Quantum Mechanics

Basic formalism

Canonical relations: $p = \frac{\hbar}{i}\frac{d}{dx}; \qquad [x,p] = i\hbar$

Schrödinger equation: $i\hbar \frac{d\,|\psi_S\rangle}{dt} = H\,|\psi_S(t)\rangle\,; \qquad U(t) = e^{-iHt/\hbar}$ for time-indep H

Heisenberg picture: $\mathcal{O}_H(t) = U^\dagger(t)\,\mathcal{O}_S\,U(t); \qquad \frac{d\mathcal{O}_H}{dt} = \frac{1}{i\hbar}\,[\mathcal{O}_H(t), H_H(t)] + \left.\frac{\partial \mathcal{O}}{\partial t}\right|_H$

Plane-wave normalization: $\langle x|p\rangle = \frac{e^{ipx/\hbar}}{\sqrt{2\pi\hbar}}$

Commutators: $[x, G(p)] = i\hbar\frac{\partial G}{\partial p}; \qquad [p, F(x)] = -i\hbar\frac{\partial F}{\partial x}$

BCH lemma: $e^{iX}Ye^{-iX} = Y + i[X,Y] - \frac{1}{2!}[X,[X,Y]] - \frac{i}{3!}[X,[X,[X,Y]]] + \ldots$

BCH formula: $e^X e^Y = e^{X+Y+\frac{1}{2}[X,Y]+\ldots}$

Quantum harmonic oscillator

Hamiltonian: $H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 = \frac{1}{2}\hbar\omega(P^2 + X^2) = \hbar\omega\left(a^\dagger\,a + \frac{1}{2}\right)$

Unitless coordinates: $X = x\left(\frac{m\omega}{\hbar}\right)^{1/2} = \frac{1}{\sqrt{2}}(a + a^\dagger)$

$P = \frac{p}{(m\omega\hbar)^{1/2}} = -\frac{i}{\sqrt{2}}(a - a^\dagger)$

Raising/lowering operators: $a = \frac{X + iP}{\sqrt{2}}; \qquad a^\dagger = \frac{X - iP}{\sqrt{2}}; \qquad [a, a^\dagger] = 1$

Eigenstates: $H\,|n\rangle = \hbar\omega\left(n + \frac{1}{2}\right);\; a\,|n\rangle = \sqrt{n}\,|n-1\rangle\,;\; a^\dagger\,|n\rangle = \sqrt{n+1}\,|n+1\rangle$

Ground state wave function: $\psi_0(X) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-X^2/2}$

Time-independent nondegenerate perturbation theory

$$\text{First-order energy correction:} \quad E_n^{(1)} = \left\langle n^{(0)} \middle| V \middle| n^{(0)} \right\rangle$$

$$\text{Second-order energy correction:} \quad E_n^{(2)} = \sum_{k \neq n} \frac{\left| \left\langle k^{(0)} \middle| V \middle| n^{(0)} \right\rangle \right|^2}{E_n^{(0)} - E_k^{(0)}}$$

$$\text{First-order ket correction:} \quad \left| n^{(1)} \right\rangle = \sum_{k \neq n} \left| k^{(0)} \right\rangle \frac{\left\langle k^{(0)} \middle| V \middle| n^{(0)} \right\rangle}{E_n^{(0)} - E_k^{(0)}}$$

$$\text{General ket:} \ \left(H_0 - E_n^{(0)}\right) \left| n^{(j)} \right\rangle = \left(E_n^{(1)} - V\right) \left| n^{(j-1)} \right\rangle + E_n^{(2)} \left| n^{(j-2)} \right\rangle + \ldots + E_n^{(j)} \left| n^{(0)} \right\rangle$$

$$\text{General energy:} \quad E_n^{(j)} = \left\langle n^{(0)} \middle| V \middle| n^{(j-1)} \right\rangle$$

Time-dependent perturbation theory

$$\text{Interaction picture:} \quad \left|\psi_I(t)\right\rangle = U_0^\dagger(t) \left|\psi_S(t)\right\rangle; \qquad \mathcal{O}_I(t) = U_0^\dagger(t) \mathcal{O}_S(t) U_0(t)$$

$$\text{Time evolution:} \quad i\hbar \frac{d\left|\psi_I\right\rangle}{dt} = V_I(t) \left|\psi_I(t)\right\rangle; \qquad \frac{d\mathcal{O}_H}{dt} = \frac{1}{i\hbar} \left[\mathcal{O}_I(t), H_0(t)\right] + \frac{\partial \mathcal{O}}{\partial t}\bigg|_I$$

$$\text{First-order transition amplitude:} \quad \langle m|U(t)|n\rangle = \delta_{mn} - \frac{i}{\hbar} \int_{t_0}^{t} dt'\, e^{i(E_m - E_n)t'/\hbar} \langle m|V_S(t')|n\rangle$$

$$\text{Fermi's golden rule (steplike):} \quad \frac{dP_m}{dt} = \frac{2\pi}{\hbar} |V_{mn}|^2 \,\delta(E_m - E_n) \approx \frac{2\pi}{\hbar^2} \rho(E_n) \left\langle |V_{mn}|^2 \right\rangle$$

$$\text{Fermi's golden rule (harmonic):} \quad \frac{dP_m^{\pm}}{dt} = \frac{2\pi}{\hbar} |V_{mn}|^2 \,\delta(E_m - (E_n \pm \hbar\omega)) \approx \frac{2\pi}{\hbar^2} \rho(E_n \pm \hbar\omega) \left\langle |V_{mn}|^2 \right\rangle$$

WKB approximation

$$\text{Quantization (free on both sides):} \quad \int_{x_L}^{x_R} dx\, \sqrt{2m(E - V(x))} = \pi\hbar \left(n + \frac{1}{2}\right),\ n \geq 0$$

$$\text{Quantization (hard wall):} \quad \int_0^{x_R} dx\, \sqrt{2m(E - V(x))} = \pi\hbar \left(n + \frac{3}{4}\right),\ n \geq 0$$

$$\text{Classically allowed region:} \quad \psi(x) \propto \frac{1}{\sqrt{k(x)}} \exp\left(\pm i \int^x dx'\, k(x')\right), \qquad E = \frac{\hbar^2 k^2}{2m} + V(x)$$

$$\text{Classically forbidden region:} \quad \psi(x) \propto \frac{1}{\sqrt{\kappa(x)}} \exp\left(\pm i \int^x dx'\, \kappa(x')\right), \qquad E = \frac{\hbar^2 \kappa^2}{2m} - V(x)$$

Scattering

General formalism

$$T\text{-matrix:}\quad T = \frac{V}{1-(E-H_0+i\eta)^{-1}V} = V + V(E-H_0+i\eta)^{-1}T$$

$$\text{Lippmann-Schwinger:}\quad |f\rangle = |i\rangle + (E-H_0+i\eta)^{-1}T\,|i\rangle = |i\rangle + (E-H_0+i\eta)^{-1}V\,|f\rangle$$

$$\text{Green's function:}\quad \frac{\hbar^2}{2m}\langle \mathbf{r}|(E-H_0+i\eta)^{-1}|\mathbf{r}'\rangle = -\frac{1}{4\pi}\frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}$$

$$\text{Scattering state (far-field approximation):}\quad f(\mathbf{r}) = \frac{1}{L^{3/2}}\left(e^{i\mathbf{k}\cdot\mathbf{r}} + f(\mathbf{k}',\mathbf{k})\frac{e^{ikr}}{r}\right)$$

$$\text{Scattering amplitude:}\quad f(\mathbf{k}',\mathbf{k}) = -\frac{mL^3}{2\pi\hbar^2}\langle \mathbf{k}'|V|f\rangle$$

$$\text{Differential cross-section:}\quad \frac{d\sigma}{d\Omega} = |f(\mathbf{k}',\mathbf{k})|^2$$

$$\text{Total cross-section:}\quad \sigma = \int d\Omega\,\frac{d\sigma}{d\Omega}$$

$$\text{Optical theorem:}\quad \sigma = \frac{4\pi}{k}\,\mathrm{Im}\,f(\mathbf{k},\mathbf{k})$$

Born approximation

$$\text{Momentum transfer:}\quad \mathbf{q} = \mathbf{k}' - \mathbf{k}; \qquad |\mathbf{q}| = 2k\sin\frac{\theta}{2}$$

$$\text{First Born approximation:}\quad f(\mathbf{k}',\mathbf{k}) \approx -\frac{m}{2\pi\hbar^2}\int d^3r'\,V(\mathbf{r}')e^{-i\mathbf{q}\cdot\mathbf{r}'}$$

$$\text{First Born approximation (isotropic } V\text{):}\quad f(\mathbf{k}',\mathbf{k}) = -\frac{2m}{q\hbar^2}\int_0^\infty dr\,r\sin(qr)V(r)$$

