln Z$.
3. The chemical potential is the free energy needed to add a particle of a certain type, i.e., the partial derivative of F with respect to N : $\mu = \left. \frac{\partial F}{\partial N} \right|_{T,V}$.

If there is a constant energy shift between the types of particles (i.e. each particle of type 1 has an energy ϵ greater than a particle of type 2), then the chemical potentials of the types of particles should have the same energy shift (the chemical potential for particles of type 1 is ϵ greater than the chemical potential for particles of type 2).

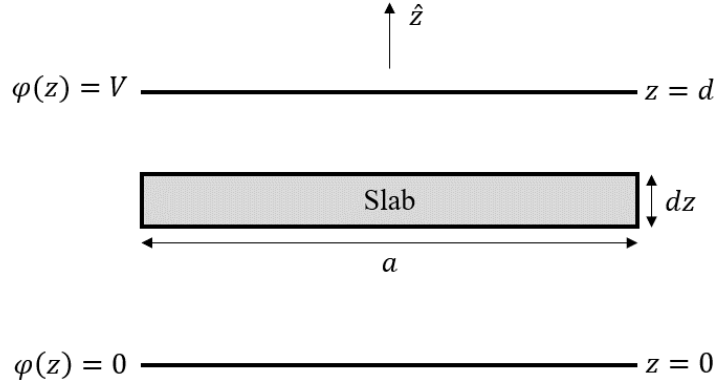
In chemical reactions, there are cases in which one particle of type 1 becomes multiple particles of type 2. In that case, the chemical potential of each type of particle must be multiplied by the multiplicity of the particle.

In this problem, we don't explicitly have two different types of particles, types 1 and 2. What we do have is a collection of "type z " particles, where $0 < z < d$: a "type z " particle is a particle at a distance z from the grounded plate of the capacitor. At equilibrium, the chemical potential of each type of particles must be constant, meaning that μ must be constant in z .

Before following steps 1-3 in the box above to find μ , we should address the fact that these steps are best suited to "noninteracting, classical particles." Electrons are neither of these, but we will assume they are for the purposes of this part. All the electrostatic interactions between the electrons will be captured by the electric potential $\varphi(z)$. Since we are in the high-temperature limit, $kT \gg |e|V$, the effects of the quantum statistics of the electrons are negligible, and we can therefore approximate the electrons by classical particles. (We will account for the spin degeneracy of the spin-1/2 particles, but it turns out it won't matter for the final answer.) We will also assume the electrons are nonrelativistic.

With all this in mind, let's calculate the partition function for a single "type z " particle Z_1 . To account for the height of the particle, we'll calculate the partition function for a single particle

in a slab-like volume of cross-sectional area a^2 and height dz , as shown in the diagram below:



Using the phase-space formalism for classical noninteracting point particles, and remembering the spin degeneracy of 2, the partition function is

$$Z_1 = 2 \int \frac{d^3p d^3x}{h^3} e^{-\beta E} \quad \text{where } \beta \equiv \frac{1}{kT} \quad (11)$$

The energy of a single particle is its nonrelativistic kinetic energy, plus the electric potential energy $-|e|\varphi(z)$

$$E = \frac{p^2}{2m} - |e|\varphi(z) \quad (12)$$

Plugging this in and carrying out the spatial integral for the partition function, we get

$$\begin{aligned} Z_1 &= 2 \int \frac{d^3p d^3x}{h^3} e^{-\beta \left(\frac{p^2}{2m} - |e|\varphi(z) \right)} \\ &= 2 \int \frac{d^3p d^3x}{h^3} e^{-p^2/(2mkT)} e^{|e|\varphi(z)/(kT)} \end{aligned}$$

