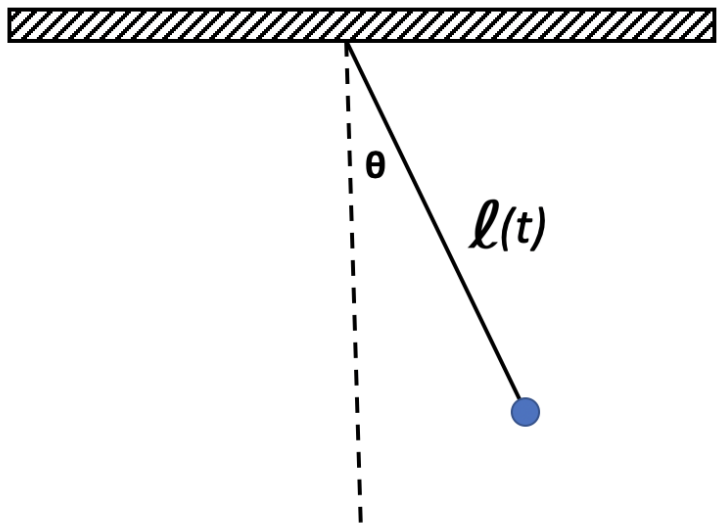
Winter 2022
Written Graduate Comprehensive Exam in Physics
UCLA
February 19, 2022

4 hours
One day
Six problems
Closed Book. Closed Notes.

Write Student Code Number:_____

1. Consider a pendulum consisting of a point mass $m$ attached to a string of slowly increasing length $\ell(t)$. The change of length versus time is $\dot{\ell}$. The motion is confined to a plane and we assume that $|\ell/\dot{\ell}|$ is much greater than the period of the oscillation.




(a) (6 pts) Find the Lagrangian $L(\theta, \dot{\theta}, t)$ and the Hamiltonian $H(\theta, p_\theta, t)$ of the system, where $\theta$ is the angle of the string relative to downward, $p_\theta$ is the conjugate momentum, and $\dot{\theta}$ is the change of $\theta$ versus time.

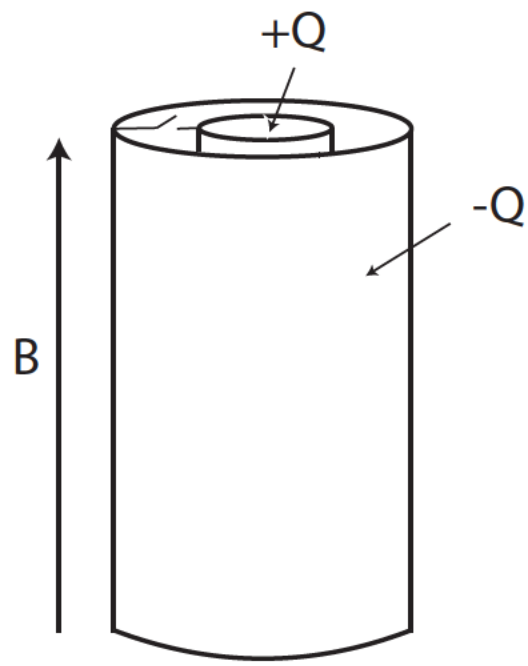
(b) (4 pts) Show why $H$ is or is not equal to the total energy $E$ of the pendulum.

(c) (4 pts) Derive the equation of motion for $\theta$ in the form of a differential equation. When the change of length versus time $\dot{\ell} = 0$, what is the angular frequency $\omega_0$ of small oscillations?

(d) (6 pts) Suppose the length varies slowly so that $|\dot{\ell}/\ell| \ll \omega_0$, the angular frequency of oscillation. Show that as the length varies, the amplitude of small oscillations varies proportionally to $\ell^a$ and find $a$, the numerical exponent. (Hint: Consider using $\int p_\theta d\theta$ as an adiabatic invariant.)