Partial waves for isotropic potentials

$$\text{Radial Schrödinger equation:}\quad -\frac{\hbar^2}{2m}\frac{d^2u_\ell}{dr^2} + \frac{\hbar^2\ell(\ell+1)}{2mr^2}u_\ell(r) + V(r)u_\ell(r) = Eu_\ell(r)$$

$$\text{Boundary conditions:}\quad u_\ell(r) = rR_\ell(r) \quad\text{and}\quad u_\ell \to 0 \text{ as } r\to 0$$

$$\text{Wave functions in free-particle zone:}\quad R_\ell(r) \propto \cos\delta_\ell\,j_\ell(kr) - \sin\delta_\ell\,n_\ell(kr)$$

$$R_\ell(r) \propto e^{2i\delta_\ell}h_\ell^{(1)}(kr) + h_\ell^{(2)}(kr)$$

$$u_0(r) \propto \cos\delta_0\sin kr + \sin\delta_0\cos kr = \sin(kr+\delta_0)$$

$$\text{Spherical Bessel functions:}\quad j_\ell(kr) \sim \frac{\sin(kr-\ell\pi/2)}{kr}; \qquad n_\ell(kr) \sim -\frac{\cos(kr-\ell\pi/2)}{kr}$$

Spherical Hankel functions: $h_\ell^{(1)}(kr) = j_\ell(kr) + i n_\ell(kr); \qquad h_\ell^{(2)}(kr) = j_\ell(kr) - i n_\ell(kr)$

Scattering amplitude: $f(\theta) = \sum_{\ell=0}^{\infty} (2\ell+1) f_\ell P_\ell(\cos\theta) \quad \text{for} \quad f_\ell = \frac{e^{2i\delta_\ell} - 1}{2ik} = \frac{e^{i\delta_\ell}\sin\delta_\ell}{k}$

$$\text{Cross-section:} \quad \sigma = \frac{4\pi}{k^2} \sum_{\ell=0}^{\infty} (2\ell+1)\sin^2\delta_\ell$$

Angular momentum

Commutation relations: $[J_i, J_j] = i\hbar\epsilon_{ijk}J_k; \qquad J_\pm = J_x \pm iJ_y; \qquad [J_+, J_-] = 2\hbar J_z$

$$\text{Eigenvalues:} \quad \mathbf{J}^2 \, |j,m\rangle = \hbar^2 \, j(j+1) \, |j,m\rangle \, ; \qquad J_z \, |j,m\rangle = \hbar \, m \, |j,m\rangle$$

$$\text{Raising and lowering operators:} \quad J_\pm \, |j,m\rangle = \hbar\sqrt{(j \mp m)(j \pm m + 1)} \, |j, m \pm 1\rangle$$

Wigner-Eckart theorem: $\langle \alpha'; \, j', \, m' | T_q^{(k)} | \alpha; \, j, \, m\rangle = \langle j, \, k; m, \, q | j, \, k; j', \, m'\rangle \cdot \frac{C}{\sqrt{2j+1}}$

Orbital angular momentum

$$\text{Definition:} \quad \mathbf{L} = \mathbf{r} \times \mathbf{p}$$

$$\text{Spherical harmonics:} \quad Y_0^0 = \sqrt{\frac{1}{4\pi}}$$

$$Y_1^0 = \sqrt{\frac{3}{4\pi}}\cos\theta = \sqrt{\frac{3}{4\pi}}\frac{z}{r}; \qquad Y_1^{\pm 1} = \mp\sqrt{\frac{3}{8\pi}} e^{\pm i\varphi}\sin\theta = \mp\sqrt{\frac{3}{8\pi}}\frac{x \pm iy}{r}$$

$$Y_\ell^0 = \sqrt{\frac{2\ell+1}{4\pi}} P_\ell(\cos\theta); \qquad Y_\ell^m(-\mathbf{r}) = (-1)^\ell Y_\ell^m(\mathbf{r}); \qquad Y_\ell^{m*} = (-1)^m Y_\ell^{-m}$$

Hydrogen atom eigenstates: $E_n = -\frac{m|e|^4}{2(4\pi\epsilon_0)^2\hbar^2}\frac{1}{n^2} \approx \frac{-13.6 \text{ eV}}{n^2} \quad \text{for} \quad m = \frac{m_e m_p}{m_e + m_p} \approx m_e$

$$\text{Bohr radius and fine structure constant:} \quad a_0 = \frac{4\pi\epsilon_0\hbar^2}{m|e|^2}; \qquad \alpha = \frac{|e|^2}{4\pi\epsilon_0\hbar c} \approx \frac{1}{137}$$

$$\text{Hydrogen atom radial wavefunctions:} \quad R_{10}(r) = \frac{2}{a_0^{3/2}} e^{-r/a_0}$$

$$R_{20}(r) = 2\left(\frac{1}{2a_0}\right)^{3/2}\left(1 - \frac{r}{2a_0}\right) e^{-r/(2a_0)}; \qquad R_{21}(r) = \frac{1}{\sqrt{3}}\left(\frac{1}{2a_0}\right)^{3/2} \frac{r}{a_0} e^{-r/(2a_0)}$$

If nuclear charge is equal to $Z|e|$, replace a_0 by a_0/Z

Spin angular momentum

$$\text{Pauli matrices:} \quad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}; \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\text{Bloch sphere:} \quad |\theta, \varphi\rangle = \cos\frac{\theta}{2}\,|\uparrow\rangle + e^{i\varphi}\sin\frac{\theta}{2}\,|\downarrow\rangle$$

$$\text{Spin 1/2 in a magnetic field (SI):} \quad H = -\boldsymbol{\mu}\cdot\mathbf{B} = -\gamma\frac{\hbar}{2}\boldsymbol{\sigma}\cdot\mathbf{B} = -\frac{g}{2}\mu_B\boldsymbol{\sigma}\cdot\mathbf{B}$$

$$\text{Gyromagnetic ratio; Bohr magneton (SI):} \quad \gamma = -\frac{g|e|}{2m}; \qquad \mu_B = \frac{|e|\hbar}{2m}; \qquad g_{\text{electron}} \approx 2$$

Classical Mechanics

Newtonian mechanics

$$\text{Nonrelativistic Doppler effect:}\quad f_r = \frac{c \pm v_r}{c \pm v_s} f_s$$

$+v_r$ if receiver moves toward source; $+v_s$ if source moves away from receiver

$$\text{Centrifugal; Coriolis; Euler forces:}\quad -m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r});\;\; -2m\boldsymbol{\omega} \times \dot{\mathbf{r}};\;\; -m\frac{d\boldsymbol{\omega}}{dt} \times \mathbf{r}$$

Formation rules

	Lagrangian	Hamiltonian
Nonrelativistic Newtonian mass:	$+\frac{1}{2}m\lvert\dot{\mathbf{r}}\rvert^2$	$+\frac{\lvert\mathbf{p}\rvert^2}{2m}$
Scalar potential:	$-V(\mathbf{r}, t)$	$+\,V(\mathbf{r}, t)$
Vector potential:	$+q\dot{\mathbf{r}} \cdot \mathbf{A}$	replace $\mathbf{p}$ by $\mathbf{p} - q\mathbf{A}$
Holonomic constraint:	$+\lambda\,(\text{constraint})$	$-\lambda(\text{constraint})$
Special relativistic mass:	$-mc^2\sqrt{1 - \frac{\lvert\dot{\mathbf{r}}\rvert^2}{c^2}}$	$+\sqrt{(mc^2)^2 + (\lvert\mathbf{p}\rvert c)^2}$

Lagrangian mechanics

Lagrangians

$$\text{Stationary action principle:}\quad \delta\left(\int dt\, L(q, \dot{q}, t)\right) = 0$$

$$\text{Euler-Lagrange equation:}\quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0$$

$$\text{Canonical momenta:}\quad p = \frac{\partial L}{\partial \dot{q}}$$

$$\text{Energy:}\quad E = \frac{\partial L}{\partial \dot{q}}\dot{q} - L$$

Lagrangian densities

$$\text{Stationary action principle:}\quad \delta\left(\int dt\, d^3x\, \mathcal{L}(\phi, \dot{\phi}, \boldsymbol{\nabla}\phi, t)\right) = 0$$

$$\text{Euler-Lagrange equation:}\quad \frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial(\partial_t \phi)}\right) + \boldsymbol{\nabla} \cdot \left(\frac{\partial \mathcal{L}}{\partial(\boldsymbol{\nabla}\phi)}\right) - \frac{\partial \mathcal{L}}{\partial \phi} = 0$$

$$\text{Energy continuity equation:}\quad \frac{\partial}{\partial t}\left[\frac{\partial \mathcal{L}}{\partial(\partial_t \phi)}\frac{\partial \phi}{\partial t} - \mathcal{L}\right] + \boldsymbol{\nabla} \cdot \left[\frac{\partial \mathcal{L}}{\partial(\boldsymbol{\nabla}\phi)}\frac{\partial \phi}{\partial t}\right] = 0 \quad \text{if} \quad \frac{\partial \mathcal{L}}{\partial t} = 0$$

$$\text{Energy-momentum tensor:}\quad \partial_\mu T^\mu{}_\nu = 0 \quad \text{for} \quad T^\mu{}_\nu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\partial_\nu \phi - \delta^\mu{}_\nu \mathcal{L}$$

