

5. Quantum Mechanics (Fall 2005)

In this problem, neglect spin and relativistic effects, and use the Born approximation.

- (a) Suppose an electron scatters off a spherically symmetric potential $V(r)$. Write down (or compute if you don't remember) the formula for the Born approximation to the scattering amplitude $f(\theta, \phi)$, in the form of a one-dimensional radial integral:

$$f(\theta, \phi) = \int_0^\infty (\text{some function of } r) \times V(r) dr$$

- (b) Now suppose that the electron scatters elastically off a spherically symmetric charge distribution, with charge density $\rho(r)$ centered at the origin. (This is not a local potential, but the answer to part (a) may still be useful.)

Calculate, in the Born approximation (that is, to first order in the potential), the scattering amplitude $f(\theta, \phi)$ and write it as

$$f(\theta, \phi) = f_R(q^2) F(q^2)$$

where \mathbf{q} is the momentum transferred between the incident and the scattered electron, and $f_R(q^2)$ is the Rutherford amplitude for scattering off a point charge:

$$f_R(q^2) = \frac{2mZ\alpha}{\hbar^2 q^2}$$

Here α is the fine-structure constant. The function $F(q^2)$ is called the "form factor". Write an explicit formula for $F(q^2)$ in terms of $\rho(r)$.

- (c) Now specialize to an electron scattering elastically off a uniformly charged sphere, centered at the origin, with radius R and total charge Ze . What is $F(q^2)$ as a function of q and R ?

Hint: You might want the definite integral $\int_0^\infty e^{-pr} \sin(pr) dr = \frac{q}{q^2 + p^2}$ and the indefinite integrals

$$\int x \sin x dx = \sin x - x \cos x \quad \text{and} \quad \int x \cos x dx = \cos x + x \sin x$$

Note: The scattering amplitude is defined so that its square is the differential cross section: $|f|^2 = \frac{d\sigma}{d\Omega}$

Rederiving the general form of $f(\theta, \phi)$ is not trivial.

See Additional Notes ^{*} at end for non-rigorous reasoning,
but the scattering amplitude should be memorized.

The Born approx follows spherical symmetry.

$$(a) f(\theta, \phi) = -\frac{m}{2\pi\hbar^2} \langle \phi | \hat{V}(\vec{r}) | \psi \rangle = -\frac{m}{2\pi\hbar^2} \int e^{-i\vec{k}\cdot\vec{r}'} V(\vec{r}') \psi(\vec{r}') d\vec{r}' [1]$$

Compute Born approx. starting w/ $f(\theta, \phi)$. The first-order approximation amounts to $|\psi\rangle = |\vec{k}_0\rangle$, s.t. $\psi(\vec{r}) = \phi_{mc}(\vec{r})$

(i.e. by a plane wave)

Defining $\vec{q} = \vec{k}_0 - \vec{k}$ and w.o.l.g., let $\vec{q} \parallel \hat{z}$ and $V(\vec{r}') = V(r')$

$$\begin{aligned} &= -\frac{m}{2\pi\hbar^2} \int dr' r'^2 V(r') \int d\theta' e^{i\vec{q}\cdot\vec{r}' \cos\theta'} \sin\theta' \\ &= -\frac{m}{\hbar^2} \int dr' r'^2 V(r') \left[\frac{1}{i\vec{q}} e^{i\vec{q}\cdot\vec{r}' \cos\theta'} \right]_{\cos\theta'=1}^1 \\ &= -\frac{m}{\hbar^2} \int dr' r' V(r') \left(\frac{1}{i\vec{q}} \right) 2i \sin(\vec{q}\cdot\vec{r}') \end{aligned}$$

$$f(\theta, \phi) = -\frac{2m}{q\hbar^2} \int_0^\infty dr' r' V(r') \sin(qr') [2]$$

(b) We need the Coulomb potential for an arbitrary, extended object

$$V(\vec{r}) = -e \phi(\vec{r}) \leftarrow \begin{array}{l} \text{electric} \\ \text{potential} \end{array}$$

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where $\phi(\vec{r}) = \int \frac{\rho(r') dr'}{4\pi\epsilon_0 |\vec{r} - \vec{r}'|}$ sph symm charge distr

[3]

Now, using the first-order Born approx (eqn [1] w/ $\psi = \phi_{inc}$) and eqn [3]

$$f(\theta, \phi) = -\frac{m}{2\pi\hbar^2} \int e^{i\vec{q} \cdot \vec{r}'} V(\vec{r}') d\vec{r}'$$

$$= -\frac{m}{2\pi\hbar^2} \left(-\frac{e}{4\pi\epsilon_0} \right) \int d\vec{r}' e^{i\vec{q} \cdot \vec{r}'} \int d\vec{r}'' \frac{\rho(r'')}{|\vec{r}' - \vec{r}''|}$$

[4]

Below I will provide two methods to get to answer:

(1) Method 1: Coordinate shift

for simplicity, why don't we shift the variables to get rid of that pesky abs value bars!