2. Consider a cylindrical capacitor of length $L$ and moment of inertia $I$ with charge $+Q$ on the inner cylindrical metallic shell of radius $a$ and charge $-Q$ on the outer cylindrical metallic shell of radius $b$. The capacitor is immersed in a uniform magnetic field $\vec{B}$, which points along the symmetry axis of the capacitor. There is a vacuum between the plates, which are held fixed to one another (e.g., by small insulating posts so that they will rotate together as a rigid body). A thin stainless steel wire with a switch is attached across the two plates of the capacitor. At $t = 0$, the switch is closed, and current flows from one plate to the other through a resistive wire (you can assume resistance R). This current creates a torque on the capacitor and it rotates.

Once the capacitor is fully discharged (the charge on each plate is zero), what is the magnitude and direction of the angular velocity, $\vec{\omega}$, of the capacitor? (For simplicity, ignore fringing fields and assume that the thin stainless wire does not break the symmetry of the system when computing fields.)

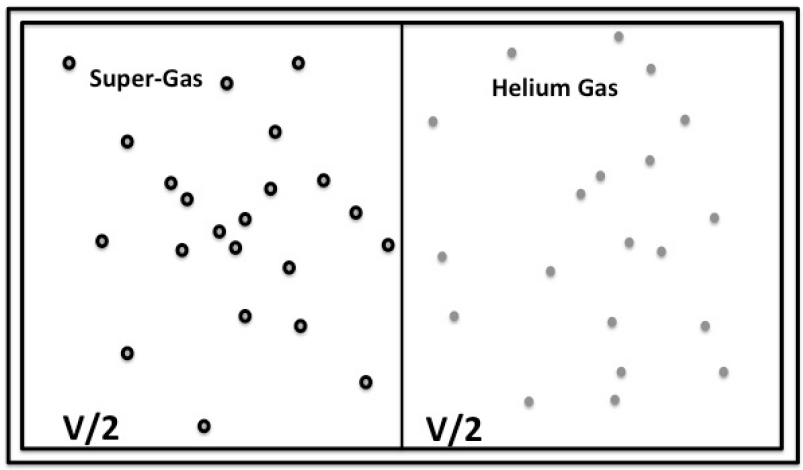


3. Physicists at a top-secret government laboratory discovered a Super-Gas that has the following number of microstates:

$$\Omega = \frac{\kappa^{10N}}{N!} \frac{\pi^{10N}\ V^{10N}\ U^{10N}}{h^{20N}\ c^{10N}}$$

as a function of internal energy $U$, volume $V$, and the number of particles $N$. The constants $\pi$, $h$, and $c$ are pi, Planck's constant, and the speed of light. The constant $\kappa$ makes the units come out correctly.

The Super-Gas is placed on the left-hand side of the container below and helium gas is placed on the right-hand side. Both have the same number of particles $N$ and occupy the same volume $V/2$. (For simplicity, neglect the heat capacity of the container and assume the container is isolated from the rest of the world. Use a simple model for the helium gas.)




a) The wall between the gas and the Super-Gas conducts heat but cannot move. At thermal equilibrium what is the ratio of the internal energy of the Super-Gas to that of the helium gas?

b) Now both sides are at temperature $T$. The wall is allowed to slide so that the volumes can change slowly. What is the ratio of the final volume of Super-Gas to that of helium?

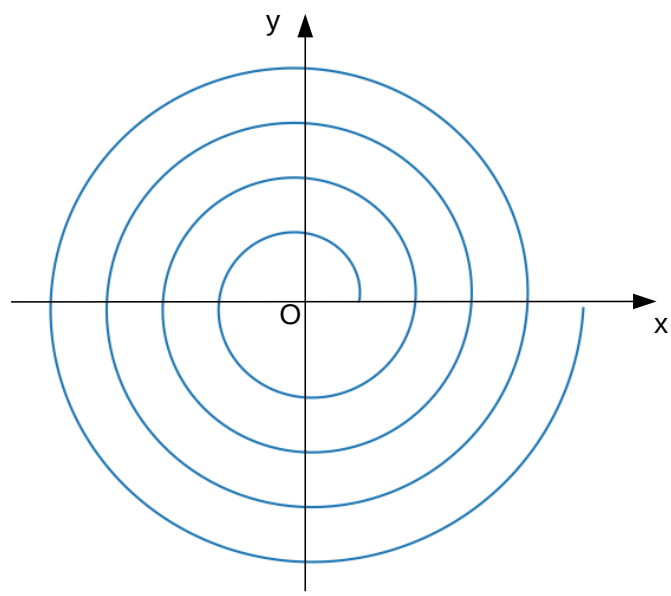
4. Part of a circuit consists of an Archimedean spiral which is given by the following equation in cylindrical coordinates:

$$\rho = a\phi,$$

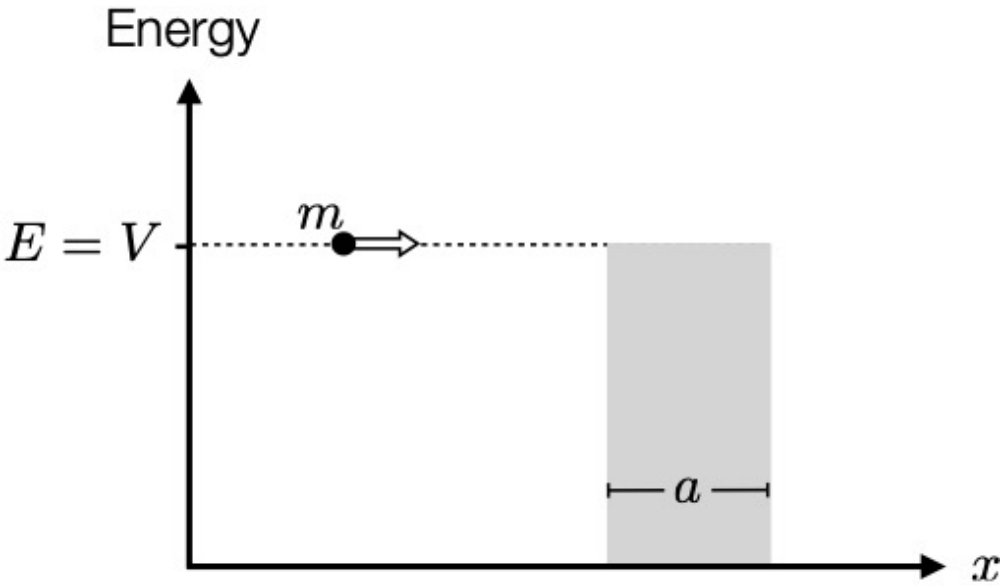
where $a > 0$, $N$ is a positive integer $> 1$, and $2\pi < \phi < 2\pi N$ as shown below for $N = 5$. The origin is marked by O.

A steady current $I$ is running counter-clockwise in the spiral. Find the contribution to the magnetic field (direction and magnitude) from the spiral at the origin as a function of $N$, $a$, $I$, and physical constants.

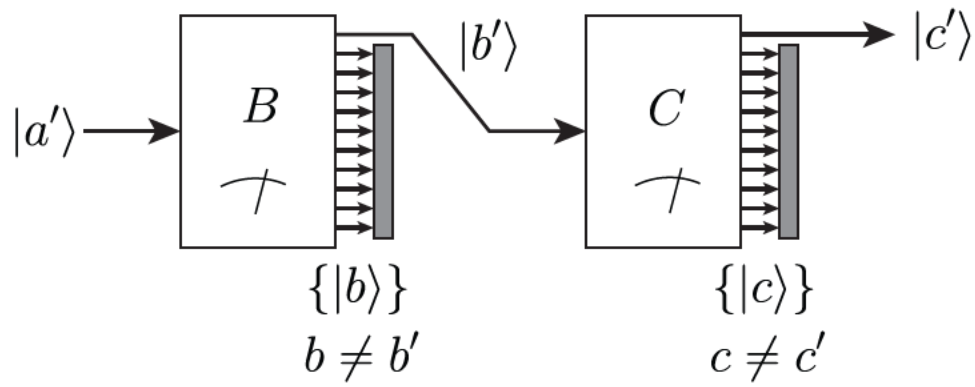
For simplicity, ignore the contribution from the rest of the circuit and assume no charge is building up at the ends.



