9. (Electromagnetism)

A conductor is often modeled with a simple Ohm's law:

$$\vec{j} - \sigma_0 \vec{E}$$

in both real and Fourier space where σ_0 is the conductivity. However, a conductor is more accurately modeled with a modified frequency dependent conductivity $\sigma = \sigma_0/(1 - i\omega \frac{\sigma_0}{\epsilon_0 \omega_p^2})$ (this frequency dependent conductivity assumes quantities vary as $e^{-i\omega t}$), which is equivalent to the modified Ohm's law

$$\frac{\sigma_0}{\epsilon_0 \omega_p^2} \frac{\partial \vec{j}}{\partial t} + \vec{j} = \sigma_0 \vec{E}$$

where ω_p is the plasma frequency of the conduction electrons, $\omega_p^2 \equiv \frac{e^2 n_0}{\epsilon_0 m}$, n_0 is the density of conduction electrons, and ϵ_0 is the permittivity of free space. Finally, at $t = 0$, a small amount of excess charge Q is uniformly distributed throughout a sphere of radius r_0 with conductivity σ_0 at $t = 0$.

(a) Using combinations of the relevant Maxwell's equations and the continuity equation, derive the equation for the charge density inside the sphere and then obtain the solution for it for the correct initial conditions. Your answer should depend on r_0, ϵ_0, σ_0, ω_p, and t.

(b) For copper, the density of conduction electrons is $n_0 = 0.85 \times 10^{29} \mathrm{m}^{-3}$ and the conductivity is $6 \times 10^7 \mathrm{S/m}$. Show that for these parameters, $r \gg \epsilon_0^2 \omega_p^2 / \sigma_0^2$. Under this condition, what is the formula and the time in seconds that it takes for the charge density to decrease to $1/e$ of its initial value at any location in the sphere?

Solution: *Solution by Jonah Hyman (jthyman@g.ucla.edu)*

(a) We are looking for a differential equation for the charge density ρ. The modified Ohm's law we are given is in terms of the charge density ρ, the current density $\mathbf{j}$, and the electric $\mathbf{E}$. The charge density and the current density are related by the continuity equation; the charge density and the electric field are related by Gauss' law. All in all, we have three formulas to use for this problem:

$$\frac{\sigma_0}{\epsilon_0\omega_p^2}\frac{\partial \mathbf{j}}{\partial t} + \mathbf{j} = \sigma_0\mathbf{E} \quad \text{(modified Ohm's law)} \tag{306}$$

$$\frac{\partial \rho}{\partial t} + \nabla\cdot\mathbf{j} = 0 \quad \text{(continuity equation)} \tag{307}$$

$$\nabla\cdot\mathbf{E} = \frac{\rho}{\epsilon_0} \quad \text{(Gauss' law)} \tag{308}$$

The continuity equation and Gauss' law include the divergences of $\mathbf{j}$ and $\mathbf{E}$. To put the modified Ohm's law in this form, take the divergence of it:

$$\frac{\sigma_0}{\epsilon_0\omega_p^2}\frac{\partial}{\partial t}\left(\nabla\cdot\mathbf{j}\right) + \nabla\cdot\mathbf{j} = \sigma_0\,\nabla\cdot\mathbf{E}$$

Then use the continuity equation and Gauss' law to substitute for $\nabla\cdot\mathbf{j}$ and $\nabla\cdot\mathbf{E}$:

$$\begin{aligned}
\frac{\sigma_0}{\epsilon_0\omega_p^2}\frac{\partial}{\partial t}\left(-\frac{\partial\rho}{\partial t}\right) + \left(-\frac{\partial\rho}{\partial t}\right) &= \sigma_0\left(\frac{\rho}{\epsilon_0}\right)\\
\frac{\sigma_0}{\epsilon_0\omega_p^2}\frac{\partial^2\rho}{\partial t^2} + \frac{\partial\rho}{\partial t} + \frac{\sigma_0}{\epsilon_0}\rho &= 0\\
\frac{\partial^2\rho}{\partial t^2} + \frac{\epsilon_0\omega_p^2}{\sigma_0}\frac{\partial\rho}{\partial t} + \omega_p^2\rho &= 0
\end{aligned} \tag{309}$$

