Name ____________________

Statistical Mechanics Problem #8

FQ06 - Sept. 21/22, 2006

(a) Consider a grand canonical ensemble of particles, at fixed temperature T and in a container of volume V. Show that the mean square fluctuation in the number of particles $\overline{(\Delta N)^2}$ is:

$$\overline{(\Delta N)^2} = k_B T \frac{\partial \overline{N}}{\partial \mu}.$$

(b) Using the relation:

$$SdT - Vdp + Nd\mu = 0$$

express the solution in terms of $\left(\frac{\partial \rho}{\partial p}\right)_{T,V}$ where p = pressure and $\rho = \frac{N}{V}$ is the density of the system.

(c) Since intensive quantities are independent of extensive quantities by definition, we can change external constraints to obtain:

$$\left(\frac{\partial \rho}{\partial p}\right)_{T,V} = \left(\frac{\partial \rho}{\partial p}\right)_{T,N}.$$

Using this relation, find an expression for $\overline{(\Delta N)^2}$ in terms of the isothermal compressibility $\kappa_T = -\frac{1}{V}\left(\frac{\partial V}{\partial p}\right)_{T,N}$.

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Statistical Mechanics Problem #9

FQ06 - Sept. 21/22, 2006

Consider an idealized "white dwarf" star made up of ionized helium only. We make several simplifying assumptions, namely:

- there is no radiation pressure
- the electrons form a completely degenerate (i.e. "$T = 0$"), ultrarelativistic ("$m_e = 0$") Fermi gas
- the density ρ is uniform.

Set up the condition for mechanical equilibrium of the star under the opposing influences of the gravitational force and the pressure of the Fermi gas. You will find that, with these approximations, equilibrium is possible for only one particular value of the mass of the star, M (this mass is called the "Chandrasekhar limit"). Give the value of M in terms of fundamental constants.

[Note: this calculation, with $m_e = 0$, overestimates the pressure; in reality equilibrium is possible for masses smaller and up to the Chandrasekhar limit).

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Electromagnetics Problem #10

FQ06 - Sept. 21/22, 2006

A spherically symmetric potential $\Phi(r)$ is given by

$$\Phi(r) = \frac{f(r)}{r}$$

where $f(r) \rightarrow A$ as $r \rightarrow 0$ and $f(r) \rightarrow B$ as $r \rightarrow \infty$. $f(r)$ is a non-singular function.

a) What is the total charge of this system ? Give the answer in terms of $A, B, f(r)$ and (possibly) derivatives of $f(r)$.

b) Identify any point charges in this system and and give their location and charge.

c) Find the charge density $\rho(r)$ for this system. Give the answer in terms of $A, B, f(r)$ and (possibly) derivatives of $f(r)$.

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Electromagnetics Problem #11

FQ06 - Sept. 21/22, 2006

Consider an electromagnetic wave incident from vacuum onto a dielectric with a dielectric constant, ϵ. The surface normal lies along the $\hat{z}$ axis.

a) Derive the reflection and transmission coefficients if the wave is incident along the $\hat{z}$ direction.

b) Derive the reflection and transmission coefficients if the wave is incident in the $\hat{x}\hat{z}$ plane with an angle θ_i with respect to the surface normal and the electric field is in the y direction. Is there an angle for which there is no reflected energy?

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Electromagnetics Problem #12

FQ06 - Sept. 21/22, 2006

Consider an single electron interacting with electric and magnetic fields obtained from the corresponding scalar and vector potentials.

a) If the fields do not depend explicitly on time then the energy is conserved. Start from the equation for the time rate of change of energy for a single charged particle and derive the relativistically correct expression for the energy?

b) Consider a one-dimensional problem where the fields only depend on one spatial variable. Suppose the fields are described by a scalar potential of the form $\phi = \phi_0 cos(kz - \omega t)$. What is the constant of the motion in the laboratory frame now? Hint: Take a linear combination of the conservation of energy equation and the conservation of momentum equation. Use this constant to determine how large ϕ_0 must be in order that an electron that starts from rest at z=0 at t=0 is trapped by the wave and to determine the maximum energy that the electron can obtain.

c) Consider a fully three-dimensional case. If both the scalar and vector potential are functions of $(x, y, z - v_\phi t)$ where v_ϕ is the phase velocity, then the energy is no longer a constant. What is the new constant? Hint: Take a linear combination of the conservation of energy equation and the component of the conservation of momentum equation in the $\hat{z}$ direction.