5. Starting with the time-independent Schrödinger equation, work out the fraction of incident particles transmitted through a rectangular one-dimensional potential barrier in the case shown below, where the energy $E$ of the incident particles is equal to the barrier height $V$. Let the particles have mass $m$ and let the barrier width be $a$.



6. Consider a set of quantum operators $A$, $B$, and $C$ that are mutually incompatible, which is to say $[A, B] \neq 0$, $[B, C] \neq 0$, and $[A, C] \neq 0$. Eigenkets and eigenvalues (assumed to be non-degenerate) of these operators will be denoted by lower-case letters for the upper-case operators (as in $B|b\rangle = b|b\rangle$, etc.), and we will consider the sequential measurement of an eigenstate of the $A$ operator, $|a'\rangle$, in bases of $B$ and $C$, as shown below.




As shown above, an input state $|a'\rangle$ is measured in the $B$ basis and sent to the next measurement only if the eigenvalue of the measurement of $B$ is $b'$, followed by a similar process in the $C$ basis. All other measurement outcomes result in the state being filtered out.

a) (4 pts) Write expressions for the expansion coefficients $c_b$ for the state $|a'\rangle$ in the $B$ basis,

$$|a'\rangle = \sum_b c_b |b\rangle$$

and $c_c$ for the state $|b'\rangle$ in the $C$ basis,

$$|b'\rangle = \sum_c c_c |c\rangle.$$

b) (4 pts) In terms of these coefficients, what is the probability that in the experimental setup above the input state ($|a'\rangle$) results in final measurement outcome $|c'\rangle$?

c) (4 pts) What is the sum of the probabilities of outcome $|c'\rangle$ if you add up contributions from the complete set of choices for the intermediate eigenket $|b'\rangle$?

d) (8 pts) If we were instead to bypass the intermediate measurement $B$ and feed $|a'\rangle$ directly into the $C$ measurement, would the probability of obtaining outcome $|c'\rangle$ have to be the same as the answer to part (c)? Briefly explain or show why or why not.

This problem was from the 2017 comp exam.

$\boxed{\text{Solution to \#1}}$

a) $L = T - V$


$$T = \frac{1}{2}m\dot{l}^2 + \frac{1}{2}I\dot{\omega}^2$$

$$= \frac{1}{2}m\dot{l}^2 + \frac{1}{2}ml^2\dot{\theta}^2$$

$$h = l - l\cos\theta$$

$$V = mgh = mgl(1-\cos\theta)$$

$$= \cancel{mgl} - mgl\cos\theta$$

ignore overall constant

$$\boxed{L = \frac{1}{2}m\dot{l}^2 + \frac{1}{2}ml^2\dot{\theta}^2 + mgl\cos\theta}$$

$$H = p_\theta\dot{\theta} - L$$

$$\left(p_\theta = \frac{\partial L}{\partial\dot{\theta}} = ml^2\dot{\theta} \Rightarrow \dot{\theta} = \frac{p_\theta}{ml^2}\right)$$

$$= ml^2\dot{\theta}^2 - \frac{1}{2}m\left(\dot{l}^2 + l^2\dot{\theta}^2\right) - mgl\cos\theta$$

$$= \frac{1}{2}m\left(l^2\dot{\theta}^2 - \dot{l}^2\right) - mgl\cos\theta$$

$$= \frac{1}{2}m\left(l^2\frac{p_\theta^2}{m^2l^4} - \dot{l}^2\right) - mgl\cos\theta$$

$$= \frac{1}{2}m\left(\frac{p_\theta^2}{m^2l^2} - \dot{l}^2\right) - mgl\cos\theta$$

$$\boxed{H = \frac{1}{2}\frac{p_\theta^2}{ml^2} - \frac{1}{2}m\dot{l}^2 - mgl\cos\theta}$$

b) $E = T + V$

$$= \frac{1}{2}m\dot{l}^2 + \frac{1}{2}ml^2\dot{\theta}^2 - mgl\cos\theta$$

$$= \frac{1}{2}m\dot{l}^2 + \frac{1}{2}ml^2\left(\frac{p_\theta^2}{m^2l^4}\right) - mgl\cos\theta$$

