6. (Classical Mechanics)

Consider a field $\Psi(\mathbf{r}, t)$; $\mathbf{r} = x, y, z$, which obeys the following equation of motion:

$$\frac{\partial^2 \Psi}{\partial t^2} - c^2 \nabla^2 \Psi = V(z)\Psi - (\operatorname{sech} \Psi)(\tanh \Psi)$$

where $V(z)$ is a real function of z, and c is a constant.

(a) Write down a Lagrangian density $\mathcal{L}(\Psi)$ whose variation (when set equal to zero) yields the above equation of motion for the field $\Psi(\mathbf{r}, t)$.

(b) Consider the energy and momentum of the field $\Psi(\mathbf{r}, t)$. Which of these quantities is conserved and why?

(c) Write down the law for the time development of the energy density for this field theory.

Solution: *Solution by Jonah Hyman (jthyman@g.ucla.edu)*

In general, it is easier to handle classical field theory using the notation of special relativity. For those unfamiliar with that notation, though, we provide two solutions: one in special relativistic notation and one in more elementary notation.

Special relativistic notation:
We use the following conventions:

$$\partial_\mu \equiv \left(\frac{1}{c}\frac{\partial}{\partial t}, \nabla\right) \quad \text{and} \quad \eta_{\mu\nu} \equiv \mathrm{diag}(+1,-1,-1,-1) \tag{199}$$

(a) The equation of motion can be written as

$$c^2 \partial^\mu \partial_\mu \Psi = V(z)\Psi - (\operatorname{sech}\Psi)(\tanh\Psi) \tag{200}$$

One way or another, you need to come up with the following ansatz for the Lagrangian density:

$$\mathcal{L} = A\,\partial^\mu \Psi \partial_\mu \Psi + Q(\Psi, \mathbf{r}, t) \tag{201}$$

where A is an undetermined constant and $Q(\Psi, \mathbf{r}, t)$ is an undetermined function. Here is how to come up with this ansatz:

- If you have some experience with the Klein-Gordon Lagrangian, you might already know that a term in the Lagrangian of the form $\partial^\mu \Psi\, \partial_\mu \Psi$ produces a term in the equation of motion of the form $\partial^\mu \partial_\mu \Psi$.
- If you don't, you might remember the process of finding the equation of motion by taking the variation of the action. That process involves integration by parts, so a term in the Lagrangian of the form $\partial^\mu \Psi\, \partial_\mu \Psi$ would produce a term in the equation of motion of the form $\partial^\mu \partial_\mu \Psi$.
- Otherwise, you might know the formula for the Euler-Lagrange equation

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \Psi)}\right) - \frac{\partial \mathcal{L}}{\partial \Psi} = 0 \tag{202}$$

  which clarifies that a term in the Lagrangian of the form $\partial^\mu \Psi\, \partial_\mu \Psi$ would produce a term in the equation of motion of the form $\partial^\mu \partial_\mu \Psi$.

Now that we have the ansatz (201), we can match the given equation of motion (200) to the Euler-Lagrange equation derived from the ansatz (201) by variation of the action and integration by parts:

$$\begin{aligned}
\delta S &= \delta\left(\int d^4x\, \mathcal{L}\right) \\
&= \delta\left(\int d^4x\, \left[A\,\partial^\mu \Psi \partial_\mu \Psi + Q(\Psi, \mathbf{r}, t)\right]\right) \\
&= \int d^4x\, \left[2A\,\partial^\mu \Psi\, \partial_\mu(\delta\Psi) + Q'(\Psi, \mathbf{r}, t)\,\delta\Psi\right] \quad \text{where } Q' \equiv \frac{\partial Q}{\partial \Psi} \\
&= \int d^4x\, \left[-2A\,\partial^\mu \partial_\mu \Psi\, \delta\Psi + Q'(\Psi, \mathbf{r}, t)\,\delta\Psi\right] + \text{boundary term} \\
&= \int d^4x\, \left[-2A\,\partial^\mu \partial_\mu \Psi + Q'(\Psi, \mathbf{r}, t)\right]\delta\Psi + \text{boundary term}
\end{aligned} \tag{203}$$