Hamiltonian mechanics

$$\text{Hamiltonian:} \quad H(p,q,t) = p\dot{q} - L = \frac{\partial L}{\partial \dot{q}}\dot{q} - L$$

$$\text{Hamilton's equations of motion:} \quad \dot{p} = -\frac{\partial H}{\partial q}; \qquad \dot{q} = \frac{\partial H}{\partial p}$$

$$\text{Hamilton-Jacobi equation:} \quad H\left(q, \frac{\partial W(q,P)}{\partial q}\right) = E$$

$$\text{Poisson bracket:} \quad \{f,g\} = \sum_i \left(\frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i}\right)$$

$$\text{Adiabatic invariant:} \quad J \propto \oint p\, dq$$

Canonical transformations

$F_1(q,Q,t)$	$p_i = \frac{\partial F_1}{\partial q_i}$;	$P_i = -\frac{\partial F_1}{\partial Q_i}$;	$H' = H + \frac{\partial F_1}{\partial t}$
$F_2(q,P,t)$	$p_i = \frac{\partial F_2}{\partial q_i}$;	$Q_i = \frac{\partial F_2}{\partial P_i}$;	$H' = H + \frac{\partial F_2}{\partial t}$
$F_3(p,Q,t)$	$q_i = -\frac{\partial F_3}{\partial p_i}$;	$P_i = -\frac{\partial F_3}{\partial Q_i}$;	$H' = H + \frac{\partial F_3}{\partial t}$
$F_4(p,P,t)$	$q_i = -\frac{\partial F_4}{\partial p_i}$;	$Q_i = \frac{\partial F_4}{\partial P_i}$;	$H' = H + \frac{\partial F_4}{\partial t}$

Fluids

$$\text{Euler hydrodynamics:} \quad \frac{\partial \rho}{\partial t} + \boldsymbol{\nabla}\cdot(\rho\mathbf{v}) = 0$$

$$\frac{\partial S}{\partial t} + (\mathbf{v}\cdot\nabla)S = 0; \qquad \rho\left[\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v}\right] = -\boldsymbol{\nabla}P(\rho,s)$$

$$\text{Speed of sound:} \quad c = \sqrt{\frac{\partial P}{\partial \rho}}$$

$$\text{Bernoulli's law:} \quad \frac{v^2}{2} + gz + \frac{P}{\rho} = \text{constant}$$

$$\text{Kelvin circulation theorem:} \quad \frac{D}{Dt}\oint \mathbf{v}\cdot d\boldsymbol{\ell} = 0 \quad \text{for} \quad \frac{D}{Dt} = \frac{\partial}{\partial t} = \mathbf{v}\cdot\nabla$$

$$\text{Vortex:} \quad \mathbf{v} = \frac{\kappa}{r}\hat{\theta}$$

Special relativity

$$\text{Sign convention:} \quad (-1, +1, +1, +1)$$

$$\text{Relativistic factors:} \quad \beta = \frac{v}{c}; \qquad \gamma = \frac{1}{\sqrt{1-\beta^2}}$$

$$\text{Lorentz boost:} \quad \Lambda = \begin{pmatrix} \gamma & -\beta\gamma \\ -\beta\gamma & \gamma \end{pmatrix}$$

$$\text{Metric invariance:} \quad \Lambda\, \eta\, \Lambda^T = \eta$$

$$\text{Time dilation; Lorentz contraction:} \quad \Delta t = \gamma\, \Delta\tau_{\text{rest}}; \qquad \ell = \frac{\ell_{\text{rest}}}{\gamma}$$

Classical mechanics

$$\text{Four-vectors:} \quad x^\mu = (ct, \mathbf{x}); \quad p^\mu = (E, c\mathbf{p}); \quad F^\mu = \left(\frac{dE}{d\tau}, c\frac{d\mathbf{p}}{d\tau}\right); \quad \partial^\mu = \left(-\frac{1}{c}\frac{\partial}{\partial t}, \nabla\right)$$

$$\text{Velocity transformation:} \quad u' = \frac{u-v}{1-uv/c^2}$$

$$\text{Energy and momentum:} \quad E = \gamma m c^2; \qquad \mathbf{p} = \gamma m \mathbf{v}; \qquad \mathbf{v} = c^2\frac{\mathbf{p}}{E}$$

$$\text{Energy-momentum relationship:} \quad E = \sqrt{(mc^2)^2 + (|\mathbf{p}|c)^2}$$

$$\text{Relativistic Doppler effect:} \quad \frac{\lambda_r}{\lambda_s} = \frac{f_s}{f_r} = \sqrt{\frac{1+\beta}{1-\beta}} \quad (\beta > 0 \text{ if source and receiver move apart})$$

Electromagnetism

$$\text{Four-vectors:} \quad J^\mu = (c\rho, \mathbf{J}); \qquad A^\mu = (\phi/c, \mathbf{A})$$

$$\text{Field transformations:} \quad \mathbf{E}'_\parallel = \mathbf{E}_\parallel; \qquad \mathbf{B}'_\parallel = \mathbf{B}_\parallel$$

$$\mathbf{E}'_\perp = \gamma\left(\mathbf{E}_\perp + \mathbf{v}\times\mathbf{B}\right); \qquad \mathbf{B}'_\perp = \gamma\left(\mathbf{B}_\perp - \frac{1}{c^2}\mathbf{v}\times\mathbf{E}\right)$$

$$\text{Electromagnetic tensor:} \quad F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu = \begin{pmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & 0 & -B_z & B_y \\ E_y/c & B_z & 0 & -B_x \\ E_z/c & -B_y & B_x & 0 \end{pmatrix}$$

$$\text{Four-vector Maxwell's equations:} \quad (\partial_\mu \partial^\mu) A^\nu = \partial_\mu F^{\mu\nu} = \mu_0 J^\nu$$

Statistical Mechanics

Elementary thermodynamics

$$\text{Euler relation:} \quad E = TS - PV + \mu N$$

$$\text{Energy:} \quad dE = \delta Q - \delta W = TdS - PdV + \mu dN$$

$$\text{Helmholtz free energy:} \quad F = E - TS; \qquad dF = -SdT - PdV + \mu dN$$

$$\text{Gibbs free energy:} \quad G = E - TS + PV; \qquad dG = -SdT + VdP + \mu dN$$

$$\text{Enthalpy:} \quad H = E + PV; \qquad dH = TdS + VdP + \mu dN$$

$$\text{Grand potential:} \quad \Phi = -PV = E - TS - \mu N$$

$$\text{Maxwell:} \quad \left.\frac{\partial T}{\partial V}\right|_S = -\left.\frac{\partial P}{\partial S}\right|_V; \qquad \left.\frac{\partial T}{\partial P}\right|_S = \left.\frac{\partial V}{\partial S}\right|_P; \qquad \left.\frac{\partial S}{\partial V}\right|_T = \left.\frac{\partial P}{\partial T}\right|_V; \qquad \left.\frac{\partial S}{\partial P}\right|_T = -\left.\frac{\partial V}{\partial T}\right|_P$$

Stat mech definitions

$$\text{Entropy and temperature:} \quad S = k \ln \Omega; \qquad \frac{1}{T} = \frac{\partial S}{\partial E}; \qquad \beta = (kT)^{-1}$$

$$\text{Stirling's formula:} \quad N! \approx N^N e^{-N} \sqrt{2\pi N}; \qquad \ln(N!) \approx N \ln N - N$$

$$\text{Heat capacity:} \quad C_X = T \left.\frac{\partial S}{\partial T}\right|_X$$

$$\text{Poisson distribution:} \quad P(N) = \frac{e^{-\lambda} \lambda^N}{N!}$$

Ensembles

$$\text{Partition function:} \quad Z = \sum_r e^{-\beta E_r}; \qquad E = -\frac{\partial \ln Z}{\partial \beta} = kT^2 \frac{\partial \ln Z}{\partial T}; \qquad F = -kT \ln Z$$

$$\text{Grand partition function:} \quad Q = \sum_r e^{-\beta(E_r - \mu N_r)}; \qquad N = kT \frac{\partial \ln Q}{\partial \mu}$$

