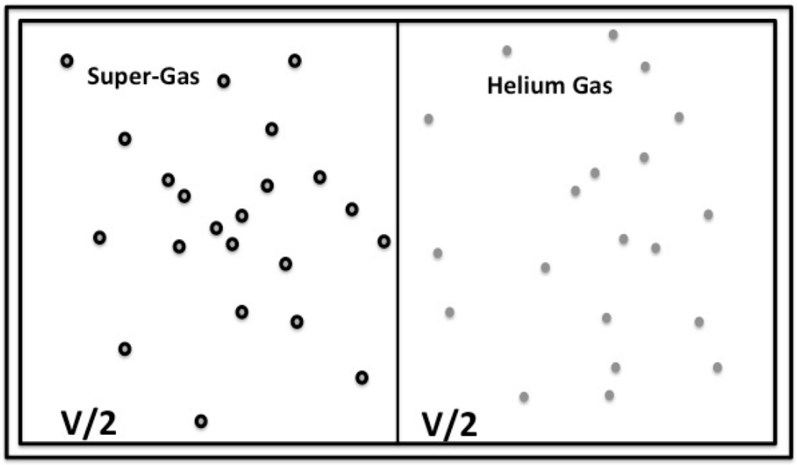
3. (Statistical Mechanics)

Physicists at a top-secret government laboratory discovered a Super-Gas that has the following number of microstates:

$$\Omega = \frac{\kappa^{10N}}{N!}\frac{\pi^{10N}V^{10N}U^{10N}}{h^{20N}c^{10N}}$$

as a function of internal energy U, volume V, and the number of particles N. The constant π, h, and c are pi, Planck's constant, and the speed of light. The constant κ makes the units come out correctly.

The Super-Gas is placed on the left-hand side of the container below and helium gas is placed on the right-hand side. Both have the same number of particles N and occupy the same volume $V/2$. (For simplicity, neglect the heat capacity of the container and assume the container is isolated from the rest of the world. Use a simple model for the helium gas.)



(a) The wall between the gas and the Super-Gas conducts heat but cannot move. At thermal equilibrium what is the ratio of the internal energy of the Super-Gas to that of the helium gas?

(b) Now both sides are at temperature T. The wall is allowed to slide so that the volumes can change slowly. What is the ratio of the final volume of Super-Gas to that of helium?

Solution: *Solution by Jonah Hyman (jthyman@g.ucla.edu)*

(a) Since both gases are at thermal equilibrium, the temperatures of both gases are equal. What we need to do is find the internal energy U of each gas in terms of the temperature T of each gas.

Let's start with the Super-Gas. All we are given is the number of microstates

$$\Omega_{\text{SG}} = \frac{1}{N!}\left(\frac{\kappa\pi V_{\text{SG}} U_{\text{SG}}}{h^2 c}\right)^{10N} \tag{22}$$

where N, V_{SG}, and U_{SG} are the number of particles, the volume, and the internal energy of the Super-Gas, respectively. The only thermodynamic quantity that directly relates to the number of microstates is the entropy, which is defined by

$$S = k \ln \Omega \quad \text{where } k \text{ is the Boltzmann constant} \tag{23}$$

Taking the logarithm of (22) and using the log rules $\ln(ab) = \ln a + \ln b$ and $\ln(x^a) = a \ln a$, we get

$$\begin{aligned}
S_{\text{SG}} &= k \ln\left[\frac{1}{N!}\left(\frac{\kappa\pi V_{\text{SG}} U_{\text{SG}}}{h^2 c}\right)^{10N}\right] \\
&= k\left[10N \ln\left(\frac{\kappa\pi V_{\text{SG}} U_{\text{SG}}}{h^2 c}\right) - \ln N!\right] \\
&= k\left[10N \ln U_{\text{SG}} + 10N \ln\left(\frac{\kappa\pi V_{\text{SG}}}{h^2 c}\right) - \ln N!\right] \\
S_{\text{SG}} &= 10Nk \ln U_{\text{SG}} + 10Nk \ln V_{\text{SG}} + 10Nk \ln\left(\frac{\kappa\pi}{h^2 c}\right) - k \ln N! \qquad (24)
\end{aligned}$$

The temperature is defined in terms of the entropy using the relation

$$\frac{1}{T} \equiv \left.\frac{\partial S}{\partial U}\right|_{V,N} \tag{25}$$

This definition can be memorized on its own, or it can be derived from the first law of thermodynamics:

$$\begin{aligned}
dU = T\,dS - p\,dV \qquad &\Longrightarrow \qquad dS = \frac{1}{T}\,dU + \frac{p}{T}\,dV \\
&\Longrightarrow \qquad \left.\frac{\partial S}{\partial U}\right|_{V,N} = \frac{1}{T} \qquad (26)
\end{aligned}$$