We may ignore the boundary term. To get the classical solution for Ψ, we set $\delta S = 0$ for all possible values of $\delta\Psi$, which gives us

$$2A\,\partial^\mu \partial_\mu \Psi = Q'(\Psi, \mathbf{r}, t) \quad \text{where } Q' \equiv \frac{\partial Q}{\partial \Psi} \tag{204}$$

Alternatively, we can use the formula for the Euler-Lagrange equation (202) to get

$$\begin{aligned}\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial \left(\partial_\mu \Psi \right)} \right) &= \frac{\partial \mathcal{L}}{\partial \Psi} \\ \partial_\mu \left(2A \partial^\mu \Psi \right) &= Q'(\Psi, \mathbf{r}, t) \\ 2A \, \partial^\mu \partial_\mu \Psi &= Q'(\Psi, \mathbf{r}, t) \quad \text{where } Q' \equiv \frac{\partial Q}{\partial \Psi}\end{aligned} \tag{205}$$

Either way, matching the calculated Euler-Lagrange equation to the equation of motion (200), we get

$$A = \frac{c^2}{2} \quad \text{and} \quad Q'(\Psi, \mathbf{r}, t) = V(z)\Psi - (\operatorname{sech} \Psi)(\tanh \Psi) \tag{206}$$

Taking the antiderivative of Q' to get Q requires knowing that

$$(\sinh x)' = \cosh x \quad \text{and} \quad (\cosh x)' = \sinh x \tag{207}$$

Note that unlike the analogous trigonometric identities, the hyperbolic trigonometric identities don't have a minus sign. We can use this information to write the antiderivative of $(\operatorname{sech} \Psi)(\tanh \Psi)$:

$$(\operatorname{sech} \Psi)(\tanh \Psi) = \frac{\sinh \Psi}{\cosh^2 \Psi} = \left(-\frac{1}{\cosh \Psi} \right)' = (-\operatorname{sech} \Psi)' \tag{208}$$

Thus, ignoring the integration constant, we can write

$$Q = \frac{1}{2} V(z) \Psi^2 + \operatorname{sech} \Psi \tag{209}$$

Therefore, putting everything together using our ansatz (201), we can write one form of the Lagrangian density:

$$\boxed{\mathcal{L} = \frac{c^2}{2} \partial^\mu \Psi \, \partial_\mu \Psi + \frac{1}{2} V(z) \Psi^2 + \operatorname{sech} \Psi = \frac{1}{2} \left(\frac{\partial \Psi}{\partial t} \right)^2 - \frac{c^2}{2} \left(\nabla \Psi \right)^2 + \frac{1}{2} V(z) \Psi^2 + \operatorname{sech} \Psi} \tag{210}$$

We can add any total derivative (including a constant) to the Lagrangian density without changing the equation of motion, and we can multiply the Lagrangian density by any constant without changing the equation of motion.

(b) In the form we have written it, the Lagrangian density has no explicit time dependence. Therefore, the energy of the field $\Psi(\mathbf{r}, t)$ is conserved.

In the form we have written it, the Lagrangian density has no explicit dependence on the coordinates x or y. Therefore, the momentum of the field $\Psi(\mathbf{r}, t)$ in the x-direction and in the y-direction, i.e., P_x and P_y, are conserved.

The Lagrangian density explicitly depends on z because of the potential $V(z)$, and such dependence cannot be eliminated by adding a total derivative to the Lagrangian. Therefore, the momentum of the field $\Psi(\mathbf{r}, t)$ in the z-direction is not conserved.

(c) You might already know (or have on your formula sheet) the definition of the energy-momentum tensor:

$$T^\mu{}_\nu \equiv \frac{\partial \mathcal{L}}{\partial \left(\partial_\mu \Psi \right)} \partial_\nu \Psi - \delta^\mu{}_\nu \mathcal{L} \tag{211}$$

Assuming that the Lagrangian density is independent of x_ν, the 4-divergence of this tensor is zero:

$$\partial_\mu T^\mu{}_\nu = 0 \tag{212}$$

If you didn't know this result, you could derive it using the following argument: Suppose that $\mathcal{L}$ does not explicitly depend on the coordinate x^ν. Then, the partial derivative of $\mathcal{L}$ with respect to x^ν depends only on the dependence of the fields Ψ and their derivatives on x^ν. Applying the chain rule, we get

$$\frac{\partial \mathcal{L}}{\partial x^\nu} = \frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \Psi\right)}\partial_\nu\left(\partial_\mu \Psi\right) + \frac{\partial \mathcal{L}}{\partial \Psi}\partial_\nu \Psi$$