Monatomic ideal gas

$$\text{Partition function:} \quad Z = \frac{V^N}{N! \lambda^{3N}} \quad \text{for} \quad \lambda = \left(\frac{h^2}{2\pi m k T}\right)^{1/2}$$

$$\text{Equation of state:} \quad PV = NkT = nRT$$

$$\text{Energy:} \quad E = \frac{3}{2} NkT$$

$$\text{Free energy:} \quad F = -NkT\left(\ln\left[\frac{V}{N}\frac{1}{\lambda^3}\right] + 1\right)$$

$$\text{Chemical potential:} \quad \mu = -kT\ln\left[\frac{V}{N}\frac{1}{\lambda^3}\right] < 0$$

$$\text{Adiabatic expansion:} \quad PV^\gamma = \text{constant} \quad \text{for} \quad \gamma = \frac{C_P}{C_V} = \frac{Nk}{C_V} + 1$$

$$\text{Maxwell velocity distribution:} \quad P(v_x, v_y, v_z) = \left(\frac{m}{2\pi kT}\right)^{3/2}\exp\left(-\frac{m(v_x^2+v_y^2+v_z^2)}{2kT}\right)$$

Heat engines

$$\text{Conservation of energy:} \quad Q_H - W = Q_C$$

$$\text{Efficiency of a heat engine:} \quad \eta = \frac{W}{Q_H} = 1 - \frac{Q_C}{Q_H} < 1; \qquad \eta = 1 - \frac{T_C}{T_H} \quad \text{for a Carnot cycle}$$

Quantum gases

Occupation numbers

$$\text{Fermi-Dirac distribution:} \quad n_k^{FD}(\epsilon) = \frac{1}{e^{\beta(\epsilon_k-\mu)}+1}$$

$$\text{Bose-Einstein distribution:} \quad n_k^{BE}(\epsilon) = \frac{1}{e^{\beta(\epsilon_k-\mu)}-1}$$

$$\text{Maxwell-Boltzmann distribution:} \quad n_k^{MB}(\epsilon) = e^{-\beta(\epsilon_k-\mu)}$$

Fermi gases

$$\text{Equation of state:} \quad P = \frac{2}{3}\frac{E}{V}$$

$$\text{Fermi energy (nonrelativistic 3-D electron gas):} \quad E_F = \frac{\hbar^2}{2m}\left(3\pi^2\frac{N}{V}\right)^{2/3}$$

$$\text{Energy at } T=0 \text{ (nonrelativistic 3-D electron gas):} \quad E = \frac{3}{5}NE_F$$

Bose gases

$$\text{Equation of state:} \quad P = \frac{2}{3}\frac{E}{V}$$

$$\text{Critical temperature:} \quad N = gV\int\frac{d^3p}{h^3}\frac{1}{e^{p^2/(2mkT_c)}-1}$$

$$\text{Number of particles in ground state:} \quad \frac{N(\epsilon=0)}{N} = 1 - \left(\frac{T}{T_c}\right)^{3/2}$$

Photon gases and blackbody radiation

$$E = \frac{4}{c}\sigma V T^4; \qquad P = \frac{1}{3}\frac{E}{V}; \qquad \frac{\text{Power radiated}}{\text{Area}} = \sigma T^4 \quad \text{for} \quad \sigma = \frac{\pi^2 k^4}{60\hbar^3 c^2}$$

Debye model

$$\omega_D = c_s \left(6\pi^2 \frac{N}{V}\right)^{1/3}$$

Other examples

Joule-Thomson process (throttling)

$$\Delta H = 0; \qquad \mu = \left.\frac{\partial T}{\partial P}\right|_H = -\frac{1}{C_P}\left(T\left.\frac{\partial S}{\partial P}\right|_T + V\right) = -\frac{1}{C_P}V(1-\alpha T) \quad \text{for} \quad \alpha = \frac{1}{V}\left.\frac{\partial V}{\partial T}\right|_P$$

Clausius-Claperyon equation

$$\frac{dP}{dT} = \frac{L}{T\Delta v}$$ L latent heat of vaporization per particle, v volume per particle

Magnetization

$$M = kT\frac{\partial \ln Z}{\partial B} = -\left.\frac{\partial F}{\partial B}\right|_{T,V,N}$$

Van der Waals gas

$$\left(P + \frac{an^2}{V^2}\right)(V - nb) = nRT$$

Electromagnetism

Maxwell's equations:
$$\boldsymbol{\nabla} \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \qquad \boldsymbol{\nabla} \cdot \mathbf{B} = 0$$
$$\boldsymbol{\nabla} \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \qquad \boldsymbol{\nabla} \times \mathbf{B} = \mu_0 \mathbf{J} + \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t}$$

Maxwell's equations in matter:
$$\boldsymbol{\nabla} \cdot \mathbf{D} = \rho_f \qquad \boldsymbol{\nabla} \cdot \mathbf{B} = 0$$
$$\boldsymbol{\nabla} \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \qquad \boldsymbol{\nabla} \times \mathbf{H} = \mathbf{J}_f + \frac{\partial \mathbf{D}}{\partial t}$$

Lorentz force law: $\mathbf{F} = q\mathbf{E} + q\mathbf{v} \times \mathbf{B}; \qquad d\mathbf{F} = \rho\mathbf{E} + \mathbf{J} \times \mathbf{B}$

Continuity equation: $\dfrac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot \mathbf{J} = 0$

Potentials: $\mathbf{E} = -\dfrac{\partial \mathbf{A}}{\partial t} - \boldsymbol{\nabla}\phi; \qquad \mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}$

Speed of light: $\mu_0 \epsilon_0 = \dfrac{1}{c^2}$

Statics

Electrostatics

Integral formulas: $\mathbf{E} = \dfrac{1}{4\pi\epsilon_0}\displaystyle\int d^3r' \, \frac{\rho(\mathbf{r}')}{\mathscr{r}^2}\hat{\boldsymbol{\mathscr{r}}}; \;\; V = \dfrac{1}{4\pi\epsilon_0}\displaystyle\int d^3r' \, \frac{\rho(\mathbf{r}')}{\mathscr{r}}; \;\; \boldsymbol{\mathscr{r}} = \mathbf{r} - \mathbf{r}'$

Electric potential: $\Delta V = -\displaystyle\int \mathbf{E} \cdot d\boldsymbol{\ell}; \qquad \mathbf{E} = -\boldsymbol{\nabla} V$

Work required to assemble a charge collection: $W = \dfrac{1}{2}\sum q_i V_i = \dfrac{1}{2}\displaystyle\int d^3r' \, \rho V$

Magnetostatics

Vector potential (Coulomb gauge): $\boldsymbol{\nabla} \cdot \mathbf{A} = 0; \;\; \nabla^2 \mathbf{A} = -\mu_0 \mathbf{J}; \;\; \mathbf{A} = \dfrac{\mu_0}{4\pi}\displaystyle\int d^3r' \, \frac{\mathbf{J}}{\mathscr{r}}$

Biot-Savart: $\mathbf{B} = \dfrac{\mu_0}{4\pi}\displaystyle\int d^3r' \, \frac{\mathbf{J} \times \hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} = \dfrac{\mu_0}{4\pi}\displaystyle\int \frac{I \, d\boldsymbol{\ell}' \times \hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}$

Multipole expansion

Electric multipole expansion: $V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\left[\frac{Q}{r} + \frac{\mathbf{p}\cdot\mathbf{r}}{r^3} + Q_{ij}\frac{3r_ir_j - r^2\delta_{ij}}{r^5} + \ldots\right]$

$$Q = \int d^3r\,\rho(\mathbf{r}); \qquad \mathbf{p} = \int d^3r\,\rho(\mathbf{r})\,\mathbf{r}; \qquad Q_{ij} = \frac{1}{2}\int d^3r\,\rho(\mathbf{r})\,r_ir_j$$

Legendre polynomial expansions: $V_{\text{out}} = \frac{1}{4\pi\epsilon_0}\sum_{\ell=0}^{\infty}\frac{1}{r^{\ell+1}}P_\ell(\cos\theta)\left[\int d^3r'\,\rho(\mathbf{r}')(r')^\ell P_\ell(\cos\theta')\right]$

$$V_{\text{in}} = \frac{1}{4\pi\epsilon_0}\sum_{\ell=0}^{\infty} r^\ell P_\ell(\cos\theta)\left[\int d^3r'\,\frac{1}{(r')^{\ell+1}}\rho(\mathbf{r}')P_\ell(\cos\theta')\right]$$

Electric dipoles

Electric dipole moment; potential $\mathbf{p} = \int d^3r'\,\rho(\mathbf{r}')\mathbf{r}'$; $V = \frac{1}{4\pi\epsilon_0}\frac{\mathbf{p}\cdot\hat{\mathbf{r}}}{r^2}$