Applying (25) to (24), we get

$$\begin{aligned}
\frac{1}{T_{\text{SG}}} &= \left.\frac{\partial S_{\text{SG}}}{\partial U_{\text{SG}}}\right|_{V,N} \\
&= \frac{\partial}{\partial U_{\text{SG}}}\left(10Nk \ln U_{\text{SG}}\right) \quad \text{since the other terms in } S_{\text{SG}} \text{ are constant in } U_{\text{SG}} \\
\frac{1}{T_{\text{SG}}} &= \frac{10Nk}{U_{\text{SG}}} \qquad (27)
\end{aligned}$$

Therefore, the internal energy of the Super-Gas is equal to

$$U_{\text{SG}} = 10NkT_{\text{SG}} \tag{28}$$

where N is the number of particles in the Super-Gas, and T_{SG} is the temperature of the Super-Gas.

We now move on to the helium gas. Since helium is a noble gas, the simplest model for the helium gas is an ideal monatomic gas. You might already know the formula of the internal energy of an ideal monatomic gas. As applied to the helium gas, it is

$$U_{\mathrm{He}} = \frac{3}{2} N k T_{\mathrm{He}}$$

If you don't know this formula, you can derive it by finding the partition function for a single particle. Using the phase-space formalism for classical noninteracting point particles, the partition function for a single particle in a monatomic ideal gas is

$$Z_1 = \int \frac{d^3p\, d^3x}{h^3} e^{-\beta E} \quad \text{where } \beta \equiv \frac{1}{kT} \tag{29}$$

The energy of a single particle is its nonrelativistic kinetic energy:

$$E = \frac{p^2}{2m} \tag{30}$$

Plugging this in and carrying out the spatial integral for the partition function, we get

$$\begin{aligned}
Z_1 &= \int \frac{d^3p\, d^3x}{h^3} e^{-\beta\left(\frac{p^2}{2m}\right)} \\
&= \int \frac{d^3p\, d^3x}{h^3} e^{-\beta p^2/(2m)} \\
&= \frac{V}{h^3} \int d^3p\, e^{-\beta p^2/(2m)} \quad \text{since} \int d^3x = V \\
&= \frac{V}{h^3} \int_{-\infty}^{\infty} dp_x \int_{-\infty}^{\infty} dp_y \int_{-\infty}^{\infty} dp_z\, e^{-\beta(p_x^2+p_y^2+p_z^2)/(2m)}] \\
&= \frac{V}{h^3} \left[\int_{-\infty}^{\infty} dp_x\, e^{-\beta p_x^2/(2m)}\right] \left[\int_{-\infty}^{\infty} dp_y\, e^{-\beta p_y^2/(2m)}\right] \left[\int_{-\infty}^{\infty} dp_z\, e^{-\beta p_z^2/(2m)}\right]
\end{aligned} \tag{31}$$

Each of these three integrals is the same Gaussian integral, which can be calculated by a change of coordinates $u \equiv (\beta/(2m))^{1/2}\, p = (2mkT)^{-1/2}\, p$:

$$\begin{aligned}
\int_{-\infty}^{\infty} dp\, e^{-\beta p^2/(2m)} &= (2mkT)^{1/2} \int_{-\infty}^{\infty} du\, e^{-u^2} \\
&= (2\pi mkT)^{1/2} \quad \text{using the result} \int_{-\infty}^{\infty} du\, e^{-u^2} = \pi^{1/2}
\end{aligned} \tag{32}$$

Plugging this back into (31), we get

$$\begin{aligned}
Z_1 &= \frac{V}{h^3} (2mkT)^{3/2} \\
&= V \left(\frac{2\pi mkT}{h^2}\right)^{3/2} \\
Z_1 &= \frac{V}{\lambda^3} \quad \text{for} \quad \lambda \equiv \left(\frac{h^2}{2\pi mkT}\right)^{1/2}
\end{aligned} \tag{33}$$

The quantity λ is called the thermal wavelength. In terms of $\beta \equiv 1/(kT)$, the partition function for a single particle in the helium gas is

$$Z_{\mathrm{He},1} = V_{\mathrm{He}} \left(\frac{2\pi m}{\beta_{\mathrm{He}} h^2}\right)^{3/2} \tag{34}$$

The internal energy of a single particle of the helium gas is given by the formula

$$U_{\mathrm{He},1} = \frac{\partial \ln(Z_{\mathrm{He},1})}{\partial \beta_{\mathrm{He}}} = \frac{1}{Z_{\mathrm{He},1}} \frac{\partial Z_{\mathrm{He},1}}{\partial \beta_{\mathrm{He}}} \tag{35}$$