The integrand is independent of x and y , so the integrals over dx and dy simplify to the cross-sectional area a^2 . Since the slab we are considering is of infinitesimal height dz , we can approximate the integral over dz by multiplying the integrand by dz (even though the integral does depend on z through $\varphi(z)$). This gives us

$$Z_1 = 2 \frac{a^2 dz}{h^3} e^{|e|\varphi(z)/(kT)} \int d^3p e^{-p^2/(2mkT)}$$

We can now simplify the integrals over the momentum p :

$$\begin{aligned} Z_1 &= 2 \frac{a^2 dz}{h^3} e^{|e|\varphi(z)/(kT)} \int_{-\infty}^{\infty} dp_x \int_{-\infty}^{\infty} dp_y \int_{-\infty}^{\infty} dp_z e^{-(p_x^2 + p_y^2 + p_z^2)/(2mkT)} \\ &= 2 \frac{a^2 dz}{h^3} e^{|e|\varphi(z)/(kT)} \left[\int_{-\infty}^{\infty} dp_x e^{-p_x^2/(2mkT)} \right] \left[\int_{-\infty}^{\infty} dp_y e^{-p_y^2/(2mkT)} \right] \left[\int_{-\infty}^{\infty} dp_z e^{-p_z^2/(2mkT)} \right] \end{aligned} \quad (13)$$

Each of these three integrals is the same Gaussian integral, which can be calculated by a change of coordinates $u \equiv (2mkT)^{-1/2} p$:

$$\begin{aligned} \int_{-\infty}^{\infty} dp e^{-p^2/(2mkT)} &= (2mkT)^{1/2} \int_{-\infty}^{\infty} du e^{-u^2} \\ &= (2\pi mkT)^{1/2} \quad \text{using the result } \int_{-\infty}^{\infty} du e^{-u^2} = \pi^{1/2} \end{aligned} \quad (14)$$

Plugging this back into (13), we get

$$\begin{aligned}
 Z_1 &= 2 \frac{a^2 dz}{h^3} e^{|e|\varphi(z)/(kT)} (2mkT)^{3/2} \\
 &= 2a^2 dz e^{|e|\varphi(z)/(kT)} \left(\frac{2\pi mkT}{h^2} \right)^{3/2} \\
 Z_1 &= \frac{2a^2 dz}{\lambda^3} e^{|e|\varphi(z)/(kT)} \quad \text{for } \lambda \equiv \left(\frac{h^2}{2\pi mkT} \right)^{1/2}
 \end{aligned} \tag{15}$$

The quantity λ is called the thermal wavelength.

Now, we should calculate the partition function for all “type z ” particles, i.e., the partition function for all particles in the slab of height dz . Let N be the number of particles in this slab. Then, since we are approximating the electrons as noninteracting, indistinguishable classical particles, we can write this partition function Z in terms of Z_1 :

$$\begin{aligned}
 Z &= \frac{Z_1^N}{N!} \\
 Z &= \frac{1}{N!} \left(\frac{2a^2 dz}{\lambda^3} \right)^N \exp \left(\frac{N|e|\varphi(z)}{kT} \right)
 \end{aligned} \tag{16}$$

The free energy can be derived from the partition function using the formula $F = -kT \ln Z$ and the logarithm rules $\ln(ab) = \ln a + \ln b$ and $\ln x^a = a \ln x$:

$$\begin{aligned}
 F &= -kT \ln \left[\frac{1}{N!} \left(\frac{2a^2 dz}{\lambda^3} \right)^N \exp \left(\frac{N|e|\varphi(z)}{kT} \right) \right] \\
 &= -kT \left[\ln \exp \left(\frac{N|e|\varphi(z)}{kT} \right) + \ln \left(\frac{2a^2 dz}{\lambda^3} \right)^N - \ln N! \right] \\
 &= -kT \left[\frac{N|e|\varphi(z)}{kT} + N \ln \left(\frac{2a^2 dz}{\lambda^3} \right) - \ln N! \right] \\
 &= -N|e|\varphi(z) - NkT \ln \left(\frac{2a^2 dz}{\lambda^3} \right) + kT \ln N! \\
 &\approx -N|e|\varphi(z) - NkT \ln \left(\frac{2a^2 dz}{\lambda^3} \right) + kT (N \ln N - N) \quad \text{by Stirling's formula} \\
 F &= -N|e|\varphi(z) - NkT \ln \left(\frac{2a^2 dz}{\lambda^3} \right) + NkT \ln N - NkT
 \end{aligned} \tag{17}$$

To find the chemical potential, take the derivative of F with respect to N :

$$\begin{aligned}
 \mu &= \left. \frac{\partial F}{\partial N} \right|_{T,V} \\
 &= -|e|\varphi(z) - kT \ln \left(\frac{2a^2 dz}{\lambda^3} \right) + kT \ln N + NkT \left(\frac{1}{N} \right) - kT \\
 &= -|e|\varphi(z) - kT \ln \left(\frac{2a^2 dz}{\lambda^3} \right) + kT \ln N \\
 \mu &= -|e|\varphi(z) + kT \ln \left(\frac{N}{dz} \frac{\lambda^3}{2a^2} \right)
 \end{aligned} \tag{18}$$

The number density of electrons at a height z is approximately equal to the number of electrons N in a slab of infinitesimal height, divided by the volume of the slab: $n(z) = N/(a^2 dz)$.