Electric field, dipole at the origin: $\mathbf{E} = \frac{p}{4\pi\epsilon_0 r^3}(2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\theta}) = \frac{1}{4\pi\epsilon_0}\frac{3(\mathbf{p}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{p}}{r^3} \quad (r \neq 0)$

Ideal electric dipoles: $\mathbf{p} = q\mathbf{d}$; $\boldsymbol{\tau} = \mathbf{p}\times\mathbf{E}$; $\mathbf{F} = (\mathbf{p}\cdot\nabla)\mathbf{E}$; $U = -\mathbf{p}\cdot\mathbf{E}$

Magnetic dipoles

Magnetic dipole moment; potential: $\mathbf{m} = \frac{1}{2}\int d^3r'\,\mathbf{r}'\times\mathbf{J}(\mathbf{r}')$; $\qquad \mathbf{A} = \frac{\mu_0}{4\pi}\frac{\mathbf{m}\times\hat{\mathbf{r}}}{r^2}$

Magnetic field, dipole at the origin: $\mathbf{B} = \frac{\mu_0 m}{4\pi}(2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\theta}) = \frac{\mu_0}{4\pi}\frac{3(\mathbf{m}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{m}}{r^3} \quad (r \neq 0)$

Ideal magnetic dipoles: $\mathbf{m} = Ia\hat{\mathbf{n}}$; $\boldsymbol{\tau} = \mathbf{m}\times\mathbf{B}$; $\mathbf{F} = (\mathbf{m}\cdot\nabla)\mathbf{B}$; $U = -\mathbf{m}\cdot\mathbf{B}$

Magnetic dipole moment from angular momentum: $\mathbf{m} = \frac{Q}{2M}\mathbf{L}$

Matter

Bound charges: $\rho_b = -\boldsymbol{\nabla}\cdot\mathbf{P}$; $\qquad \sigma_b = \mathbf{P}\cdot\hat{\mathbf{n}}$

Electric displacement: $\mathbf{D} = \epsilon_0\mathbf{E} + \mathbf{P}$

Linear isotropic homogeneous dielectrics: $\mathbf{P} = \epsilon_0\chi_e\mathbf{E}$; $\qquad \mathbf{D} = \epsilon_0(1+\chi_e)\mathbf{E} = \epsilon\mathbf{E}$

Bound currents: $\mathbf{J}_b = \boldsymbol{\nabla}\times\mathbf{M}$; $\qquad \mathbf{K}_b = \mathbf{M}\times\hat{\mathbf{n}}$

Fictitious magnetic charges: $\rho_m = -\boldsymbol{\nabla}\cdot\mathbf{M}$; $\qquad \sigma_m = \mathbf{M}\cdot\hat{\mathbf{n}}$

H field: $\mathbf{H} = \frac{\mathbf{B}}{\mu_0} - \mathbf{M}$

Linear isotropic homogeneous magnetic materials: $\mathbf{M} = \chi_m\mathbf{H}$; $\qquad \mathbf{B} = \mu_0(1+\chi_m)\mathbf{H} = \mu\mathbf{H}$

Potential theory

Solutions to Laplace's equation

Spherical—azimuthal symmetry: $$V = \sum_{\ell=0}^{\infty}\left(A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}}\right)P_\ell(\cos\theta)$$

Cylindrical—azimuthal symmetry ($D_0 = 0$ in a simply connected region):

$$V = (A_0 + B_0 \ln s)(C_0 + D_0\varphi) + \sum_{k=1}^{\infty}\left(A_k s^k + \frac{B_k}{s^k}\right)(C_k \sin(k\varphi) + D_k\cos(k\varphi))$$

Cylindrical—Bessel functions:

$$s\frac{\partial}{\partial s}\left(s\frac{\partial R}{\partial s}\right) + \left(s^2\kappa^2 - \alpha^2\right)R = 0; \qquad \frac{\partial^2\Phi}{\partial\varphi^2} = -\alpha^2\Phi; \qquad \frac{\partial^2 Z}{\partial z^2} = \kappa^2 Z$$

$$R_\alpha^\kappa = \begin{cases} A_0 + B_0\ln s & \text{for} \quad \kappa = 0 \quad \text{and} \quad \alpha = 0 \\ A_\alpha s^\alpha + B_\alpha s^{-\alpha} & \text{for} \quad \kappa = 0 \quad \text{and} \quad \alpha \neq 0 \\ A_\alpha^\kappa J_\alpha(\kappa s) + B_\alpha^\kappa Y_\alpha(\kappa s) & \text{for} \quad \kappa^2 > 0 \quad \text{(exponential in } z\text{, oscillatory in } s\text{)} \\ A_\alpha^\kappa I_\alpha(\kappa s) + B_\alpha^\kappa K_\alpha(\kappa s) & \text{for} \quad \kappa^2 < 0 \quad \text{(oscillatory in } z\text{, exponential in } s\text{)} \end{cases}$$

$$Y_\alpha, K_\alpha \to \infty \quad \text{as} \quad s \to 0$$

Mathematical relationships

Sines and cosines: $$\int_0^a dx\, \sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi x}{a}\right) = \frac{a}{2}\delta_{nm}$$

$$\int_0^{2\pi} d\varphi\, e^{im\varphi}e^{-in\varphi} = 2\pi\delta_{mn}$$

Legendre polynomials: $$\int_{-1}^{1} dx\, P_\ell(x)P_m(x) = \frac{2}{2\ell+1}\delta_{\ell m}$$

$$P_0(x) = 1; \qquad P_1(x) = x; \qquad P_2(x) = \frac{3x^2-1}{2}; \qquad P_3(x) = \frac{5x^3-3x}{2}; \qquad P_\ell(1) = 1$$

Boundary conditions

$$\hat{\mathbf{n}}\cdot(\mathbf{E}_2 - \mathbf{E}_1) = \frac{\sigma}{\epsilon_0}; \qquad \hat{\mathbf{n}}\cdot(\mathbf{D}_2 - \mathbf{D}_1) = \sigma_f; \qquad \hat{\mathbf{n}}\times(\mathbf{E}_2 - \mathbf{E}_1) = 0$$

$$\hat{\mathbf{n}}\cdot(\mathbf{B}_2 - \mathbf{B}_1) = 0; \qquad \hat{\mathbf{n}}\times(\mathbf{B}_2 - \mathbf{B}_1) = \mu_0\mathbf{K}; \qquad \hat{\mathbf{n}}\times(\mathbf{H}_2 - \mathbf{H}_1) = \mathbf{K}_f$$

$\hat{\mathbf{n}}$ points from medium 1 to medium 2

Green's functions

$$\text{Definition:}\quad \nabla^2 G(\mathbf{r},\mathbf{r}') = -\frac{1}{\epsilon_0}\delta(\mathbf{r}-\mathbf{r}')$$

$$\text{Dirichlet Green's fn.:}\quad \phi(\mathbf{r}) = \int d^3r'\,\rho(\mathbf{r}')G_D(\mathbf{r},\mathbf{r}') - \epsilon_0\int_S dS'\,\phi_S(\mathbf{r}')\frac{\partial G_D}{\partial n'};\quad G_D(\mathbf{r}_S,\mathbf{r}') = 0$$

$$\text{Delta functions:}\quad \delta(z-z') = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\, e^{ik(z-z')} = \frac{1}{\pi}\int_0^{\infty} dk\,\cos(k(z-z'))$$

$$\delta(\varphi-\varphi') = \frac{1}{2\pi}\sum_{m=-\infty}^{\infty} e^{im(\varphi-\varphi')}$$

Energy and momentum

Energy

$$\text{Mechanical power:}\quad \frac{\partial u_{mech}}{\partial t} = \mathbf{J}\cdot\mathbf{E}$$

$$\text{Electromagnetic energy density:}\quad u_{EM} = \frac{1}{2}(\mathbf{E}\cdot\mathbf{D}+\mathbf{B}\cdot\mathbf{H}) = \frac{\epsilon_0}{2}E^2 + \frac{1}{2\mu_0}B^2 \quad \text{in a vacuum}$$

$$\text{Electromagnetic energy density (time-harmonic field):}\quad \langle u_{EM}\rangle = \frac{1}{2}\frac{\partial}{\partial\omega}(\epsilon\omega)E^2$$

$$\text{Poynting vector:}\quad \mathbf{S} = \mathbf{E}\times\mathbf{H} = \frac{1}{\mu_0}\mathbf{E}\times\mathbf{B} \quad \text{in a vaccum}$$

$$\text{Poynting's theorem:}\quad \frac{\partial u_{mech}}{\partial t} + \frac{\partial u_{EM}}{\partial t} + \boldsymbol{\nabla}\cdot\mathbf{S} = 0$$