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Electromagnetics Problem #13

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Consider a charge q moving on a circle of radius a (centered at the origin) on the x-y plane, with constant angular velocity ω.

a) In the dipole approximation, calculate the power radiated per unit solid angle in the direction defined by the azimuthal angle θ (i.e. θ is the angle with the z axis).

b) Still in the dipole approximation, what is the state of polarization of the radiation emitted in the direction $\theta = 0$? And $\theta = \pi/2$?

c) Going beyond the dipole approximation, show that radiation is emitted also at frequencies other than ω (what frequencies ?). You may want to follow the steps below:

- show that if $\rho(\mathbf{x}, t)$ is periodic in time (but not necessarily of the form $\rho(\mathbf{x})e^{-i\omega t}$) with period $T = 2\pi/\omega$, then one can write:

$$\rho(x,t) = \frac{1}{2}\rho_0(x) + \sum_{n=1}^{\infty}\mathrm{Re}\left[\rho_n(x)e^{-in\omega t}\right] \text{where } \rho_n(x) = \frac{2}{T}\int_0^T dt\rho(x,t)e^{in\omega t} \quad (n \geq 1)$$

- recall that the mutipole moments are:

$$q_{lm} = \int d^3x Y_{lm}(\theta,\varphi) r^l \rho(x)$$

Write $\rho(\mathbf{x})$ in spherical coordinates for this problem, and using the expression above, find the frequencies at which the different multipole terms radiate.

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Electromagnetics Problem #14

FQ06 - Sept. 21/22, 2006

Consider a rotating sphere with radius R. A charge Q is distributed homogeneously over the sphere. The sphere rotates counter clockwise around the z-axis with angular velocity ω. [See figure below].

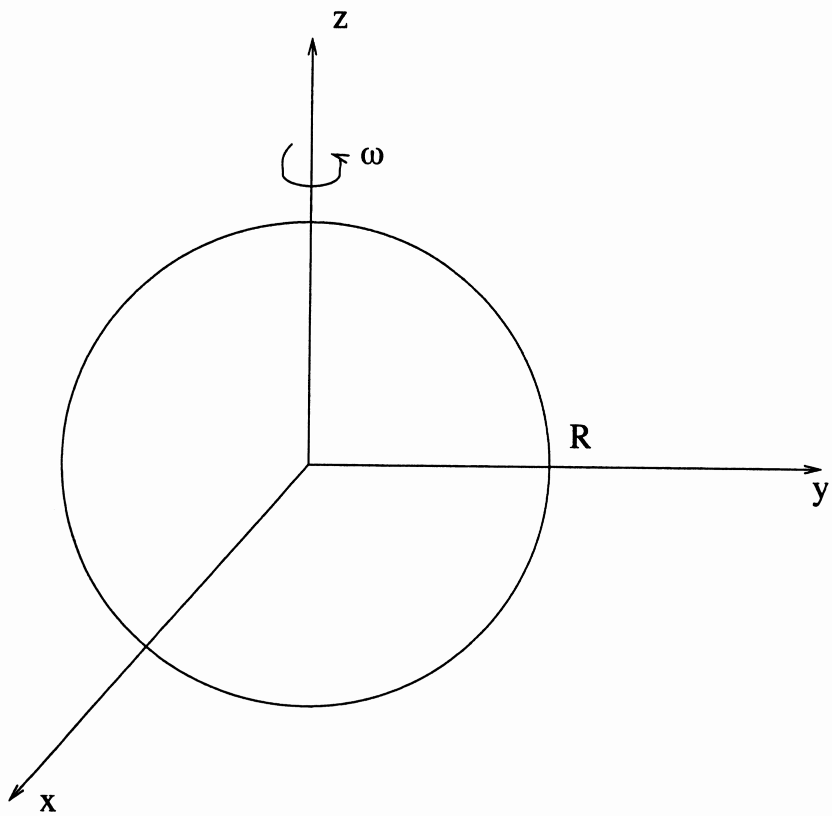

Figure for problem EM5

a) Find the charge density ρ and the current density $\vec{j}$ in terms of delta functions. Show that $\vec{\nabla} \cdot \vec{j} = 0$.

b) Find the vector potential $\vec{A}(\vec{x})$ in the Coulomb gauge ($\vec{\nabla} \cdot \vec{A} = 0$). Hint: To do the integral it is advantageous to choose $\vec{r} = r\vec{e}_z$ and choose $\vec{\omega}$ to be arbitrary.

c) Calculate the magnetic field $\vec{B}$ from the vector potential $\vec{A}$.