$$E = \frac{p_\theta^2}{2ml^2} + \frac{1}{2}m\dot{l}^2 - mgl\cos\theta$$

$$H = \frac{p_\theta^2}{2ml^2} - \frac{1}{2}m\dot{l}^2 - mgl\cos\theta$$

$$H \neq E \quad (\text{unless } \dot{l} = 0)$$

---

c) Lagrange's equation:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = 0$$

$$\frac{d}{dt}\left(ml^2\dot{\theta}\right) + mgl\sin\theta = 0$$

$$ml^2\ddot{\theta} + 2ml\dot{l}\dot{\theta} + mgl\sin\theta = 0$$

$$\boxed{\ddot{\theta} + 2\frac{\dot{l}}{l}\dot{\theta} + \frac{g}{l}\sin\theta = 0}$$

For small oscillations and $\dot{l} = 0$

$$\ddot{\theta} + \frac{g}{l}\theta = 0 \qquad \text{physics 1A} \Rightarrow \omega_0 = \sqrt{g/l}$$

d) $I = \int_{cycle} d\theta\, p_\theta$ is adiabatic invariant = const.

For small $\theta$ and $\dot{\ell}$ small, trial solution is

$$\theta(t) = A(t)\cos(\omega_0 t)$$

$$I = \int_{cycle} d\theta \left( m\ell^2 \dot{\theta} \right) = \text{const}$$

$\theta = \dot{\theta} t$

$d\theta = \dot{\theta}\, dt$

$$= \int dt\, \dot{\theta}\, m\ell^2 \dot{\theta}$$

$$= \int dt\, m\ell^2 \dot{\theta}^2$$

$\dot{\theta}(t) = A\omega_0 \sin(\omega_0 t)$

$$= \int_{cycle} dt\, m\ell^2 A^2 \omega_0^2 \sin^2(\omega_0 t)$$

$$= \frac{1}{2} m\ell^2 A^2 \omega_0^2 (T) \qquad T = 2\pi/\omega$$

$$= \pi m \ell^2 A^2 \omega_0$$

... next page -->

$$\pi m \ell^2 A^2 \omega_0 = \text{const}$$

For small oscillations $\omega_0 = \sqrt{g/\ell}$

$$\ell^2 A^2 \ell^{-1/2} = \text{const}$$

$$A^2 \ell^{3/2} = \text{const}$$

$$A^2 \propto \ell^{-3/2}$$

$$\boxed{A \propto \ell^{-3/4}}$$

## Solution for problem 2

Calculate the angular momentum stored in the fields before the current starts to flow; this will be the mechanical angular momentum after the capacitor is drained.

$$\vec{L}_{\text{em}} = \mu_o \epsilon_o \int \vec{r} \times \vec{S} dV = \epsilon_o \int \vec{r} \times \vec{E} \times \vec{B} dV$$

The electric field between the plates is:

$$\vec{E} = \frac{Q}{2\pi L \epsilon_o r} \hat{r}$$

And the Poynting vector in between the plates is then ($B = B\hat{z}$):

$$\vec{S} = -\frac{QB}{2\pi L \mu_o \epsilon_o r} \hat{\phi}$$

And $L_{\text{em}}$ becomes (After integrating):

$$\vec{L}_{\text{em}} = -\frac{QB}{2\pi L} \int_a^b 2\pi r dr L = -\frac{QB}{2} \left(b^2 - a^2\right) \hat{z}$$

The final angular velocity of the capacitor is just this divided by the moment of inertia:

$$\vec{\omega} = \frac{\vec{L}_{\text{em}}}{I} = -\frac{QB}{2I} \left(b^2 - a^2\right) \hat{z}$$

**Problem 3 Solutions:**

a) At thermal equilibrium the temperatures on each side are equal: $T_{He} = T_{SG}$