Now, integrating the first term on the right-hand side by parts and keeping the boundary term, we get

$$\frac{\partial \mathcal{L}}{\partial x^\nu} = \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \Psi\right)}\partial_\nu \Psi\right) - \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \Psi\right)}\right)\partial_\nu \Psi + \frac{\partial \mathcal{L}}{\partial \Psi}\partial_\nu \Psi$$

By the Euler-Lagrange equation, $\partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \Psi)}\right) - \frac{\partial \mathcal{L}}{\partial \Psi} = 0$, so the last two terms cancel:

$$\frac{\partial \mathcal{L}}{\partial x^\nu} = \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \Psi\right)}\partial_\nu \Psi\right)$$

Writing $\frac{\partial \mathcal{L}}{\partial x^\nu} = \frac{\partial}{\partial x^\mu}\left(\delta^\mu{}_\nu \mathcal{L}\right)$ and simplifying, this becomes

$$\partial_\mu\left(\frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \Psi\right)}\partial_\nu \Psi - \delta^\mu{}_\nu \mathcal{L}\right) = 0 \tag{213}$$

This is equal to $\partial_\mu T^\mu{}_\nu = 0$ after we apply the definition of the energy-momentum tensor (211). Breaking up $\partial_\mu T^\mu{}_\nu = 0$ into a sum of spatial and time components, we get

$$0 = \partial_\mu T^\mu{}_\nu = \frac{1}{c}\frac{\partial T^0{}_\nu}{\partial t} + \sum_{i=1}^{3}\partial_i T^i{}_\nu \tag{214}$$

The law for the time dependence of the energy is the component of this equation with $\mu = 0$:

$$0 = \frac{1}{c}\frac{\partial T^0{}_0}{\partial t} + \sum_{i=1}^{3}\partial_i T^i{}_0 \tag{215}$$

Using the definition (211) and the fact that $\partial_\mu \equiv \left(\frac{1}{c}\frac{\partial}{\partial t}, \nabla\right)$, we get that

$$T^0{}_0 = \frac{\partial \mathcal{L}}{\partial\left(\partial_t \Psi\right)} - \mathcal{L} \quad \text{and} \quad T^i{}_0 = \frac{1}{c}\frac{\partial \mathcal{L}}{\partial\left(\nabla \Psi\right)}\frac{\partial \Psi}{\partial t} \tag{216}$$

so (215) becomes (after canceling a factor of $1/c$) a continuity equation for the energy density:

$$0 = \frac{\partial}{\partial t}\underbrace{\left(\frac{\partial \mathcal{L}}{\partial\left(\partial_t \Psi\right)} - \mathcal{L}\right)}_{\text{energy density}} + \nabla\cdot\underbrace{\left(\frac{\partial \mathcal{L}}{\partial\left(\nabla \Psi\right)}\frac{\partial \Psi}{\partial t}\right)}_{\text{energy current density}} \tag{217}$$

Plugging in our Lagrangian from part (a) (210), we get that the energy density is

$$\begin{aligned} u &\equiv \frac{\partial \mathcal{L}}{\partial\left(\partial_t \Psi\right)} - \mathcal{L} \\ &= \frac{\partial \Psi}{\partial t} - \left(\frac{1}{2}\left(\frac{\partial \Psi}{\partial t}\right)^2 - \frac{c^2}{2}\left(\nabla \Psi\right)^2 + \frac{1}{2}V(z)\Psi^2 + \operatorname{sech}\Psi\right) \\ &= \frac{1}{2}\left(\frac{\partial \Psi}{\partial t}\right)^2 + \frac{c^2}{2}\left(\nabla \Psi\right)^2 - \frac{1}{2}V(z)\Psi^2 - \operatorname{sech}\Psi \end{aligned} \tag{218}$$

and that the energy current density is

$$\mathbf{S} \equiv \frac{\partial \mathcal{L}}{\partial \left(\nabla \Psi\right)} \frac{\partial \Psi}{\partial t} = -c^2 \nabla \Psi \frac{\partial \Psi}{\partial t}$$