Plugging in this result, we get

$$\mu = -|e|\varphi(z) + kT \ln \left(n(z) \frac{\lambda^3}{2} \right) \quad (19)$$

Note that the chemical potential reflects the energy shift $-|e|\varphi(z)$ between particles of “type z ” for different z : The chemical potential, which is the free energy to add a particle of a certain type, must reflect the electric potential energy of that particle.

At equilibrium, the chemical potential is constant in z . We can use this information to solve for $n(z)$ in terms of $\varphi(z)$:

$$\begin{aligned} \ln \left(n(z) \frac{\lambda^3}{2} \right) &= \frac{\mu}{kT} + \frac{|e|\varphi(z)}{kT} \\ n(z) \frac{\lambda^3}{2} &= \exp \left(\frac{\mu}{kT} + \frac{|e|\varphi(z)}{kT} \right) \\ n(z) &= \frac{2}{\lambda^3} e^{\mu/(kT)} \exp \left(\frac{|e|\varphi(z)}{kT} \right) \end{aligned} \quad (20)$$

$$n(z) = \frac{2}{\lambda^3} e^{\mu/(kT)} \exp \left(\frac{|e|\varphi(z)}{kT} \right) \quad (21)$$

This is another equation that relates $n(z)$ and $\varphi(z)$, and it should be considered alongside the differential equation we got from electromagnetism in part (a):

$$\frac{d^2\varphi}{dz^2} = \frac{|e|}{\epsilon_0} n(z)$$

Substituting for $n(z)$ in (21), we get an second-order ordinary differential equation for φ :

$$\frac{d^2\varphi}{dz^2} = \frac{2|e|}{\epsilon_0\lambda^3} e^{\mu/(kT)} \exp \left(\frac{|e|\varphi(z)}{kT} \right) \quad (22)$$

This differential equation is too hard to solve by hand, but we can simplify it considerably by using the weak potential assumption, $\frac{|e|V}{kT} \ll 1$. Since $\varphi(z) < V$, this allows us to expand the exponential to first order in $|e|\varphi/(kT)$:

$$\frac{d^2\varphi}{dz^2} \approx \frac{2|e|}{\epsilon_0\lambda^3} e^{\mu/(kT)} \left(1 + \frac{|e|\varphi(z)}{kT} \right) \quad (23)$$

This differential equation is solvable by hand. To start, use the trick of defining

$$u(z) \equiv 1 + \frac{|e|\varphi(z)}{kT} \quad (24)$$

Then, $u''(z) = \frac{|e|}{kT} \varphi''(z)$, so (23) becomes

$$\frac{d^2u}{dz^2} \approx \frac{2|e|^2}{\epsilon_0 kT \lambda^3} e^{\mu/(kT)} u(z) \quad (25)$$

So far, we do not have enough information to determine what the constant value of the chemical potential μ is. Since a , $|e|$, and λ are positive and μ is real, it will be useful to fold the constants in the prefactor of (25) into a single nonnegative constant, which we will call γ^2 :

$$\frac{d^2u}{dz^2} \approx \gamma^2 u \quad \text{for} \quad \gamma^2 \equiv \frac{2|e|^2}{\epsilon_0 kT \lambda^3} e^{\mu/(kT)} \quad (26)$$

As of now, we don't know the value of γ . But we do know that γ^2 is positive, so we can write the general solution to this differential equation for u , which is a sum of exponentials:

$$u(z) = A'e^{\gamma z} + B'e^{-\gamma z} \quad \text{for constants } A' \text{ and } B' \quad (27)$$

Plugging in the definition of $u(z)$ (24), we get

$$\begin{aligned} \varphi(z) &= \frac{kT}{|e|} (u(z) - 1) \\ &= \frac{kT}{|e|} (A'e^{\gamma z} + B'e^{-\gamma z} - 1) \end{aligned}$$

