a) We have for Fermi – Dirac statistics,

$$n_k = \frac{1}{e^{\beta(\epsilon_k - \mu)} + 1} \quad \text{in momentum space} \quad \text{— (1)}$$

Total no:, $N = \sum_k n_k$ — (2)

In any d-dimensions, $\sum_k = \overset{\text{spin degeneracy}}{2}\left(\frac{L}{2\pi}\right)^d \left[\int k^{d-1} dk\right] S_d$ — (3)

where S_d is the d-dimensional ~~space~~ surface area which is,

$$S_d = \frac{\pi^{d/2}}{\Gamma(d/2+1)} \quad \text{— (4)}. \qquad \epsilon_k = \frac{\hbar^2 k^2}{2m}$$

The average energy,

$$E = \sum_k \epsilon_k n_k = 2\left(\frac{L}{2\pi}\right)^d \frac{\pi^{d/2}}{\Gamma(d/2+1)} \int_0^\infty k^{d-1} \frac{\hbar^2 k^2}{2m} \frac{dk}{e^{\beta\left(\frac{\hbar^2 k^2}{2m} - \mu\right)} + 1}$$

$$= 2\left(\frac{L}{2\pi}\right)^d \frac{\pi^{d/2}}{\Gamma(d/2+1)} \int \frac{\left(\frac{2m}{\hbar^2}\right)^{\frac{d-1}{2}} \epsilon^{\frac{d-1}{2}} \epsilon \left(\frac{2m}{\hbar^2}\right)^{+1/2} \frac{\epsilon^{-1/2}}{2} d\epsilon}{e^{\beta(\epsilon - \mu)} + 1}$$

$$= \left(\frac{L}{2\pi}\right)^d \frac{\pi^{d/2}}{\Gamma(d/2+1)} \left(\frac{2m}{\hbar^2}\right)^{d/2} \int_0^\infty \frac{\epsilon^{d/2} d\epsilon}{e^{\beta(\epsilon - \mu)} + 1}$$

$$\epsilon = \frac{\hbar^2 k^2}{2m}.$$

$$k = \left(\frac{2m}{\hbar^2}\right)^{1/2} \epsilon^{1/2}$$

$$dk = \left(\frac{2m}{\hbar^2}\right)^{1/2} \frac{1}{2} \epsilon^{-1/2} d\epsilon$$

we have from Sommerfield expansion,

$$\int_0^\infty \frac{f(\epsilon)\,d\epsilon}{e^{\beta(\epsilon-\mu)}+1} \approx \int_0^\mu f(\epsilon)\,d\epsilon + \frac{\pi^2}{6}T^2 f'(\mu) \quad \text{for } T \ll \mu \ll \epsilon_F$$

$$\therefore \; E \approx \left(\frac{L}{2\pi}\right)^d \frac{\pi^{d/2}}{\Gamma(d/2+1)} \left(\frac{2m}{\hbar^2}\right)^{d/2} \left[\frac{\mu^{d/2+1}}{d/2+1} + \frac{\pi^2 T^2}{6} \times \frac{d}{2}\, \mu^{d/2-1} \right]$$

Specific heat, $C_V = \left(\frac{\partial E}{\partial T}\right)_V$

$$C_V \approx \left(\frac{L}{2\pi}\right)^d \frac{\pi^{d/2}}{\Gamma(d/2+1)} \left(\frac{2m}{\hbar^2}\right)^{d/2} \left(\frac{\pi^2 d}{6}\right) T.$$

$$= \alpha T.$$

$2+d/2-d.$

$\frac{4+d-2d}{2}$

where $\alpha = \left(\frac{L}{2\pi}\right)^d \frac{\pi^{d/2}}{\Gamma(d/2+1)} \left(\frac{2m}{\hbar^2}\right)^{d/2} \frac{\pi^2 d}{6}.$

$$= \underline{\underline{\frac{d}{6}\left(\frac{L}{2}\right)^d \frac{(2m/\hbar^2)^{d/2}}{\Gamma(d/2+1)} \pi^{2-d/2}}}.$$

b) Similarly for Bosons,

$$n_k = \frac{1}{e^{\beta(\epsilon_k-\mu)}-1}$$

The average energy,

$$E = g_s \left(\frac{L}{2\pi}\right)^d \frac{\pi^{d/2}}{\Gamma(d/2+1)} \int_0^\infty \frac{k^{d-1}\,dk\;\epsilon.}{e^{\beta(\epsilon-\mu)}-1.}, \quad \text{where } g_s = \text{spin degeneracy}$$