The differential equation for the charge density is therefore

$$\boxed{\frac{\partial^2\rho}{\partial t^2} + \gamma\frac{\partial\rho}{\partial t} + \omega_p^2\rho = 0 \quad \text{for} \quad \gamma \equiv \frac{\epsilon_0\omega_p^2}{\sigma_0}} \tag{310}$$

This is the equation of a damped harmonic oscillator. It can be solved by using the ansatz for an linear ordinary differential equation with constant coefficients. (Note that even though this is a partial differential equation in the time variable, there are no spatial derivatives in it, so we can treat it as an ordinary differential equation.) This ansatz is

$$\rho(t) \propto e^{\alpha t} \quad \text{for a constant } \alpha \tag{311}$$

To find possible values of α, plug this ansatz into the differential equation to get

$$\begin{aligned}
\left(\alpha^2 + \gamma\alpha + \omega_p^2\right)e^{\alpha t} &= 0\\
\alpha^2 + \gamma\alpha + \omega_p^2 &= 0
\end{aligned} \tag{312}$$

This equation can be solved using the quadratic formula:

$$\begin{aligned}
\alpha_\pm &= \frac{-\gamma \pm \sqrt{\gamma^2 - 4\omega_p^2}}{2}\\
&= -\frac{\gamma}{2} \pm \sqrt{\left(\frac{\gamma}{2}\right)^2 - \omega_p^2}\\
&= -\frac{\gamma}{2} \pm \eta \quad \text{for} \quad \eta \equiv \sqrt{\left(\frac{\gamma}{2}\right)^2 - \omega_p^2}
\end{aligned} \tag{313}$$

The general solution for this differential equation is a linear combination of the solutions with $\alpha_\pm$:

$$\begin{aligned} \rho(t) &= Ae^{\alpha_+ t} + Be^{\alpha_- t} \\ \rho(t) &= e^{-(\gamma/2)t}\left(Ae^{\eta t} + Be^{-\eta t}\right) \end{aligned} \tag{314}$$

To solve for A and B, use the initial conditions. The initial charge density is equal to the total charge, divided by the volume of the sphere:

$$\rho(0) = \rho_0 \equiv \frac{Q}{\frac{4}{3}\pi r_0^3} \tag{315}$$

Plugging in this initial condition gives us

$$\begin{aligned} \rho_0 &= e^{-(\gamma/2)t}\left(Ae^{\eta t} + Be^{-\eta t}\right)\big|_{t=0} \\ \rho_0 &= A + B \end{aligned} \tag{316}$$

We may also assume that $\frac{d\rho}{dt}\big|_{t=0} = 0$, i.e., that the charge is "held" for a moment and then released. This implies that

$$\begin{aligned} 0 = \frac{d\rho}{dt}\bigg|_{t=0} &= A\left(-\frac{\gamma}{2}+\eta\right)e^{-(\gamma/2)t+\eta t} + B\left(-\frac{\gamma}{2}+-\eta\right)e^{-(\gamma/2)t-\eta t}\bigg|_{t=0} \\ 0 &= A\left(-\frac{\gamma}{2}+\eta\right) + B\left(-\frac{\gamma}{2}-\eta\right) \end{aligned} \tag{317}$$

Equations (316) and (317) are a system of two equations for two unknowns (A and B). We can use any method to solve for A and B:

$$\begin{aligned} \rho_0\left(\frac{\gamma}{2}+\eta\right) &= A\left(\frac{\gamma}{2}+\eta\right) + B\left(\frac{\gamma}{2}+\eta\right) \\ 0 &= A\left(-\frac{\gamma}{2}+\eta\right) + B\left(-\frac{\gamma}{2}-\eta\right) \\ \implies \rho_0\left(\frac{\gamma}{2}+\eta\right) &= 2A\eta \quad \text{adding the two equations} \\ \implies A &= \frac{\rho_0}{2}\left(1+\frac{\gamma/2}{\eta}\right) \end{aligned} \tag{318}$$

$$\begin{aligned} \rho_0\left(\frac{\gamma}{2}-\eta\right) &= A\left(\frac{\gamma}{2}-\eta\right) + B\left(\frac{\gamma}{2}-\eta\right) \\ 0 &= A\left(-\frac{\gamma}{2}+\eta\right) + B\left(-\frac{\gamma}{2}-\eta\right) \\ \implies \rho_0\left(\frac{\gamma}{2}-\eta\right) &= -2B\eta \quad \text{adding the two equations} \\ \implies B &= \frac{\rho_0}{2}\left(1-\frac{\gamma/2}{\eta}\right) \end{aligned} \tag{319}$$