Thus, the law for the time development of the energy density u of this theory is

$$\boxed{\frac{\partial u}{\partial t} + \nabla \cdot \left(-c^2 \nabla \Psi \frac{\partial \Psi}{\partial t}\right) = 0 \quad \text{for} \quad u \equiv \frac{1}{2}\left(\frac{\partial \Psi}{\partial t}\right)^2 + \frac{c^2}{2}\left(\nabla \Psi\right)^2 - \frac{1}{2} V(z) \Psi^2 - \operatorname{sech} \Psi} \tag{219}$$

Nonrelativistic notation:

(a) The equation of motion is given by

$$\frac{\partial^2 \Psi}{\partial t^2} - c^2 \nabla^2 \Psi = V(z)\Psi - (\operatorname{sech}\Psi)(\tanh\Psi) \tag{220}$$

One way or another, you need to come up with the following ansatz for the Lagrangian density:

$$\mathcal{L} = B\left(\frac{\partial \Psi}{\partial t}\right)^2 + C(\nabla \Psi)^2 + Q(\Psi, \mathbf{r}, t) \tag{221}$$

where B and C are an undetermined constants and $Q(\Psi, \mathbf{r}, t)$ is an undetermined function. Here is how to come up with this ansatz:

- You might remember the process of finding the equation of motion by taking the variation of the action. That process involves integration by parts, so a term in the Lagrangian of the form $\left(\frac{\partial \Psi}{\partial t}\right)^2$ would produce a term in the equation of motion of the form $\frac{\partial^2 \psi}{\partial t^2}$, and a term in the Lagrangian of the form $(\nabla \Psi)^2$ would produced a term in the equation of motion of the form $\nabla^2 \Psi$.
- Otherwise, you might know the formula for the Euler-Lagrange equation

$$\partial_t \left(\frac{\partial \mathcal{L}}{\partial \left(\partial_t \Psi\right)}\right) + \nabla \cdot \left(\frac{\partial \mathcal{L}}{\partial \left(\nabla \Psi\right)}\right) - \frac{\partial \mathcal{L}}{\partial \Psi} = 0 \tag{222}$$

  This clarifies that a term in the Lagrangian of the form $\left(\frac{\partial \Psi}{\partial t}\right)^2$ would produce a term in the equation of motion of the form $\frac{\partial^2 \psi}{\partial t^2}$, and a term in the Lagrangian of the form $(\nabla \Psi)^2$ would produced a term in the equation of motion of the form $\nabla^2 \Psi$.

Now that we have the ansatz (221), we can match the given equation of motion (220) to the Euler-Lagrange equation derived from the ansatz (221) by variation of the action and integration by parts:

$$\begin{aligned}
\delta S &= \delta\left(\int d^4x\,\mathcal{L}\right) \\
&= \delta\left(\int d^4x\,\left[B\left(\frac{\partial\Psi}{\partial t}\right)^2 + C(\nabla\Psi)^2 + Q(\Psi,\mathbf{r},t)\right]\right) \\
&= \int d^4x\,\left[2B\,\frac{\partial\Psi}{\partial t}\,\frac{\partial}{\partial t}\,(\delta\Psi) + 2C\nabla\Psi\cdot\nabla(\delta\Psi) + Q'(\Psi,\mathbf{r},t)\,\delta\Psi\right] \quad \text{where } Q' \equiv \frac{\partial Q}{\partial\Psi} \\
&= \int d^4x\,\left[-2B\,\frac{\partial^2\Psi}{\partial t^2}\delta\Psi - 2C\nabla^2\Psi\,\delta\Psi + Q'(\Psi,\mathbf{r},t)\,\delta\Psi\right] + \text{boundary term} \\
&= \int d^4x\,\left[-2B\,\frac{\partial^2\Psi}{\partial t^2} - 2C\nabla^2\Psi + Q'(\Psi,\mathbf{r},t)\right]\delta\Psi + \text{boundary term}
\end{aligned} \tag{223}$$