Taking the derivative of (34), we get

$$\begin{aligned}
U_{\mathrm{He},1} &= \frac{1}{V_{\mathrm{He}} \left(\frac{2\pi m}{\beta_{\mathrm{He}} h^2}\right)^{3/2}} \frac{\partial}{\partial \beta_{\mathrm{He}}} \left[V_{\mathrm{He}} \left(\frac{2\pi m}{\beta_{\mathrm{He}} h^2} \right)^{3/2} \right] \\
&= \frac{1}{V_{\mathrm{He}} \left(\frac{2\pi m}{\beta_{\mathrm{He}} h^2}\right)^{3/2}} \left[\frac{3}{2} V_{\mathrm{He}} \left(\frac{2\pi m}{h^2} \right)^{3/2} \frac{1}{\beta_{\mathrm{He}}^{5/2}} \right] \\
&= \frac{3}{2} \frac{1}{\beta_{\mathrm{He}}} \\
U_{\mathrm{He},1} &= \frac{3}{2} k T_{\mathrm{He}} \qquad (36)
\end{aligned}$$

This is the internal energy of a single particle of the helium gas. To get the internal energy of the entire gas, multiply by the number of particles of the helium gas N:

$$U_{\mathrm{He}} = \frac{3}{2} N k T_{\mathrm{He}} \tag{37}$$

We now have the internal energies of both the Super-Gas (28) and the helium gas (37). Dividing (28) by (37), we get that

$$\frac{U_{\mathrm{SG}}}{U_{\mathrm{He}}} = \frac{10 N k T_{\mathrm{SG}}}{\frac{3}{2} N k T_{\mathrm{He}}} \tag{38}$$

We are given the the number of particles in the Super-Gas are equal to the number of particles in the helium gas. At thermal equilibrium, the temperature of both gases is the same: $T \equiv T_{\mathrm{SG}} = T_{\mathrm{He}}$. Therefore,

$$\frac{U_{\mathrm{SG}}}{U_{\mathrm{He}}} = \frac{10 N k T}{\frac{3}{2} N k T}$$

or

$$\boxed{\frac{U_{\mathrm{SG}}}{U_{\mathrm{He}}} = \frac{20}{3}} \tag{39}$$

(b) Once the wall is allowed to slide, the system reaches mechanical equilibrium. (It is already in thermal equilibrium, since the temperature of both gases is the same.) In mechanical equilibrium, the wall is at rest, so the force exerted on either side of the wall is equal. The pressure of each gas is the force that gas exerts on the wall per unit area of the wall. Therefore, in mechanical equilibrium, the pressure of each gas is the same. We need to find an expression for the pressure p of each gas.

Let's start with the Super-Gas. From the first law of thermodynamics, we can write an expression for the pressure p in terms of a partial derivative of the entropy S_{SG} that we found in part (a).

$$\begin{aligned}
dU = T\,dS - p\,dV \qquad &\Longrightarrow \qquad dS = \frac{1}{T}\,dU + \frac{p}{T}\,dV \\
&\Longrightarrow \qquad \left.\frac{\partial S}{\partial V}\right|_{U,N} = \frac{p}{T} \qquad (40)
\end{aligned}$$

The pressure of the Super-Gas is therefore given by

$$p_{\text{SG}} = T_{\text{SG}} \left.\frac{\partial S_{\text{SG}}}{\partial V_{\text{SG}}}\right|_{U_{\text{SG}},N} \tag{41}$$

The volume-dependent part of the entropy of the Super-Gas S_{SG} (24) is equal to $10Nk\ln V_{\text{SG}}$. Taking the derivative of this part, we get

$$\begin{aligned} p_{\text{SG}} &= T_{\text{SG}} \frac{\partial}{\partial V_{\text{SG}}} \left(10Nk\ln V_{\text{SG}}\right) \\ &= T_{\text{SG}} \frac{10Nk}{V_{\text{SG}}} \\ p_{\text{SG}} &= \frac{10NkT_{\text{SG}}}{V_{\text{SG}}} \qquad (42) \end{aligned}$$

The helium gas is an ideal gas, so its pressure is given by the ideal gas law:

$$\begin{aligned} p_{\text{He}}V_{\text{He}} &= NkT_{\text{He}} \\ \Longrightarrow \qquad p_{\text{He}} &= \frac{NkT_{\text{He}}}{V_{\text{He}}} \qquad (43) \end{aligned}$$

Setting p_{SG} (42) equal to p_{He} (43), we get

$$\frac{10NkT_{\text{SG}}}{V_{\text{SG}}} = \frac{NkT_{\text{He}}}{V_{\text{He}}} \tag{44}$$

We are given the the number of particles in the Super-Gas are equal to the number of particles in the helium gas. We are also given that the temperature of the two gases is the same. With this information, we can solve for the ratio of the final volume of the Super-Gas to that the helium gas:

$$\begin{aligned} \frac{10NkT}{V_{\text{SG}}} &= \frac{NkT}{V_{\text{He}}} \\ \frac{10}{V_{\text{SG}}} &= \frac{1}{V_{\text{He}}} \end{aligned}$$

or

$$\boxed{\frac{V_{\text{SG}}}{V_{\text{He}}} = 10} \tag{45}$$