Plugging in our answers for A and B into the ansatz (314), we get the answer

$$\boxed{\rho(t) = \frac{\rho_0}{2}e^{-(\gamma/2)t}\left[\left(1+\frac{\gamma/2}{\eta}\right)e^{\eta t} + \left(1-\frac{\gamma/2}{\eta}\right)e^{-\eta t}\right]} \tag{320}$$

with variables defined as follows:

$$\rho_0 \equiv \frac{Q}{\frac{4}{3}\pi r_0^3}; \qquad \gamma \equiv \frac{\epsilon_0\omega_p^2}{\sigma_0}; \qquad \eta \equiv \sqrt{\left(\frac{\gamma}{2}\right)^2 - \omega_p^2} \tag{321}$$

Two notes are in order. First, η is imaginary if $\gamma/2 < \omega_p$, i.e., if $\frac{\epsilon_0\omega_p}{2\sigma_0} < 1$. In this case, $\rho(t)$ is given by the real part of the above equation. Second, this equation tells us that the charge density remains spatially uniform at all times.

(b) Before plugging in the numbers, let's simplify the expression:

$$\frac{\epsilon_0^2\omega_p^2}{\sigma_0^2} = \frac{\epsilon_0^2\left(\frac{e^2 n_0}{\epsilon_0 m}\right)}{\sigma_0^2} = \frac{\epsilon_0 e^2 n_0}{m\sigma_0^2} \tag{322}$$

The problem gives us some of these values:

$$n_0 = 0.85\cdot 10^{29}\,\mathrm{m}^{-3} \quad \text{and} \quad \sigma_0 = 6\cdot 10^7\,\mathrm{S/m} \tag{323}$$

This problem was given on an open-book, open-note exam, so the other values could be looked up. Here, e is the charge of an electron, and m is the mass of an electron.

$$\epsilon_0 = 8.854\cdot 10^{-12}\,\mathrm{F/m}; \qquad e = 1.602\cdot 10^{-19}\,\mathrm{C}; \qquad m = 9.109\cdot 10^{-31}\,\mathrm{kg} \tag{324}$$

Plugging all these values into a calculator, we get

$$\frac{\epsilon_0^2\omega_p^2}{\sigma_0^2} = \frac{\epsilon_0 e^2 n_0}{m\sigma_0^2} \approx 5.9\cdot 10^{-6} \ll 4 \tag{325}$$

Before proceeding, it might be useful to think about how we would go about doing an approximate calculation on a closed-book exam without a formula sheet. Instead of memorizing the values in (324), it is probably easier and more useful to memorize the following values:

$$\mu_0 \approx 4\pi\cdot 10^{-7}\,\mathrm{H/m} \approx 10^{-6}\,\mathrm{H/m}; \qquad e \approx 1.6\cdot 10^{-19}\,\mathrm{C} \approx 2\cdot 10^{-19}\,\mathrm{C}; \qquad m_{\text{electron}} \approx 511\,\mathrm{keV}/c^2 \tag{326}$$

Note that one electron-volt (eV) is equal to $1.6\cdot 10^{-19}$ J, so you don't need to memorize a new conversion factor, and we can rewrite the mass of the electron as

$$m_{\text{electron}} \approx 511\cdot 10^3\,\mathrm{J}\,e/c^2 \approx \frac{1}{2}\cdot 10^6\,\mathrm{J}\,e/c^2 \tag{327}$$

where, in a slight abuse of notation, we use e to mean the number $1.6\cdot 10^{-19}$, without any units.