<u>helium</u>: Since ideal gas:

$$U_{He} = \frac{3}{2} N k T_{He} \qquad k = k_{Boltzmann}$$

$$T_{He} = \frac{2 U_{He}}{3 N k}$$

<u>Supergas:</u>

$$S_{SG} = k \ln \Omega$$

$$S_{SG} = k \left[ \ln U_{SG}^{10N} + \ln \begin{pmatrix} \text{other} \\ \text{stuff} \\ f(N,V) \end{pmatrix} \right]$$

$$\frac{1}{T_{SG}} = \left( \frac{\partial S_{SG}}{\partial U_{SG}} \right) \Bigg|_{V,N}$$

$$\frac{1}{T_{SG}} = 10 N k \left( \frac{1}{U_{SG}} \right)$$

$$T_{SG} = \frac{U_{SG}}{10 N k}$$

<u>Using $T_{He} = T_{SG}$:</u>

$$\frac{U_{He}}{3/2} = \frac{U_{SG}}{10} \Rightarrow \boxed{\frac{U_{SG}}{U_{He}} = \frac{20}{3}}$$

b) mechanical equilibrium $\Rightarrow$ Final pressures are equal

<u>helium:</u>

$$P_{He} V_{He} = NkT \quad \text{(ideal gas)}$$

$$P_{He} = \frac{NkT}{V_{He}}$$

<u>supergas:</u>

$$P_{SG} = T\left(\frac{\partial S_{SG}}{\partial V_{SG}}\right)\Bigg|_{U,N}$$

$$= T\left(\frac{k(10N)}{V_{SG}}\right)$$

$$P_{He} = P_{SG}$$

$$\frac{NkT}{V_{He}} = \frac{10NkT}{V_{SG}}$$

$$\boxed{\frac{V_{SG}}{V_{He}} = 10}$$

**Solution to Problem #4**

$$B = \frac{\mu_0 I}{4\pi} \int d\vec{\ell} \times \frac{(\vec{r} - \vec{r}\,')}{|\vec{r} - \vec{r}\,'|^3}$$

$$\vec{r} = 0 \qquad\qquad |r - r'|^2 = a^2 \varphi'^2$$

$$\vec{r}\,' = \rho' \hat{\rho}\,' = a\varphi' \hat{\rho}\,'$$


$$d\vec{\ell} = d\rho' \hat{\rho}\,' + \rho' d\varphi' \hat{\varphi}\,' =$$

$$= a\, d\varphi' \hat{\rho}\,' + a\varphi' d\varphi' \hat{\varphi}\,'$$

$$d\ell \times r' = \cancel{a^2 \varphi' d\varphi' \hat{\rho}\,' \times \hat{\rho}\,'} + a^2 \varphi'^2 d\varphi' \hat{\varphi}\,' \times \hat{\rho}\,' =$$

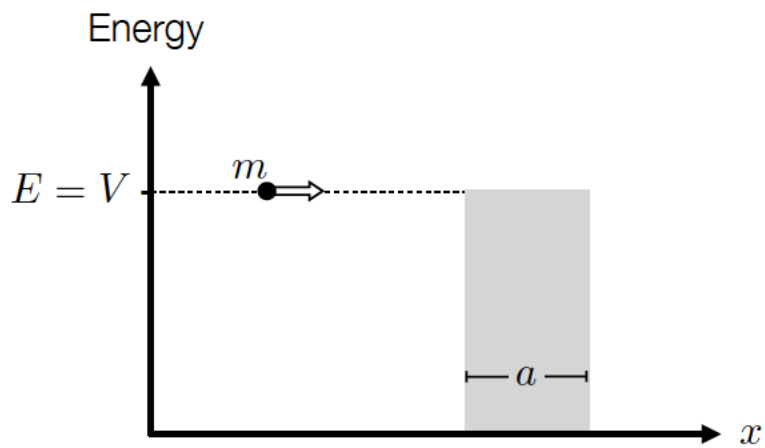
$$= -a^2 \varphi'^2 d\varphi' \hat{z}$$

$$B = \frac{\mu_0 I \hat{z}}{4\pi} \int \frac{a^2 \varphi'^2 d\varphi'}{a^3 \varphi'^3} = \frac{\mu_0 I \hat{z}}{4\pi a} \int_{2\pi}^{2\pi N} \frac{d\varphi'}{\varphi'} =$$

$$\boxed{B = \frac{\mu_0 I}{4\pi a} \ln N \; \hat{z}}$$

# Solution: QM Problem 5

Starting with the 1D time-independent Schroedinger equation, work out the fraction of incident particles transmitted through a rectangular 1D potential barrier in the case that the energy $E$ of the incident particles is equal to the barrier height $V$. Let the particles have mass $m$ and let the barrier width be $a$.