$$= \frac{g_s}{2}\left(\frac{L}{2\pi}\right)^d \frac{\pi^{d/2}}{\Gamma(d/2+1)} \left(\frac{2m}{\hbar^2}\right)^{d/2} \int_0^\infty \frac{\epsilon^{d/2}\,d\epsilon}{e^{\beta(\epsilon-\mu)}-1}$$

$$= \frac{g_s}{2}\left(\frac{L}{2\pi}\right)^d \frac{\pi^{d/2}}{\Gamma(d/2+1)}\left(\frac{2m}{\hbar^2}\right)^{d/2} \int_0^\infty \frac{\epsilon^{d/2}\, e^{-\beta(\epsilon-\mu)}}{1-e^{-\beta(\epsilon-\mu)}}\, d\epsilon$$

$$= \frac{g_s}{2}\left(\frac{L}{2\pi}\right)^d \frac{\pi^{d/2}}{\Gamma(d/2+1)}\left(\frac{2m}{\hbar^2}\right)^{d/2} \int_0^\infty d\epsilon\, e^{\beta\mu}\, \epsilon^{d/2} e^{-\beta\epsilon} \sum_{n=0}^\infty \left(e^{-\beta(\epsilon-\mu)}\right)^n$$

$$= \frac{g_s}{2}\left(\frac{L}{2\pi}\right)^d \frac{\pi^{d/2}}{\Gamma(d/2+1)}\left(\frac{2m}{\hbar^2}\right)^{d/2} e^{\beta\mu} \sum_{n=0}^\infty e^{n\beta\mu} \int_0^\infty d\epsilon\, e^{-\beta\epsilon} e^{-\beta\epsilon n} \epsilon^{d/2}$$

$$= \frac{g_s}{2}\left(\frac{L}{2\pi}\right)^d \frac{\pi^{d/2}}{\Gamma(d/2+1)}\left(\frac{2m}{\hbar^2}\right)^{d/2} e^{\beta\mu} \sum_{n=0}^\infty e^{n\beta\mu} \int_0^\infty \epsilon^{d/2} e^{-\beta(n+1)\epsilon}\, d\epsilon$$

$$= \frac{g_s}{2}\left(\frac{L}{2\pi}\right)^d \frac{\pi^{d/2}}{\Gamma(d/2+1)}\left(\frac{2m}{\hbar^2}\right)^{d/2} e^{\beta\mu} \sum_{n=0}^\infty e^{n\beta\mu} \frac{1}{[\beta(n+1)]^{d/2+1}}$$

$$= \frac{g_s}{2}\left(\frac{L}{2\pi}\right)^d \frac{\pi^{d/2}}{\Gamma(d/2+1)}\left(\frac{2m}{\hbar^2}\right)^{d/2} \frac{1}{\beta^{d/2+1}} \sum_{n=0}^\infty \frac{e^{(n+1)\beta\mu}}{(n+1)^{d/2+1}}.$$

$$= \frac{g_s}{2}\left(\frac{L}{2\pi}\right)^d \frac{\pi^{d/2}}{\Gamma(d/2+1)}\left(\frac{2m}{\hbar^2}\right)^{d/2} \beta^{-(d/2+1)} \sum_{n=1}^\infty \frac{e^{\beta\mu n}}{n^{d/2+1}}$$

In the limit $\mu=0$. → Riemann ζ-fundn.

$$\Rightarrow E = \frac{g_s}{2}\left(\frac{L}{2\pi}\right)^d \frac{\pi^{d/2}}{\Gamma(d/2+1)}\left(\frac{2m}{\hbar^2}\right)^{d/2} (k_BT)^{d/2+1}\, \zeta(d/2+1)$$

Specific heat

$$C_V = \left(\frac{\partial E}{\partial T}\right)_V$$

$$C_V = \frac{g_s}{2}\left(\frac{L}{2\pi}\right)^d \frac{\pi^{d/2}}{\Gamma(d/2+1)}\left(\frac{2m}{\hbar^2}\right)^{d/2} (d/2+1)\,(k_BT)^{d/2}\, \zeta(d/2+1).$$

$$\therefore C_V = \alpha T^{d/2}$$

where $\alpha = \frac{g_s}{2}\left(\frac{L}{2\pi}\right)^d \frac{\pi^{d/2}}{\Gamma(d/2+1)}\left(\frac{2m}{\hbar^2}\right)^{d/2} (d/2+1)\, k_B$