Momentum

$$\text{Mechanical force density:}\quad \mathbf{f} = \rho\mathbf{E} + \mathbf{J}\times\mathbf{B}$$

$$\text{Electromagnetic momentum density:}\quad \mathbf{g} = \frac{\mathbf{S}}{c^2} = \epsilon_0\mathbf{E}\times\mathbf{B} \quad \text{in a vacuum}$$

$$\text{Stress-energy tensor (vaccum):}\quad T_{ij} = \epsilon_0\left(E_iE_j - \frac{1}{2}\delta_{ij}E^2\right) + \frac{1}{\mu_0}\left(B_iB_j - \frac{1}{2}\delta_{ij}B^2\right)$$

$$\text{Momentum conservation:}\quad \mathbf{f} + \frac{\partial\mathbf{g}}{\partial t} - \boldsymbol{\nabla}\cdot\mathbf{T} = 0$$

Circuits

$$\text{Capacitors:}\quad C = \frac{Q}{V};\qquad E = \frac{1}{2}CV^2 = \frac{1}{2}\frac{Q^2}{C};\qquad Z = \frac{1}{i\omega C}$$

$$\text{Inductors:}\quad \varepsilon = -\int \mathbf{E}\cdot d\boldsymbol{\ell} = -L\frac{dI}{dt};\qquad W = \frac{1}{2}LI^2;\qquad Z = i\omega L$$

$$\text{Resistors:}\quad V = IR;\qquad P = I^2R = IV;\qquad Z = R$$

Waves

$$\text{Wave equation (vacuum):}\quad \nabla^2\mathbf{E} - \frac{1}{c^2}\frac{\partial^2\mathbf{E}}{\partial t^2} = 0$$

$$\text{Plane waves:}\quad \omega = \frac{c}{n}k; \qquad n = \sqrt{\frac{\mu\epsilon}{\mu_0\epsilon_0}}; \qquad \langle\mathbf{S}\rangle = \frac{1}{2}\epsilon_0 c E^2; \qquad \mathbf{B} = \frac{\mathbf{k}\times\mathbf{E}}{\omega}$$

$$\text{Phase velocity; group velocity:}\quad \mathbf{v}_p = \frac{\omega}{k}\hat{\mathbf{k}}; \qquad \mathbf{v}_g = \frac{d\omega}{d\mathbf{k}}$$

$$\text{Impedance:}\quad Z = \frac{E}{H} = \sqrt{\frac{\mu}{\epsilon}}$$

$$\text{Conductors:}\quad \tilde{\epsilon} = \epsilon + i\,\frac{\sigma}{\omega}; \qquad k^2 = \tilde{\epsilon}\mu\omega^2$$

$$\text{Plasma frequency:}\quad \omega_p = \sqrt{\frac{q^2 n}{m\epsilon_0}}; \qquad \epsilon = \epsilon_0\left(1 - \frac{\omega_p^2}{\omega^2}\right)$$

$$\text{Time-averaging theorem:}\quad \langle \mathrm{Re}(A)\,\mathrm{Re}(B)\rangle = \frac{1}{2}\,\mathrm{Re}(AB^*) = \frac{1}{2}\,\mathrm{Re}(A^*B)$$

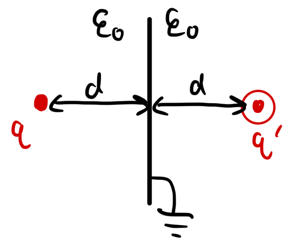
Waveguides

Transverse Electric (TE)	Transverse Magnetic (TM)
$E_z = 0$	$H_z = 0$
$(\nabla_\perp^2 + \gamma^2)\,H_z = 0$	$(\nabla_\perp^2 + \gamma^2)\,E_z = 0$
$\mathbf{H}_\perp = \frac{ih}{\gamma^2}\nabla_\perp H_z$	$\mathbf{E}_\perp = \frac{ih}{\gamma^2}\nabla_\perp E_z$
$\mathbf{E}_\perp = -\frac{\omega\mu}{h}\hat{\mathbf{z}}\times\mathbf{H}_\perp$	$\mathbf{H}_\perp = \frac{\omega\epsilon}{h}\hat{\mathbf{z}}\times\mathbf{E}_\perp$

$$\text{Structural dispersion:}\quad \gamma^2 = \mu\epsilon\omega^2 - h^2; \qquad \gamma^2 > 0$$

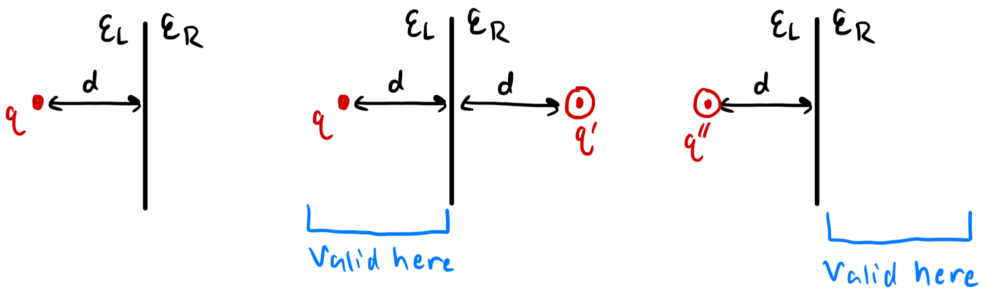
Method of images

Conducting plane



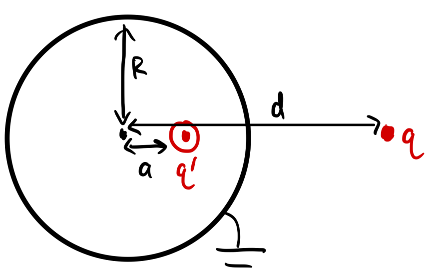
$$q' = -q$$

Dielectric plane



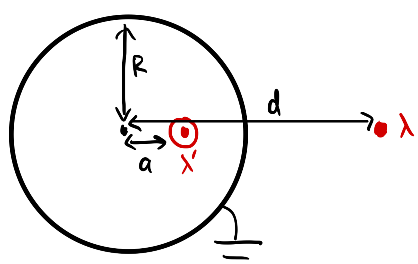
$$q' = \frac{\epsilon_L - \epsilon_R}{\epsilon_L + \epsilon_R} q; \qquad q'' = \frac{2\epsilon_R}{\epsilon_L + \epsilon_R} q$$

Conducting sphere



$$q' = -\frac{R}{d} q; \qquad a = \frac{R^2}{d}$$

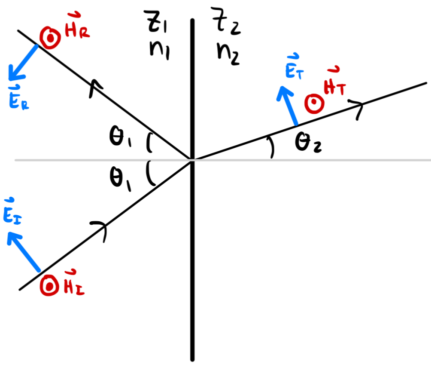
Conducting cylinder



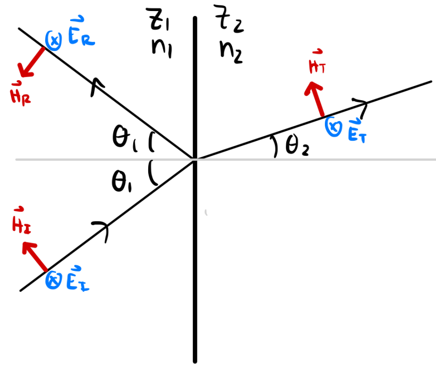
$$\lambda' = -\lambda; \qquad a = \frac{R^2}{d}$$

Fresnel relations

$$\text{Snell's law:} \quad n_1 \sin\theta_1 = n_2 \sin\theta_2$$



p-polarization



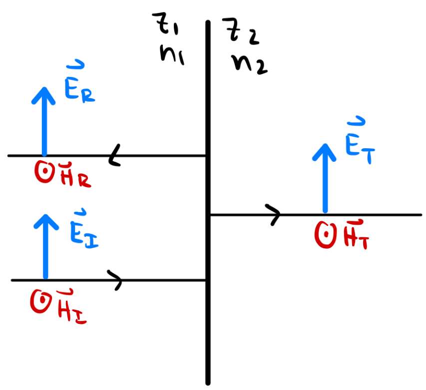
s-polarization

$$r_p = \frac{Z_1 \cos\theta_1 - Z_2 \cos\theta_2}{Z_1 \cos\theta_1 + Z_2 \cos\theta_2} \qquad r_s = \frac{Z_2 \cos\theta_1 - Z_1 \cos\theta_2}{Z_2 \cos\theta_1 + Z_1 \cos\theta_2}$$

$$r_p = \frac{n_2 \cos\theta_1 - n_1 \cos\theta_2}{n_2 \cos\theta_1 + n_1 \cos\theta_2} \qquad r_s = \frac{n_1 \cos\theta_1 - n_2 \cos\theta_2}{n_1 \cos\theta_1 + n_2 \cos\theta_2}$$

$$t_p = \frac{2Z_2 \cos\theta_1}{Z_1 \cos\theta_1 + Z_2 \cos\theta_2} \qquad t_s = \frac{2Z_2 \cos\theta_1}{Z_2 \cos\theta_1 + Z_1 \cos\theta_2}$$

$$t_p = \frac{2n_1 \cos\theta_1}{n_2 \cos\theta_1 + n_1 \cos\theta_2} \qquad t_s = \frac{2n_1 \cos\theta_1}{n_1 \cos\theta_1 + n_2 \cos\theta_2}$$