## Solution

We require solutions to the time-independent Schrodinger equation in three regions: (I) to the left of the barrier, (II) in the barrier where $E = V$, and (III) to the right of the barrier. The wavefunctions in these regions are

$$\psi_I = Ae^{ikx} + Be^{-ikx}\,, \qquad \psi_{II} = Cx + D\,, \qquad \psi_{III} = Fe^{ikx} \tag{1}$$

where $k = \sqrt{2mE/\hbar^2}$. These wavefunctions and their derivatives are continuous at the boundaries of the barrier. Let the left boundary be at $x = 0$ and the right boundary be at $x = a$. Matching gives the equations

$$A + B = D\,, \qquad ik(A - B) = C\,, \qquad Ca + D = Fe^{ika}\,, \qquad C = ikFe^{ika} \tag{2}$$

We want to find $F/A$. Using these equations we get,

$$\frac{F}{A} = \frac{e^{-ika}}{1 - ika/2} \tag{3}$$

The transmission coefficient is then

$$T = \left|\frac{F}{A}\right|^2 = \frac{1}{1 + \frac{mEa^2}{2\hbar^2}} \tag{4}$$

**Solution: Problem 6**

(Source: This is almost verbatim from the discussion in Ch. 1 of Sakurai that concludes with his statement, "Here lies the heart of quantum mechanics.")

(a) Operating on the left with any arbitrary $B$ eigenstate $\langle b''|$ gives

$$\begin{aligned}\langle b''|a'\rangle &= \langle b''|\sum_b c_b|b\rangle \\ &= c_{b''}\langle b''|b''\rangle \\ &= c_{b''}\end{aligned}$$

and so $\boxed{c_b = \langle b|a'\rangle}$. The same procedure applied to the second case shows that $\boxed{c_c = \langle c|b'\rangle}$.

(b) The probability of outcome $|b'\rangle$ given $|a'\rangle$ is $P(b'|a') = |c_{b'}|^2$. The probability of outcome $|c'\rangle$ given $|b'\rangle$ is $P(c'|b') = |c_{c'}|^2$. Therefore, we have $\boxed{P(c'|a' \,\&\, b') = |c_{b'}|^2\,|c_{c'}|^2}$.

(c)

$$\begin{aligned}\sum_{b'} P(c'|a' \,\&\, b') &= \boxed{\sum_{b'} |c_{b'}|^2\,|c_{c'}|^2} \\ &= \sum_{b'} \langle a'|b'\rangle\langle b'|c'\rangle\langle c'|b'\rangle\langle b'|a'\rangle \qquad (3)\end{aligned}$$

(d) $P(c'|a') = |\langle a'|c'\rangle|^2 = \langle a'|c'\rangle\langle c'|a'\rangle$. To compare this to Eq. (3), we can write $\langle a'|$ in the $B$ basis,

$$\langle a'| = \sum_b \langle a'|b\rangle\langle b|$$

which gives us

$$\begin{aligned}P(c'|a') &= \left|\sum_b \langle a'|b\rangle\langle b|c'\rangle\right|^2 \\ &= \sum_{b,b''} \langle a'|b\rangle\langle b|c'\rangle\langle c'|b''\rangle\langle b''|a'\rangle. \qquad (4)\end{aligned}$$

Comparing Eq. (3) to Eq. (4), we see that these will $\boxed{\text{not}}$ have to be the same probability.

There are other ways to arrive at this answer as well. For instance, if $|a'\rangle$ and $|c'\rangle$ are orthogonal, clearly $P(c'|a') = 0$, but there may be $B$ eigenstates that are not orthogonal to either one, in which case result (3) is nonzero.