8. (Electromagnetism)

An electron is bound to a spring with spring constant k, free to move in three dimensions.

(a) Calculate the scattering cross section for linearly-polarized EM waves of frequency ω incident on the electron.

(b) In what limit for the incident wave frequency should the total cross section equal the Thomson scattering cross section? Take this limit and confirm that it results in the Thomson scattering cross section.

(c) In what limit should the total cross section yield Rayleigh scattering? Take this limit and confirm that the cross section is consistent with Rayleigh scattering (what is the frequency dependence you expect?).

Solution: *Solution by Audrey Farrell*

(a) In order to determine the cross section, we need the incident and radiated power

$$\sigma = \frac{\langle P_{rad}\rangle}{\langle S_{inc}\rangle}, \quad \langle S_{inc}\rangle = \frac{1}{2}c\epsilon_0 E_0^2, \quad \langle P_{rad}\rangle = \left\langle \frac{\mu_0}{6\pi c}|\ddot{\vec{p}}|^2\right\rangle$$

In order to find the radiated power, we need the trajectory of the electron inside the wave.

There are a few key assumptions in order to do this problem quickly: 1) The incident waves are time-harmonic and linearly polarized in $\hat{x}$, 2) the electron velocity is small enough that magnetic force is negligible, 3) damping is negligible, and 4) the electron response is at the same frequency as the incident wave.

With these assumptions in place we have a 1D equation of motion rather than 3D, with the spring and Lorentz forces in the $\hat{x}$ direction

$$m\ddot{x} = -kx - qE_0 e^{-i\omega t} \rightarrow \ddot{x} + \omega_0^2 x + \frac{qE_0}{m}\, e^{-i\omega t} = 0$$

where $q = e$ is the electron charge, I'll leave it as q for now to disambiguate from exponentials.

Since we've assumed the electron responds at the same frequency ω as the incident wave (physically this corresponds to enough time having passed that transient frequencies have died off), we can solve this quickly using Fourier analysis, $\ddot{x} \rightarrow -\omega^2 x$

$$x(t) = -\frac{qE_0}{m(\omega_0^2 - \omega^2)}\, e^{-i\omega t}$$

This position corresponds to an oscillating dipole moment

$$\vec{p}\,(t) = q\vec{x}\,(t) = -\frac{q^2 E_0}{m(\omega_0^2 - \omega^2)}\, e^{-i\omega t}\hat{x} \quad \rightarrow \quad \ddot{\vec{p}}\,(t) = -\omega^2 \vec{p}\,(t)$$

giving a radiated power of

$$\langle P_{rad}\rangle = \left\langle \frac{\mu_0}{6\pi c}|\ddot{\vec{p}}|^2\right\rangle = \frac{\mu_0}{6\pi c}\left(\frac{q^4\omega^4 E_0^2}{2m^2(\omega_0^2 - \omega^2)^2}\right)$$

and cross section

$$\sigma = \frac{\mu_0}{6\pi c^2 \epsilon_0}\left(\frac{q^4\omega^4}{m^2(\omega_0^2 - \omega^2)^2}\right) = \frac{1}{6\pi\epsilon_0^2 c^4}\,\frac{q^4\omega^4}{m^2(\omega_0^2 - \omega^2)^2}$$

Using the classical electron radius

$$r_e = \frac{e^2}{4\pi\epsilon_0 mc^2} \quad \rightarrow \quad \boxed{\sigma = \frac{8\pi}{3} r_e^2 \frac{\omega^4}{(\omega_0^2 - \omega^2)^2}}$$

(b) The Thomson limit corresponds to $\omega \gg \omega_0$, so the electron behaves like a free particle and the spring plays very little role in the dynamics of the system. In this limit $(\omega_0^2 - \omega^2) \rightarrow \omega^2$ and

$$\boxed{\sigma_{\omega \gg \omega_0} \rightarrow \frac{8\pi}{3} r_e^2 = \sigma_T}$$

which is the Thomson cross section for unpolarized incident light.

(c) In the Rayleigh limit $\omega \ll \omega_0$, and the cross section is

$$\boxed{\sigma_{\omega \ll \omega_0} \rightarrow \frac{8\pi}{3} r_e^2 \frac{\omega^4}{\omega_0}}$$

This frequency-dependent cross section is what we expect in Rayleigh scattering (high frequencies have a higher cross section, and the sky sure is blue).