$\vec{r}' \rightarrow \vec{r}' + \vec{r}''$, $d\vec{r}' \rightarrow d\vec{r}'$ & combining terms \Rightarrow

$$= \left(\frac{m \omega_{pe}}{e 2\pi\hbar^2} \right) \int d\vec{r}' \int d\vec{r}'' \left\{ \frac{e^{i\vec{q} \cdot \vec{r}''}}{|\vec{r}'|} \right\} e^{i\vec{q} \cdot \vec{r}''} \rho(r'')$$

[5]

Resolving the $d\vec{r}'$ integral:

$$\int d\vec{r}' \left(\frac{e^{i\vec{q} \cdot \vec{r}''}}{|\vec{r}'|} \right) = \lim_{\mu \rightarrow 0} \int d\vec{r}' \frac{1}{r'} e^{i\vec{q} \cdot \vec{r}'' - \mu r'}$$

[6]

w.o.l.g we let $\vec{q} \parallel \hat{z}$

$$\begin{aligned} &= \lim_{\mu \rightarrow 0} 2\pi \int r' e^{i\vec{q} \cdot \vec{r}''} \sin\theta' dr' d\theta' \cancel{\int} \frac{1}{r'} e^{i\vec{q} \cdot \vec{r}'' - \mu r'} \\ &= 2\pi \lim_{\mu \rightarrow 0} \int dr' r' e^{-\mu r'} \int_{-1}^1 d\cos\theta' e^{i\vec{q} \cdot \vec{r}'' \cos\theta'} \\ &= 2\pi \lim_{\mu \rightarrow 0} \int dr' r' e^{-\mu r'} \left(\frac{1}{i\vec{q} \cdot \vec{r}''} \right) (2 \sin q r'') \\ &= \frac{4\pi}{\vec{q} \cdot \vec{r}''} \lim_{\mu \rightarrow 0} \int dr' e^{-\mu r'} \sin qr' \quad \leftarrow \text{Trick: } \sin x = \text{Im}[e^{ix}] \text{ we are given the integral but easily solved w/ trick} \\ &= \frac{4\pi}{\vec{q} \cdot \vec{r}''} \lim_{\mu \rightarrow 0} \text{Im} \left[\left(\frac{1}{-\mu + i\vec{q} \cdot \vec{r}''} \right) e^{-\mu r'} e^{i\vec{q} \cdot \vec{r}''} \right]_0^\infty = -\left(\frac{4\pi}{\vec{q} \cdot \vec{r}''} \right) \lim_{\mu \rightarrow 0} \text{Im} \left[\frac{1}{(-\mu + i\vec{q} \cdot \vec{r}'')} \frac{(-\mu - i\vec{q} \cdot \vec{r}'')}{(-\mu - i\vec{q} \cdot \vec{r}'')} \right] \\ &= \lim_{\mu \rightarrow 0} -\left(\frac{4\pi}{\vec{q} \cdot \vec{r}''} \right) \text{Im} \left[\frac{-\mu - i\vec{q} \cdot \vec{r}''}{\mu^2 + \vec{q}^2} \right] = \frac{4\pi}{\vec{q} \cdot \vec{r}''} \end{aligned}$$

Returning to scattering amplitude (eqn [5])

$$\begin{aligned} f(\theta, \phi) &= \left(\frac{m \omega_{pe}}{e 2\pi\hbar^2} \right) \int d\vec{r}'' e^{i\vec{q} \cdot \vec{r}''} \rho(r'') \left(\frac{4\pi}{\vec{q}^2} \right) \\ &= \left(\frac{2m \omega_{pe}}{q^2 \hbar^2} \right) \left\{ \frac{\hbar c}{ze} \int d\vec{r}' e^{i\vec{q} \cdot \vec{r}'} \rho(r') \right\} \end{aligned}$$

[7]

$$= f_n(q^2) \left\{ \frac{\hbar c}{ze} 2\pi \int dr' \left(\frac{r'^2}{iqr'} \right) 2 \sin qr' g(r') \right\}$$

$$\boxed{f(\theta, \phi) = f_n(q^2) \left\{ \frac{4\pi \hbar c}{ze q^2} \int dr' r' \rho(r') \sin qr' \right\}}$$

[8]

$$f(\theta, \phi) = f_n(q^2) \left\{ \frac{4\pi \alpha c}{Z e q} \int dr' r' \rho(r') \sin qr' \right\}$$

$\uparrow F(q^2)$ in curly brackets

[8]