Normal incidence

$$r = \frac{Z_2 - Z_1}{Z_1 + Z_2} = \frac{n_1 - n_2}{n_1 + n_2}; \qquad t = \frac{2Z_2}{Z_1 + Z_2} = \frac{2n_1}{n_1 + n_2}$$

$$\text{Power reflection coefficients:} \quad R = |r|^2; \qquad T = \left| \frac{\langle \mathbf{S}_T \rangle \cdot \hat{\mathbf{n}}}{\langle \mathbf{S}_I \rangle \cdot \hat{\mathbf{n}}} \right| = \frac{Z_1 \cos\theta_2}{Z_2 \cos\theta_1} |t|^2$$

Electrodynamics

Lorenz gauge condition: $\boldsymbol{\nabla} \cdot \mathbf{A}_L + \frac{1}{c^2}\frac{\partial \phi_L}{\partial t} = 0$

Inhomogenous wave equations: $\nabla^2 \phi_L - \frac{1}{c^2}\frac{\partial^2 \phi_L}{\partial t^2} = -\frac{\rho}{\epsilon_0}$; $\nabla^2 \mathbf{A}_L - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}_L}{\partial t^2} = -\mu_0 \mathbf{J}$

Retarded potentials: $\phi(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0}\int d^3r' \, \frac{\rho(\mathbf{r}', t - \mathscr{r}/c)}{\mathscr{r}}$; $\mathbf{A}(\mathbf{r}, t) = \frac{\mu_0}{4\pi}\int d^3r' \, \frac{\mathbf{J}(\mathbf{r}', t - \mathscr{r}/c)}{\mathscr{r}}$

Radiation and scattering

Generally applicable—radiation

$$\text{Radiation vector:}\quad \boldsymbol{\alpha}(\mathbf{r},t) = \frac{\partial}{\partial t}\int d^3r'\,\mathbf{j}\left(\mathbf{r}', t - \frac{r}{c} + \frac{\hat{\mathbf{r}}\cdot\mathbf{r}'}{c}\right) \quad (20.143)$$

$$\text{Fields:}\quad c\mathbf{B}_{\rm rad} = -\hat{\mathbf{r}}\times\frac{\partial \mathbf{A}_{\rm rad}}{\partial t} = -\frac{\mu_0}{4\pi r}\hat{\mathbf{r}}\times\boldsymbol{\alpha} \quad (20.107)$$

$$\mathbf{E}_{\rm rad} = -\hat{\mathbf{r}}\times c\mathbf{B}_{\rm rad} = \hat{\mathbf{r}}\times\left(\hat{\mathbf{r}}\times\frac{\partial \mathbf{A}_{\rm rad}}{\partial t}\right) \quad (20.108)$$

$$\hat{\mathbf{r}}\times\mathbf{E}_{\rm rad} = c\mathbf{B}_{\rm rad};\qquad \mathbf{E}_{\rm rad}\times c\mathbf{B}_{\rm rad} = \hat{\mathbf{r}}\left|\mathbf{E}_{\rm rad}\right|^2;\qquad \left|\mathbf{E}_{\rm rad}\right| = c\left|\mathbf{B}_{\rm rad}\right| \quad (20.109)$$

$$\text{Power radiated:}\quad \frac{dP}{d\Omega} = r^2\hat{\mathbf{r}}\cdot\mathbf{S}_{\rm rad} = \frac{r^2}{c\mu_0}\left|\mathbf{E}_{\rm rad}\right|^2 = \frac{r^2 c}{\mu_0}\left|\mathbf{B}_{\rm rad}\right|^2 = \frac{\mu_0}{(4\pi)^2 c}\left|\hat{\mathbf{r}}\times\boldsymbol{\alpha}\right|^2 \quad (20.81)\ (20.110)$$

$$\mathbf{S}_{\rm rad} = \frac{1}{\mu_0 c}(\mathbf{E}_{\rm rad}\times c\mathbf{B}_{\rm rad}) = \frac{\left|\mathbf{E}_{\rm rad}\right|^2}{\mu_0 c} = \frac{c}{\mu_0}\left|\mathbf{B}_{\rm rad}\right|^2$$

$$\text{For}\quad \hat{\mathbf{j}}(\mathbf{k},\omega) = \int d^3r'\,\mathbf{j}(\mathbf{r}',\omega)e^{-i\mathbf{k}\cdot\mathbf{r}'},\qquad \left\langle\frac{dP}{d\Omega}\right\rangle = \frac{\mu_0\omega^2}{2(4\pi)^2 c}\left|\hat{\mathbf{r}}\times\hat{\mathbf{j}}(\mathbf{k},\omega)\right|^2 \quad (20.120)$$

Generally applicable—scattering

$$\left|\langle\mathbf{S}_{\rm inc}\rangle\right| = \frac{1}{2}\epsilon_0 c|E_0|^2 = \frac{1}{2}\frac{1}{\mu_0 c}|E_0|^2$$

$$\frac{d\sigma}{d\Omega} = \frac{\langle dP/d\Omega\rangle}{\left|\langle\mathbf{S}_{\rm inc}\rangle\right|} = \frac{r^2\left\langle\mathbf{S}_{\rm rad}\cdot\hat{\mathbf{r}}\right\rangle}{\left|\langle\mathbf{S}_{\rm inc}\rangle\right|} = r^2\frac{\left|\mathbf{E}_{\rm rad}\right|^2}{\left|\mathbf{E}_{\rm inc}\right|^2} \quad (21.4)$$

Radiation—specific cases

Nonrelativistic particle (Larmor formula)

$$\mathbf{j}\left(\mathbf{r}, t - r/c + (\hat{\mathbf{r}}\cdot\mathbf{r}_0)/c\right) \approx \mathbf{j}\left(\mathbf{r}, t - r/c\right) = q\,\mathbf{v}\left(t - r/c\right)\,\delta\left(\mathbf{r} - \mathbf{r}_0\left(t - r/c\right)\right) \quad (20.111)$$

$$\boldsymbol{\alpha} = q\,\mathbf{a}(t - r/c) = q\,\mathbf{a}_{\rm ret}$$

$$\mathbf{E}_{\rm rad} = \frac{\mu_0 q}{4\pi r}\,\hat{\mathbf{r}}\times\left[\hat{\mathbf{r}}\times\mathbf{a}(t - r/c)\right] \quad (20.113)$$

$$\frac{dP}{d\Omega} = \frac{\mu_0}{(4\pi)^2 c}\,q^2|\mathbf{a}_{\rm ret}|^2\sin^2\theta \quad (20.114)$$

$$P = \frac{\mu_0}{4\pi c}\,\frac{2}{3}\,q^2|\mathbf{a}_{\rm ret}|^2 = \frac{1}{4\pi\epsilon_0}\,\frac{2}{3}\,\frac{q^2\left|\mathbf{a}_{\rm ret}\right|^2}{c^3} \quad (20.115)$$

Electric dipole radiation $\mathbf{p} = \int d^3r' \, \rho(\mathbf{r}')\mathbf{r}'$

$$\boldsymbol{\alpha} = \ddot{\mathbf{p}}_{\rm ret} \quad (20.151)$$

$$c\mathbf{B}_{\rm rad} = -\frac{\mu_0}{4\pi} \frac{\hat{\mathbf{r}} \times \ddot{\mathbf{p}}_{\rm ret}}{r} \quad (20.152)$$

$$\mathbf{E}_{\rm rad} = \frac{\mu_0}{4\pi} \frac{\hat{\mathbf{r}}(\hat{\mathbf{r}} \cdot \ddot{\mathbf{p}}_{\rm ret}) - \ddot{\mathbf{p}}_{\rm ret}}{r} \quad (20.153)$$

$$\frac{dP}{d\Omega} = \frac{\mu_0}{(4\pi)^2 c} \, |\hat{\mathbf{r}} \times \ddot{\mathbf{p}}_{\rm ret}|^2 \quad (20.154)$$

$$P = \frac{\mu_0}{4\pi c} \frac{2}{3} \, |\ddot{\mathbf{p}}_{\rm ret}|^2 \quad (20.155)$$