We may ignore the boundary term. To get the classical solution for Ψ, we set $\delta S = 0$ for all possible values of $\delta\Psi$, which gives us

$$2B\,\frac{\partial^2\Psi}{\partial t^2} + 2C\nabla^2\Psi = Q'(\Psi,\mathbf{r},t) \quad \text{where } Q' \equiv \frac{\partial Q}{\partial\Psi} \tag{224}$$

Alternatively, we can use the formula for the Euler-Lagrange equation (222) to get

$$\begin{aligned}
\partial_t\left(\frac{\partial\mathcal{L}}{\partial\left(\partial_t\Psi\right)}\right) + \nabla\cdot\left(\frac{\partial\mathcal{L}}{\partial\left(\nabla\Psi\right)}\right) &= \frac{\partial\mathcal{L}}{\partial\Psi} \\
\partial_t\left(2B\frac{\partial\Psi}{\partial t}\right) + \nabla\cdot\left(2C\nabla\Psi\right) &= Q'(\Psi,\mathbf{r},t) \\
2B\,\frac{\partial^2\Psi}{\partial t^2} + 2C\nabla^2\Psi &= Q'(\Psi,\mathbf{r},t) \quad \text{where } Q' \equiv \frac{\partial Q}{\partial\Psi}
\end{aligned} \tag{225}$$

Either way, matching the calculated Euler-Lagrange equation to the equation of motion (220), we get

$$B = \frac{1}{2}; \qquad C = -\frac{c^2}{2}; \qquad Q'(\Psi,\mathbf{r},t) = V(z)\Psi - (\operatorname{sech}\Psi)\,(\tanh\Psi) \tag{226}$$

Taking the antiderivative of Q' to get Q requires knowing that

$$(\sinh x)' = \cosh x \quad \text{and} \quad (\cosh x)' = \sinh x \tag{227}$$

Note that unlike the analogous trigonometric identities, the hyperbolic trigonometric identities don't have a minus sign. We can use this information to write the antiderivative of $(\operatorname{sech}\Psi)\,(\tanh\Psi)$:

$$(\operatorname{sech}\Psi)\,(\tanh\Psi) = \frac{\sinh\Psi}{\cosh^2\Psi} = \left(-\frac{1}{\cosh\Psi}\right)' = (-\operatorname{sech}\Psi)' \tag{228}$$

Thus, ignoring the integration constant, we can write

$$Q = \frac{1}{2}V(z)\Psi^2 + \operatorname{sech}\Psi \tag{229}$$

Therefore, putting everything together using our ansatz (221), we can write one form of the Lagrangian density:

$$\boxed{\mathcal{L} = \frac{1}{2}\left(\frac{\partial\Psi}{\partial t}\right)^2 - \frac{c^2}{2}\left(\nabla\Psi\right)^2 + \frac{1}{2}V(z)\Psi^2 + \operatorname{sech}\Psi} \tag{230}$$

We can add any total derivative (including a constant) to the Lagrangian density without changing the equation of motion, and we can multiply the Lagrangian density by any constant without changing the equation of motion.

(b) In the form we have written it, the Lagrangian density has no explicit time dependence. Therefore, the energy of the field $\Psi(\mathbf{r}, t)$ is conserved.

In the form we have written it, the Lagrangian density has no explicit dependence on the coordinates x or y. Therefore, the momentum of the field $\Psi(\mathbf{r}, t)$ in the x-direction and in the y-direction, i.e., P_x and P_y, are conserved.

The Lagrangian density explicitly depends on z because of the potential $V(z)$, and such dependence cannot be eliminated by adding a total derivative to the Lagrangian. Therefore, the momentum of the field $\Psi(\mathbf{r}, t)$ in the z-direction is not conserved.

(c) You might already know (or have on your formula sheet) the energy continuity equation:

$$0 = \frac{\partial}{\partial t}\underbrace{\left(\frac{\partial \mathcal{L}}{\partial\left(\partial_t \Psi\right)} - \mathcal{L}\right)}_{\text{energy density}} + \nabla \cdot \underbrace{\left(\frac{\partial \mathcal{L}}{\partial\left(\nabla \Psi\right)} \frac{\partial \Psi}{\partial t}\right)}_{\substack{\text{energy current}\\ \text{density}}} \tag{231}$$

