8. (Electromagnetism)

The purpose of this problem is to determine the current density and the magnetic field created by two spheres immersed in a medium with homogeneous and isotropic conductivity σ. Consider two spheres of equal radii a whose centers are separated by a distance $2d \gg a$, which are held at constant potentials $+V$ and $-V$ respectively with $V > 0$. In the midplane between the two spheres, consider points at an equal distance $R \gg d$ from the two centers.

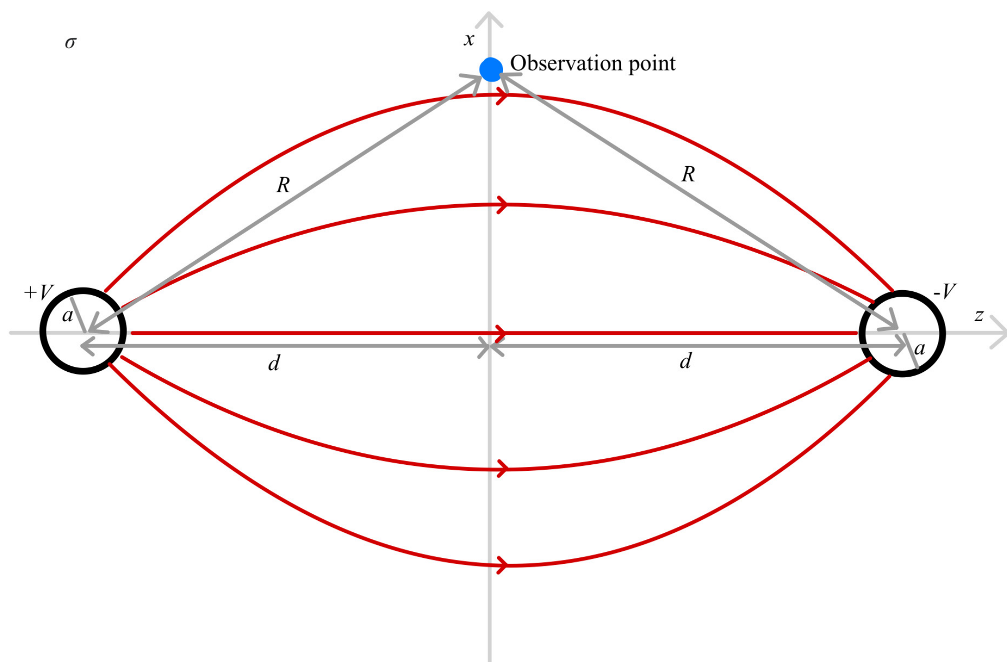
(a) Compute the current density vector $\boldsymbol{J}$ in terms of σ, V, a, d, R.

(b) Compute the magnetic field vector $\boldsymbol{B}$ in terms of σ, V, a, d, R.

Hint: place the center of the spheres at $(0, 0, \pm d)$ so the midplane is the xy-plane.

Solution: *Solution by Jonah Hyman (jthyman@g.ucla.edu)*

Here is a diagram of the setup. The electric field lines are marked in red:



(a) For a medium with homegeneous and isotropic conductivity σ, Ohm's law applies:

$$\mathbf{J} = \sigma \mathbf{E} \tag{290}$$

Therefore, in order to find the current density $\mathbf{J}$, we should first find the electric field $\mathbf{E}$. Since the spheres (which we can assume to be conducting) are separated by a large distance $2d \gg a$, we can assume the charge density on each sphere is spherically symmetric. We can also assume that near each sphere, the electric potential due to the other sphere is negligible.

Let $\pm Q$ be the total charge on the sphere held at potential $\pm V$. If we have spherical symmetry, the electric field and electric potential outside a sphere of total charge q are the same as the electric field of a point charge q:

$$\mathbf{E}(\mathbf{r}) = \frac{q}{4\pi\epsilon_0 \mathscr{r}^2}\hat{\boldsymbol{\mathscr{r}}} = \frac{q}{4\pi\epsilon_0 \mathscr{r}^3}\boldsymbol{\mathscr{r}} \quad \text{and} \quad V(\mathbf{r}) = \frac{q}{4\pi\epsilon_0 \mathscr{r}} \tag{291}$$