(2) Method 2: Using F.T. theorem (more general method) [see additional notes]*

We can apply the convolution theorem to eqn [4] by

$$\text{identifying the } d\vec{r}' \text{ integral as } h(\vec{r}') = \int d\vec{r}' \frac{e^{i\vec{q} \cdot \vec{r}'}}{|\vec{r}' - \vec{r}|} = \int d\vec{r}' u(\vec{r}') v(\vec{r}' - \vec{r}')$$

$$\begin{aligned} &= u * v \\ &= \mathcal{F}^{-1}(\tilde{u}(p) \cdot \tilde{v}(p)) \end{aligned}$$

$$\Rightarrow f(\theta, \phi) = \left(\frac{m \alpha \hbar c}{e 2\pi \hbar c} \right) \int d\vec{r}'' g(r'') h(\vec{r}'')$$

$$\begin{aligned} u(\vec{r}') &= e^{i\vec{q} \cdot \vec{r}'} \xrightarrow{\text{F.T.}} \tilde{u}(p) = \int e^{-i\vec{p} \cdot \vec{r}'} e^{i\vec{q} \cdot \vec{r}'} d\vec{r}' = (2\pi)^3 \delta^{(3)}(\vec{q} - \vec{p}) \\ v(\vec{r}') &= \frac{1}{|\vec{r}'|} \quad \tilde{v}(p) = \int e^{-i\vec{p} \cdot \vec{r}'} \frac{1}{|\vec{r}'|} d\vec{r}' \quad (\text{see eqn [6]}) \\ &= 4\pi/p^2 \end{aligned}$$

$$\tilde{h}(\vec{p}) = \tilde{u} \cdot \tilde{v} = \frac{4\pi}{p^2} (2\pi)^3 \delta^{(3)}(\vec{q} - \vec{p})$$

$$\begin{aligned} h(\vec{r}'') &= \frac{1}{(2\pi)^3} \int d\vec{p} \frac{4\pi}{p^2} \delta^{(3)}(\vec{q} - \vec{p}) e^{i\vec{p} \cdot \vec{r}''} \\ &= \frac{4\pi}{q^2} e^{i\vec{q} \cdot \vec{r}''} \end{aligned}$$

[10]

Substituting [10] \rightarrow [9] we get eqn [7] identically!
(repeated here)

$$\Rightarrow f(\theta, \phi) = \left(\frac{m \alpha \hbar c}{e 2\pi \hbar c^2} \right) \int d\vec{r}'' g(r'') \left(\frac{4\pi}{q^2} \right) e^{i\vec{q} \cdot \vec{r}''}$$

The rest of the simplification follows as before

(c) $\rho(r) \rightarrow$ uniformly charged sphere w/ rad R and total $Q = Ze$

$$\rho(R) = \frac{Ze}{4/3 \pi R^3}$$

Calculating $F(q^2) \propto \int_0^R dr' r' \rho(r') \sin qr'$

$$- \left(\frac{3}{4} \right) \frac{Ze}{\pi R^3} \int_0^R dr' r' \sin qr' \quad \text{given to us}$$

$$x = qr', dx = q dr'$$

$$= \left(\frac{3}{4} \right) \frac{Ze}{\pi R^3} \left(\frac{1}{q^2} \right) \left[\sin x - x \cos x \right]_0^{qR}$$

$$\begin{aligned}
 &= \left(\frac{3}{4}\right) \frac{2e}{\pi R^3} \left(\frac{1}{q^2}\right) [\sin x - x \cos x]_0 \\
 &= \left(\frac{3}{4}\right) \frac{2e}{\pi R^3} \left(\frac{1}{q^2}\right) [\sin qR - qR \cos qR] \\
 F(q^2) &= \left(\frac{4\pi \hbar c}{3e q_b}\right) \frac{3}{4} \left(\frac{2e}{\pi R^3}\right) \left(\frac{1}{q^2}\right) [\sin qR - qR \cos(qR)]
 \end{aligned}$$

$$F(q^2) = \frac{3 \hbar c}{(qR)^3} [\sin qR - qR \cos(qR)]$$

[12]

Note 1: Reasoning form of $f(\theta, \phi)$

\Rightarrow We know for scattering there is an incoming plane wave ϕ_{inc} and an outgoing spherical wave ϕ_{sc} (scattered) (from homogenous & inhom. sol'n to TISE, respectively)

$$\therefore \psi = \phi_{inc} + \phi_{sc}$$

From the asymptotic limit, \vec{r} (detector) $\gg \vec{r}'$ (target size)

$$\psi \propto e^{ikr} + f(\theta, \phi) \left(\frac{e^{ikr}}{r} \right)$$

outgoing sph
wave
(eng. dep) \Rightarrow Thus should be familiar
scattering amp from classical scattering

Finding $f(\theta, \phi)$ from scratch is arduous & time-consuming, starting here we can make an educated guess @ form.