If you didn't know this result, you could derive it using the following argument: Suppose that $\mathcal{L}$ does not explicitly depend on the coordinate t. Then, the partial derivative of $\mathcal{L}$ with respect to t depends only on the dependence of the fields Ψ and their derivatives on t. Applying the chain rule, we get

$$\frac{\partial \mathcal{L}}{\partial t} = \frac{\partial \mathcal{L}}{\partial\left(\partial_t \Psi\right)}\partial_t\left(\partial_t \Psi\right) + \frac{\partial \mathcal{L}}{\partial\left(\nabla \Psi\right)} \cdot \nabla(\partial_t \Psi) + \frac{\partial \mathcal{L}}{\partial \Psi}\partial_t \Psi$$

Now, integrating the first two terms on the right-hand side by parts and keeping the boundary term, we get

$$\frac{\partial \mathcal{L}}{\partial t} = \frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial\left(\partial_t \Psi\right)}\partial_t \Psi\right) + \nabla \cdot \left(\frac{\partial \mathcal{L}}{\partial\left(\nabla \Psi\right)} \cdot \partial_t \Psi\right) - \frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial\left(\partial_t \Psi\right)}\right)\partial_t \Psi - \nabla \cdot \left(\frac{\partial \mathcal{L}}{\partial\left(\nabla \Psi\right)}\right)\partial_t \Psi + \frac{\partial \mathcal{L}}{\partial \Psi}\partial_t \Psi$$

By the Euler-Lagrange equation, $\partial_t\left(\frac{\partial \mathcal{L}}{\partial(\partial_t \Psi)}\right) + \nabla \cdot \left(\frac{\partial \mathcal{L}}{\partial(\nabla \Psi)}\right) - \frac{\partial \mathcal{L}}{\partial \Psi} = 0$, so the last three terms cancel:

$$\frac{\partial \mathcal{L}}{\partial t} = \frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial\left(\partial_t \Psi\right)}\partial_t \Psi\right) + \nabla \cdot \left(\frac{\partial \mathcal{L}}{\partial\left(\nabla \Psi\right)} \cdot \partial_t \Psi\right)$$

Simplifying, this becomes

$$0 = \frac{\partial}{\partial t}\underbrace{\left(\frac{\partial \mathcal{L}}{\partial\left(\partial_t \Psi\right)} - \mathcal{L}\right)}_{\text{energy density}} + \nabla \cdot \underbrace{\left(\frac{\partial \mathcal{L}}{\partial\left(\nabla \Psi\right)} \frac{\partial \Psi}{\partial t}\right)}_{\substack{\text{energy current}\\ \text{density}}} \tag{232}$$

Plugging in our Lagrangian from part (a) (230), we get that the energy density is

$$\begin{aligned} u &\equiv \frac{\partial \mathcal{L}}{\partial\left(\partial_t \Psi\right)} - \mathcal{L} \\ &= \frac{\partial \Psi}{\partial t} - \left(\frac{1}{2}\left(\frac{\partial \Psi}{\partial t}\right)^2 - \frac{c^2}{2}\left(\nabla \Psi\right)^2 + \frac{1}{2}V(z)\Psi^2 + \operatorname{sech}\Psi\right) \\ &= \frac{1}{2}\left(\frac{\partial \Psi}{\partial t}\right)^2 + \frac{c^2}{2}\left(\nabla \Psi\right)^2 - \frac{1}{2}V(z)\Psi^2 - \operatorname{sech}\Psi \end{aligned} \tag{233}$$

and that the energy current density is

$$\begin{aligned}\mathbf{S} &\equiv \frac{\partial \mathcal{L}}{\partial\left(\nabla\Psi\right)}\frac{\partial\Psi}{\partial t}\\ &= -c^2\nabla\Psi\frac{\partial\Psi}{\partial t}\end{aligned}$$

Thus, the law for the time development of the energy density u of this theory is

$$\boxed{\frac{\partial u}{\partial t} + \nabla\cdot\left(-c^2\nabla\Psi\,\frac{\partial\Psi}{\partial t}\right) = 0 \quad \text{for} \quad u \equiv \frac{1}{2}\left(\frac{\partial\Psi}{\partial t}\right)^2 + \frac{c^2}{2}\left(\nabla\Psi\right)^2 - \frac{1}{2}V(z)\Psi^2 - \operatorname{sech}\Psi} \tag{234}$$