Equation (322) can be reshuffled to make use of the easier-to-memorize numerical factors:

$$\begin{aligned}\frac{\epsilon_0^2\omega_p^2}{\sigma_0^2} &= \frac{\epsilon_0 e^2 n_0}{m\sigma_0^2}\\ &= \frac{\epsilon_0 c^2 e^2 n_0}{mc^2\sigma_0^2}\\ &= \frac{e n_0}{\mu_0(\frac{mc^2}{e})\sigma_0^2} \quad \text{using } c^2 = \frac{1}{\epsilon_0\mu_0}\end{aligned} \tag{328}$$

Substituting in the approximate numerical values (and ignoring units now that everything is in SI units), we get the following:

$$\begin{aligned}\frac{\epsilon_0^2\omega_p^2}{\sigma_0^2} &\approx \frac{\left(2\cdot 10^{-19}\right)\left(10^{29}\right)}{10^{-6}\cdot\left(\frac{1}{2}\cdot 10^6\right)\cdot\left(\frac{1}{2}\cdot 10^8\right)^2}\\ &\approx 16\cdot 10^{-6}\\ &\approx 10^{-5}\end{aligned} \tag{329}$$

which is enough to tell us that $\frac{\epsilon_0^2\omega_p^2}{\sigma_0^2} \ll 4$.

Recall from part (a) that $\gamma \equiv \frac{\epsilon_0 \omega_p^2}{\sigma_0}$. So the condition that $\frac{\epsilon_0^2 \omega_p^2}{\sigma_0^2} \ll 4$ is equivalent to the condition that

$$\frac{\gamma^2}{\omega_p^2} \ll 4$$

This is in turn equivalent to the condition that

$$\frac{\gamma}{2} \ll \omega_p \tag{330}$$

Recall from part (a) that $\eta \equiv \sqrt{\left(\frac{\gamma}{2}\right)^2 - \omega_p^2}$. Given the condition $\frac{\gamma}{2} \ll \omega_p$, we can approximate η by

$$\eta \approx i\omega_p \quad \text{with} \quad |\eta| \gg \frac{\gamma}{2} \tag{331}$$

Plugging this condition into our part (a) answer (320), we get

$$\rho(t) = \frac{\rho_0}{2} e^{-(\gamma/2)t} \left[e^{i\omega_p t} + e^{-i\omega_p t} \right]$$

$$\boxed{\rho(t) = \rho_0 e^{-(\gamma/2)t} \cos(\omega_p t) \quad \text{with} \quad \rho_0 \equiv \frac{Q}{\frac{4}{3}\pi r_0^3}; \qquad \gamma \equiv \frac{\epsilon_0 \omega_p^2}{\sigma_0}} \tag{332}$$

Because of the initial exponential factor, the time it takes for the charge density to decrease to $1/e$ of its initial value is equal to the time constant in that exponential factor, which is $\tau \equiv 2/\gamma$. Plugging in the known expressions for γ and ω_p, we get

$$\begin{aligned}
\tau &= \frac{2}{\gamma} \\
&= \frac{2\sigma_0}{\epsilon_0 \omega_p^2} \\
&= \frac{2\sigma_0}{\epsilon_0 \left(\frac{e^2 n_0}{\epsilon_0 m}\right)}
\end{aligned}$$

$$\boxed{\tau = \frac{2\sigma_0 m}{e^2 n_0}} \tag{333}$$

We can plug in the exact numbers to get

$$\boxed{\tau \approx 5.0 \cdot 10^{-14}\,\mathrm{s}} \tag{334}$$

Alternatively, we can do the approximate calculation as follows, using the memorized values mentioned above:

$$\begin{aligned}
\tau &= \frac{2\sigma_0 \left(\frac{mc^2}{e}\right)}{e n_0 c^2} \\
&\approx \frac{2\left(\frac{1}{2}\cdot 10^8\right)\left(\frac{1}{2}\cdot 10^6\right)}{\left(2\cdot 10^{-19}\right)\left(10^{29}\right)\left(2\cdot 10^8\right)^2}\ \mathrm{s} \\
&\approx \frac{1}{16}\cdot 10^{-12}\,\mathrm{s} \\
&\approx 0.06\cdot 10^{-12}\,\mathrm{s} \\
&\approx 6\cdot 10^{-14}\,\mathrm{s}
\end{aligned} \tag{335}$$

which matches the exact result pretty well. In either case, the point is that any excess volume charge placed on a conducting sphere becomes neutral almost instantaneously. This justifies the typical electrostatics assumption that all charge on a conductor moves to the surface.