Here, $\boldsymbol{\mathscr{r}}$ points from the center of the sphere to the observation point. In this problem, we have two spheres, so we have two values of $\boldsymbol{\mathscr{r}}$. Without loss of generality, we can rotate our coordinate axes so that the observation point is in the xz-plane. With this in mind, we can write

$$\mathbf{r} = \sqrt{R^2 - d^2}\,\hat{\mathbf{x}} \quad \text{(observation point)}$$
$$\mathbf{r}'_+ = -d\,\hat{\mathbf{z}} \quad \text{(center of positively charged sphere)}$$
$$\mathbf{r}'_- = +d\,\hat{\mathbf{z}} \quad \text{(center of negatively charged sphere)}$$
$$\boldsymbol{\mathscr{r}}_+ \equiv \mathbf{r} - \mathbf{r}'_+ = \sqrt{R^2 - d^2}\,\hat{\mathbf{x}} + d\,\hat{\mathbf{z}} \tag{292}$$
$$\boldsymbol{\mathscr{r}}_- \equiv \mathbf{r} - \mathbf{r}'_- = \sqrt{R^2 - d^2}\,\hat{\mathbf{x}} - d\,\hat{\mathbf{z}} \tag{293}$$
$$\mathscr{r}_+ = \mathscr{r}_- = R \tag{294}$$

The electric field at the observation point is the sum of the electric fields due to each of the spheres:

$$
\begin{aligned}
\mathbf{E} &= \mathbf{E}_+ + \mathbf{E}_- \\
&= \frac{Q}{4\pi\epsilon_0 \mathscr{r}_+^3}\, \boldsymbol{\mathscr{r}}_+ + \frac{-Q}{4\pi\epsilon_0 \mathscr{r}_-^3}\, \boldsymbol{\mathscr{r}}_- \\
&= \frac{Q}{4\pi\epsilon_0 R^3}\left(\sqrt{R^2-d^2}\,\hat{\mathbf{x}} + d\,\hat{\mathbf{z}}\right) + \frac{-Q}{4\pi\epsilon_0 R^3}\left(\sqrt{R^2-d^2}\,\hat{\mathbf{x}} - d\,\hat{\mathbf{z}}\right) \\
&= \frac{2Qd}{4\pi\epsilon_0 R^3}\hat{\mathbf{z}}
\end{aligned} \tag{295}
$$

The problem asks us to write the answer in terms of V and a, not Q. V is defined as the electric potential on the sphere. Assuming the spheres are conducting, the electric potential is the same ($\pm V$) everywhere on the sphere. From (291), we know that since th radius of each sphere is a, the electric potential on the edge of the positively charged sphere is

$$
V = \frac{Q}{4\pi\epsilon_0 a} \tag{296}
$$

This allows us to solve for substitute Q for V and a in (295), which gives us

$$
\mathbf{E} = \frac{2Vad}{R^3}\hat{\mathbf{z}} \tag{297}
$$

Using Ohm's law $\mathbf{J} = \sigma\mathbf{E}$, we find the volume current density at the observation point

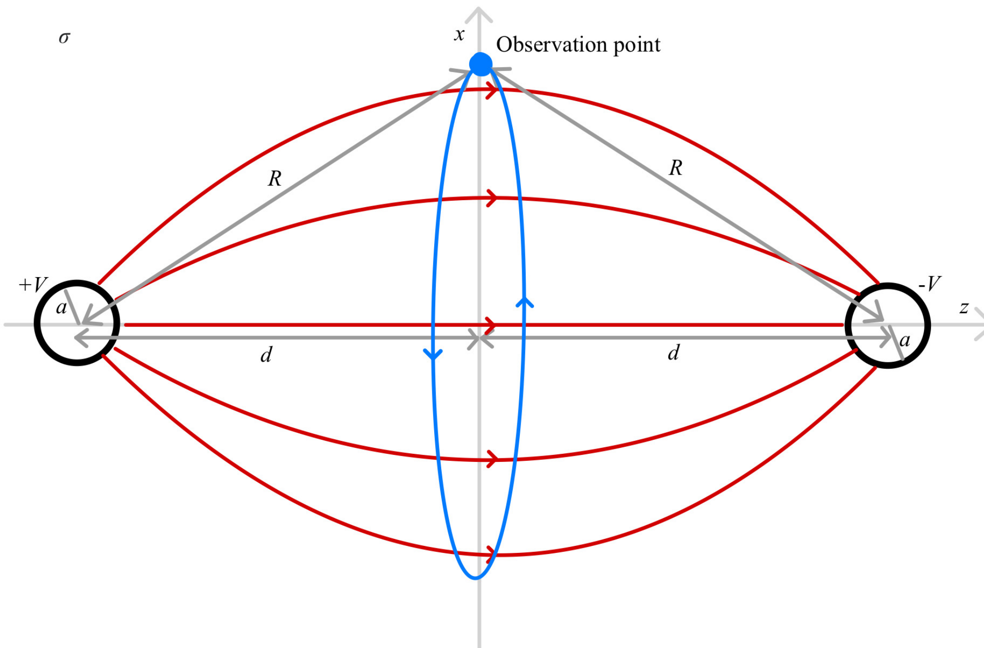
$$
\boxed{\mathbf{J} = \frac{2\sigma V ad}{R^3}\hat{\mathbf{z}}} \tag{298}
$$