Since ϕ_{sc} is the inhom. sol'n to TISE, we can represent as int egn

$$\phi_{sc}(\vec{r}) \propto \int d\vec{r}' \left(\frac{e^{ik|\vec{r}-\vec{r}'|}}{|\vec{r}-\vec{r}'|} \right) V(\vec{r}') \psi(\vec{r}')$$

Green fn (if you care...)
primed coord
refer to target

\rightarrow taking $\vec{r} \gg \vec{r}'$ asymptotic limit

$$\begin{aligned}
 \phi_{sc} &\propto \left(\frac{e^{ikr}}{r} \right) \int d\vec{r}' e^{-i\vec{k} \cdot \vec{r}'} V(\vec{r}') \psi(\vec{r}') \\
 &\quad \text{↑ desired form}
 \end{aligned}$$

$$\begin{aligned}
 |\vec{r}-\vec{r}'| &= \sqrt{\vec{r}^2 - 2\vec{r} \cdot \vec{r}' + \vec{r}'^2} \\
 \frac{1}{|\vec{r}-\vec{r}'|} &= \frac{1}{r} \frac{1}{\sqrt{1 - \frac{2\vec{r} \cdot \vec{r}'}{r^2}}} = r \left(1 - \frac{1}{2} \frac{2\vec{r} \cdot \vec{r}'}{r^2}\right)^{-\frac{1}{2}} \\
 &= r - \vec{r} \cdot \vec{r}'^{-1}
 \end{aligned}$$

Finally, by reasonable "deduction"

$$= \frac{1}{r} \left(1 + \frac{\vec{r} \cdot \vec{r}'}{r^2}\right) \approx \frac{1}{r}$$

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$$f(\theta, \phi) \propto \int d\vec{r}' e^{-i\vec{k} \cdot \vec{r}'} V(\vec{r}') \psi(\vec{r}')$$

We now proceed w/ the correct constant [c.f. Zetilli 11.62 for proper derivation]

$$\therefore f(\theta, \phi) = -\frac{m}{2\pi\hbar^2} \int e^{-i\vec{k} \cdot \vec{r}'} \hat{V}(\vec{r}') \psi(\vec{r}') d\vec{r}' \quad \text{Scattering Amp}$$

$$= -\frac{m}{2\pi\hbar^2} \int d\vec{r}' \underbrace{\langle \phi | \vec{r}' \rangle}_{e^{i\vec{k} \cdot \vec{r}'}} \hat{V}(\vec{r}') \underbrace{\psi(\vec{r}')}_{\psi(\vec{r})}$$

$$= -\frac{m}{2\pi\hbar^2} \int d\vec{r}' \langle \phi | \hat{V}(\vec{r}) | \vec{r}' \rangle \langle \vec{r}' | \psi \rangle$$

$$f(\theta, \phi) = f(\vec{k}_0, \vec{k})$$

$$= -\frac{m}{2\pi\hbar^2} \langle \phi | \hat{V}(\vec{r}) | \psi \rangle$$

I would memorize this form
and then expand

o Note 2: Convolution thm + F.T.

Solved in one-dimension \Rightarrow can be easily generalized to n-dims

$$\text{First: } \tilde{g}(p) = \int g(r) e^{-ipr} dr \quad (\text{fourier transform})$$

$$= \int dr \left[\int dr' e^{-ipr} u(r') v(r-r') \right]$$

$$\text{Next: let } u(r) = \frac{1}{2\pi} \int ds e^{isr} \tilde{u}(s) \quad (\text{inv fourier transform})$$

$$v(r) = \frac{1}{2\pi} \int dt e^{itr} \tilde{v}(t)$$

$$\tilde{g}(p) = \int dr \left[\int dr' e^{-ipr} \left\{ \frac{1}{2\pi} \int ds e^{isr'} \tilde{u}(s) \right\} \left\{ \frac{1}{2\pi} \int dt e^{it(r-r')} \tilde{v}(t) \right\} \right]$$

Third: and using $\delta(\vec{k}-\vec{k}') = \frac{1}{2\pi} \int dx e^{i(\vec{k}-\vec{k}') \cdot \vec{x}}$ we can simplify

$$= \frac{1}{2\pi} \int dr \underbrace{e^{ir(t-p)}}_{\delta(t-p)} \int ds \tilde{u}(s) \int dt \tilde{v}(t) \left\{ \frac{1}{2\pi} \int dr' e^{ir'(s-t)} \right\} \underbrace{\delta(s-t)}_{\hat{\delta}(s-t)}$$

$$\text{Finally: } \left\{ \tilde{g}(p) = \tilde{u}(p) \cdot \tilde{v}(p) \right\}$$

$$\therefore g(r) = \mathcal{F}^{-1}(\tilde{g}(p)) = \mathcal{F}^{-1}(\tilde{u} \cdot \tilde{v}) = u * v \quad (\text{as expected})$$