Redefining the constants $A \equiv \frac{kT}{|e|} A'$ and $B \equiv \frac{kT}{|e|} B'$, this boils down to

$$\varphi(z) = Ae^{\gamma z} + Be^{-\gamma z} - \frac{kT}{|e|} \quad (28)$$

Now, it is time to impose the three boundary conditions from part (a):

$$\varphi(0) = 0, \quad \varphi(d) = V, \quad \left. \frac{dn}{dz} \right|_{z=0} = 0$$

The last boundary condition, which is a condition on $n(z)$, can be made into a condition on $\varphi(z)$ by using (21), which relates $n(z)$ to $\varphi(z)$:

$$\begin{aligned} 0 &= \left. \frac{dn}{dz} \right|_{z=0} \\ &= \left. \frac{d}{dz} \left[\frac{2a^2}{\lambda^3} e^{\mu/(kT)} \exp \left(\frac{|e|\varphi(z)}{kT} \right) \right] \right|_{z=0} \\ &= \left. \frac{|e|}{kT} \frac{2a^2}{\lambda^3} e^{\mu/(kT)} \frac{d\varphi}{dz} \exp \left(\frac{|e|\varphi(z)}{kT} \right) \right|_{z=0} \\ 0 &= \left. \frac{|e|}{kT} \frac{2a^2}{\lambda^3} e^{\mu/(kT)} \frac{d\varphi}{dz} \right|_{z=0} \quad \text{since } \varphi(0) = 0 \\ \implies 0 &= \left. \frac{d\varphi}{dz} \right|_{z=0} \end{aligned} \quad (29)$$

Therefore, the three boundary conditions on $\varphi(z)$ are

$$\varphi(0) = 0, \quad \varphi(d) = V, \quad \left. \frac{d\varphi}{dz} \right|_{z=0} = 0 \quad (30)$$

We can now explain why we need three boundary conditions: Two of the boundary conditions are needed to fix A and B in (28), and the other one is needed to fix the constant γ (which is related to the constant, but currently unknown, chemical potential μ).

Applying these boundary conditions to (28), we get three conditions on A , B , and γ :

$$\begin{aligned} \varphi(0) = 0 \quad \implies \quad 0 &= Ae^{\gamma(0)} + Be^{-\gamma(0)} - \frac{kT}{|e|} \quad \text{by (28)} \\ 0 &= A + B - \frac{kT}{|e|} \\ A + B &= \frac{kT}{|e|} \end{aligned} \quad (31)$$

$$\begin{aligned}
\varphi(d) = V &\implies V = Ae^{\gamma d} + Be^{-\gamma d} - \frac{kT}{|e|} \quad \text{by (28)} \\
Ae^{\gamma d} + Be^{-\gamma d} &= V + \frac{kT}{|e|}
\end{aligned} \tag{32}$$

$$\begin{aligned}
\left. \frac{d\varphi}{dz} \right|_{z=0} = 0 &\implies 0 = \gamma Ae^{\gamma(0)} - \gamma Be^{-\gamma(0)} \quad \text{by (28)} \\
A - B &= 0
\end{aligned} \tag{33}$$

By (33), $A = B$; applying this fact to (31), we get

$$A = B = \frac{kT}{2|e|} \tag{34}$$

Plugging this into (32), we can find γ :

$$\begin{aligned}
\frac{kT}{2|e|}e^{\gamma d} + \frac{kT}{2|e|}e^{-\gamma d} &= V + \frac{kT}{|e|} \\
\frac{kT}{|e|} \left(\frac{e^{\gamma d} + e^{-\gamma d}}{2} \right) &= V + \frac{kT}{|e|} \\
\frac{kT}{|e|} \cosh(\gamma d) &= V + \frac{kT}{|e|} \quad \text{by the definition } \cosh x \equiv \frac{e^x + e^{-x}}{2} \\
\cosh(\gamma d) &= \frac{|e|V}{kT} + 1
\end{aligned} \tag{35}$$

$$\gamma d = \cosh^{-1} \left(\frac{|e|V}{kT} + 1 \right) \tag{36}$$

We can simplify this further by noting that if $\frac{|e|V}{kT}$ is zero, then $\gamma d = 0$ (since $\cosh^{-1} 1 = 0$). Therefore, since $\frac{|e|V}{kT}$ is small, γd is very close to zero. This means that in (35), we can expand $\cosh(\gamma d)$ to lowest order in γd . Using the definition of the hyperbolic cosine in terms of exponentials, we can expand $\cosh x$ in a Taylor series:

$$\begin{aligned}
\cosh x &= \frac{e^x + e^{-x}}{2} \\
&= \frac{(1 + x + \frac{x^2}{2} + \dots) + (1 - x + \frac{x^2}{2} + \dots)}{2} \\
&= \frac{2 + x^2 + \dots}{2} \\
\cosh x &= 1 + \frac{x^2}{2} + \dots
\end{aligned} \tag{37}$$

Therefore, to lowest nontrivial order in γd , we can rewrite (35) using this Taylor expansion and solve for γ to lowest order in $\frac{|e|V}{kT}$:

$$\begin{aligned}
1 + \frac{(\gamma d)^2}{2} &\approx \frac{|e|V}{kT} + 1 \\
(\gamma d)^2 &\approx \frac{2|e|V}{kT} \\
\gamma &\approx \frac{1}{d} \sqrt{\frac{2|e|V}{kT}}
\end{aligned} \tag{38}$$

Now, we can put everything together. Using (34) for the values of A and B and (38) for the value of γ , we can plug into (28) to get

$$\begin{aligned}
 \varphi(z) &= \frac{kT}{2|e|} e^{\gamma z} + \frac{kT}{2|e|} e^{-\gamma z} - \frac{kT}{|e|} \\
 &= \frac{kT}{|e|} \left[\left(\frac{e^{\gamma z} + e^{-\gamma z}}{2} \right) - 1 \right] \\
 &= \frac{kT}{|e|} [\cosh(\gamma z) - 1] \\
 \varphi(z) &= \frac{kT}{|e|} \left[\cosh \left(\sqrt{\frac{2|e|V}{kT}} \frac{z}{d} \right) - 1 \right] \quad \text{to lowest order in } \frac{|e|V}{kT}
 \end{aligned} \tag{39}$$

To find $n(z)$, use the electromagnetic relation between $\varphi(z)$ and $n(z)$ from part (a):

$$\frac{d^2 \varphi}{dz^2} = \frac{|e|}{\epsilon_0} n(z) \quad \implies \quad n(z) = \frac{\epsilon_0}{|e|} \frac{d^2 \varphi}{dz^2}$$

Taking two derivatives of (39), and using the facts that $(\sinh x)' = \cosh x$ and $(\cosh x)' = \sinh x$, we get

$$\begin{aligned}
 n(z) &= \frac{\epsilon_0}{|e|} \frac{d^2}{dz^2} \left[\frac{kT}{|e|} \left[\cosh \left(\sqrt{\frac{2|e|V}{kT}} \frac{z}{d} \right) - 1 \right] \right] \\
 &= \frac{\epsilon_0}{|e|} \frac{kT}{|e|} \left(\frac{2|e|V}{kT} \frac{1}{d^2} \right) \cosh \left(\sqrt{\frac{2|e|V}{kT}} \frac{z}{d} \right) \\
 n(z) &= \frac{2\epsilon_0 V}{|e| d^2} \cosh \left(\sqrt{\frac{2|e|V}{kT}} \frac{z}{d} \right)
 \end{aligned} \tag{40}$$

Using the definition of γ in (26), it is possible to check directly that this satisfies the stat mech relation between $n(z)$ and $\varphi(z)$ (21) to lowest order in $\frac{|e|V}{kT}$. It is also possible to check directly (using the Taylor series expansion of $\cosh x$ when necessary) that all the boundary conditions on $\varphi(z)$ and $n(z)$ are satisfied.

To summarize, we have to lowest order in $\frac{|e|V}{kT}$

$$\boxed{\varphi(z) = \frac{kT}{|e|} \left[\cosh \left(\sqrt{\frac{2|e|V}{kT}} \frac{z}{d} \right) - 1 \right] \quad \text{and} \quad n(z) = \frac{2\epsilon_0 V}{|e| d^2} \cosh \left(\sqrt{\frac{2|e|V}{kT}} \frac{z}{d} \right)} \tag{41}$$

The stat mech parts of this problem bear some relation to the “isothermal atmosphere” problem.