where $\hat{\mathbf{z}}$ points from the positively charged sphere to the negatively charged sphere.

(b) The source of the magnetic field is the current density calculated in part (a). This is a magnetostatic situation, so **E** and **B** are constant in time. Therefore, we can use Ampere's law to calculate the magnetic field.

$$
\nabla \times \mathbf{B} = \mu_0 \mathbf{J} \tag{299}
$$

This problem is azimuthally symmetric, so the magnetic field depends only on the distance from the z-axis. Therefore, we should choose an Amperian loop of radius $s \equiv \sqrt{R^2-d^2}$ in the xy-plane, centered at $z = 0$:



Because of the azimuthal symmetry, the circulation of the magnetic field around the Amperian loop is $\int \mathbf{B} \cdot d\boldsymbol{\ell} = 2\pi s|\mathbf{B}|$. Using the integral version of Ampere's law, we get

$$\begin{aligned} 2\pi s|\mathbf{B}| &= \int_{\text{loop}} \mathbf{B} \cdot d\boldsymbol{\ell} \\ &= \int_{\text{loop interior}} \mu_0 \mathbf{J} \cdot d\mathbf{a} \\ &= 2\pi\mu_0 \int_{s'=0}^{s'=s} J(s')\, s'\, ds' \quad \text{since } \mathbf{J} \text{ is azimuthally symmetric} \end{aligned} \tag{300}$$

From part (a), since $R' = \sqrt{(s')^2 + d^2}$, the current density a distance s' from the z-axis and R' from the center of each sphere is

$$\mathbf{J}(s') = \frac{2\sigma V a d}{R'^3}\hat{\mathbf{z}} = \frac{2\sigma V a d}{\left((s')^2 + d^2\right)^{3/2}} \tag{301}$$

Plugging this into the integral, we have

$$\begin{aligned} 2\pi s|\mathbf{B}| &= 2\pi\mu_0 \int_{s'=0}^{s'=s} \frac{2\sigma V a d}{\left((s')^2 + d^2\right)^{3/2}}\, s'\, ds' \\ &= 4\pi\mu_0 \sigma V a d \int_{s'=0}^{s'=s} ds' \frac{s'}{\left((s')^2 + d^2\right)^{3/2}} \\ &= 4\pi\mu_0 \sigma V a d \left[-\frac{1}{\left((s')^2 + d^2\right)^{1/2}} \right]_{s'=0}^{s'=s} \\ &= 4\pi\mu_0 \sigma V a d \left[\frac{1}{d} - \frac{1}{\left(s^2 + d^2\right)^{1/2}} \right] \\ &= 4\pi\mu_0 \sigma V a d \left[\frac{1}{d} - \frac{1}{R} \right] \quad \text{since } R = \sqrt{s^2 + d^2} \end{aligned} \tag{302}$$

Therefore, solving for $|\mathbf{B}|$, we get

$$|\mathbf{B}| = \frac{2\mu_0 \sigma V a d}{s}\left[\frac{1}{d} - \frac{1}{R}\right] = \frac{2\mu_0 \sigma V a d}{\sqrt{R^2 - d^2}}\left[\frac{1}{d} - \frac{1}{R}\right] \tag{303}$$

Since $R \gg d$, this answer can be simplified to

$$|\mathbf{B}| \approx \frac{2\mu_0 \sigma V a}{R} \tag{304}$$

Adding in the direction of the magnetic field, chosen according to the right-hand rule, we get

$$\boxed{\mathbf{B} \approx \frac{2\mu_0 \sigma V a}{R}\hat{\varphi}} \tag{305}$$

where $\hat{\varphi}$ points counterclockwise with respect to $\hat{\mathbf{z}}$.