Magnetic dipole radiation $\mathbf{m} = \frac{1}{2} \int d^3r' \, \mathbf{r}' \times j(\mathbf{r}')$

$$\boldsymbol{\alpha} = \frac{1}{c} \ddot{\mathbf{m}}_{\rm ret} \times \hat{\mathbf{r}} \quad (20.172)$$

$$\mathbf{E}_{\rm rad} = \frac{\mu_0}{4\pi c} \frac{\hat{\mathbf{r}} \times \ddot{\mathbf{m}}_{\rm ret}}{r} \quad (20.174)$$

$$c\mathbf{B}_{\rm rad} = \frac{\mu_0}{4\pi c} \frac{\hat{\mathbf{r}}(\hat{\mathbf{r}} \cdot \ddot{\mathbf{m}}_{\rm ret}) - \ddot{\mathbf{m}}_{\rm ret}}{r} \quad (20.173)$$

$$\frac{dP}{d\Omega} = \frac{\mu_0}{(4\pi)^2 c^3} \, |\hat{\mathbf{r}} \times \ddot{\mathbf{m}}_{\rm ret}|^2 \quad (20.179)$$

$$P = \frac{\mu_0}{4\pi c^3} \frac{2}{3} \, |\ddot{\mathbf{m}}_{\rm ret}|^2 \quad (20.180)$$

Scattering—specific cases

Thomson scattering

(EM plane wave incident on a free electron $-e$; long-wavelength approximation)

$\mathbf{j}(\mathbf{r}, t) = \dfrac{ie^2 E_0}{m\omega} \exp\left[i\left(\mathbf{k}_0 \cdot \mathbf{r}_0 - \omega t\right)\right] \, \delta(\mathbf{r}(t) - \mathbf{r}_0(t)) \, \hat{\mathbf{e}}_0 \quad (21.10)$, computed via eqn. of motion

$$\mathbf{p}(t) = -\frac{e^2 E_0}{m\omega^2} \hat{\mathbf{e}}_0 e^{-i\omega t} \quad \text{for} \quad r_0 \approx 0 \quad (21.13)$$

$$\frac{d\sigma}{d\Omega} = r_e^2 |\hat{\mathbf{k}} \times \hat{\mathbf{e}}_0|^2 = r_e^2 \left(1 - |\hat{\mathbf{k}} \cdot \hat{\mathbf{e}}_0|^2\right) \quad \text{for} \quad r_e = \frac{e^2}{4\pi\epsilon_0 mc^2} \quad (21.11)$$

$$\sigma = \frac{8\pi}{3} r_e^2 \quad \text{for linearly polarized or unpolarized} \qquad (21.18)$$

Rayleigh scattering

(EM plane wave incident on a *small* object; $\mathbf{p}$ and $\mathbf{m}$ must be calculated; $k_0 = \omega/c$)

$$\frac{d\sigma}{d\Omega} = \left(\frac{k_0^2}{4\pi\epsilon_0 E_0 c}\right)^2 |\hat{\mathbf{k}} \times \mathbf{m} + \hat{\mathbf{k}} \times (\hat{\mathbf{k}} \times c\hat{\mathbf{p}})|^2 \quad (21.20)$$

$$\text{If } \mathbf{p} = \alpha\epsilon_0 E_0 \hat{\mathbf{e}}_0 \text{ and } \mathbf{m} = 0, \quad \frac{d\sigma}{d\Omega} = \left(\frac{k_0^2\alpha}{4\pi}\right)^2 \left(1 - |\mathbf{k}\cdot\hat{\mathbf{e}}_0|^2\right) = \left(\frac{k_0^2}{4\pi\epsilon_0 E_0}\right)^2 |\hat{\mathbf{k}}\times\mathbf{p}|^2 \quad (21.21)$$

$$\hat{\mathbf{j}} = -i\omega\mathbf{p} \quad \text{for a small dipole at the origin}$$

Born approximation $\chi_e \ll 1$

(EM plane wave—weak dielectric response)

$$\mathbf{j} = \frac{\partial \mathbf{P}}{\partial t} \approx -i\omega\epsilon_0\chi \left[\hat{\mathbf{e}}_0 E_0 e^{i(\mathbf{k}_0\cdot\mathbf{r}-\omega t)}\right] \quad (21.46)$$

$$\frac{d\sigma}{d\Omega} = \left(\frac{k_0^2}{4\pi}\right)^2 |\hat{\mathbf{k}} \times \hat{\mathbf{e}}_0|^2 \left|\int d^3r'\,\chi e^{-i\mathbf{q}\cdot\mathbf{r}'}\right|^2 \quad \text{for} \quad \mathbf{q} = \mathbf{k} - \mathbf{k}_0 \quad (21.47)$$

Mathematics

Trigonometry

$$\sin x = \frac{e^{ix} - e^{-ix}}{2i}; \qquad \cos x = \frac{e^{ix} + e^{-ix}}{2}$$

$$\sin^2 x + \cos^2 x = 1; \qquad 1 + \tan^2 x = \sec^2 x; \qquad 1 + \cot^2 x = \csc^2 x$$

$$\sin\Big(x \pm \frac{\pi}{2}\Big) = \pm \cos x; \qquad \cos\Big(x \pm \frac{\pi}{2}\Big) = \mp \sin x$$

$$\sin(x + \pi) = -\sin x; \qquad \cos(x + \pi) = -\cos x$$

$$\sin(a \pm b) = \sin a \cos b \pm \cos a \sin b$$

$$\cos(a \pm b) = \cos a \cos b \mp \sin a \sin b$$

$$\tan(a \pm b) = \frac{\tan a \pm \tan b}{1 \mp \tan a \tan b}$$

$$\cos a \cos b = \frac{1}{2}\left[\cos(a - b) + \cos(a + b)\right]$$

$$\sin a \sin b = \frac{1}{2}\left[\cos(a - b) - \cos(a + b)\right]$$

$$\sin a \cos b = \frac{1}{2}\left[\sin(a + b) + \sin(a - b)\right]$$

$$\sin(2x) = 2 \sin x \cos x$$

$$\cos(2x) = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$$

$$\tan(2x) = \frac{2 \tan x}{1 - \tan^2 x}$$

$$\sin^2 x = \frac{1 - \cos(2x)}{2}; \qquad \cos^2 x = \frac{1 + \cos(2x)}{2}$$

$$(\sin x)' = \cos x; \qquad (\cos x)' = -\sin x; \qquad (\tan x)' = \sec^2 x$$

$$(\csc x)' = -\csc x \cot x; \qquad (\sec x)' = \sec x \tan x; \qquad (\cot x)' = -\csc^2 x$$

$$(\ln(\sec x + \tan x))' = \sec x; \qquad (\ln(\csc x + \cot x))' = -\csc x$$

$$\Big(\arcsin\Big(\frac{x}{c}\Big)\Big)' = \frac{1}{\sqrt{c^2 - x^2}}; \qquad \Big(\arctan\Big(\frac{x}{c}\Big)\Big)' = \frac{c}{c^2 + x^2}$$

Thrigonometry

$$\sinh x = \frac{e^x - e^{-x}}{2}; \qquad \cosh x = \frac{e^x + e^{-x}}{2}; \qquad \tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{e^{2x} - 1}{e^{2x} + 1}$$

$$\cosh^2 x - \sinh^2 x = 1; \qquad \operatorname{sech}^2 x = 1 - \tanh^2 x; \qquad \operatorname{csch}^2 x = \coth^2 x - 1$$

$$\sinh(a \pm b) = \sinh a \cosh b \pm \cosh a \sinh b$$

$$\cosh(a \pm b) = \cosh a \cosh b \pm \sinh a \sinh b$$

$$\tanh(a \pm b) = \frac{\tanh a \pm \tanh b}{1 \pm \tanh a \tanh b}$$

$$\sinh(2x) = 2 \sinh x \cosh x$$

$$\cosh(2x) = \sinh^2 x + \cosh^2 x = 2\sinh^2 x + 1 = 2\cosh^2 x - 1$$

$$\tanh(2x) = \frac{2 \tanh x}{1 + \tanh^2 x}$$

$$\sinh^2 x = \frac{\cosh(2x) - 1}{2}; \qquad \cosh^2 x = \frac{\cosh(2x) + 1}{2}$$

$$\operatorname{arcsinh}(x) = \ln\left(x + \sqrt{x^2 + 1}\right); \qquad \operatorname{arccosh}(x) = \ln\left(x + \sqrt{x^2 - 1}\right)$$

$$(\sinh x)' = \cosh x; \qquad (\cosh x)' = \sinh x; \qquad (\tanh x)' = \operatorname{sech}^2 x$$

$$\left(\operatorname{arcsinh}\left(\frac{x}{c}\right)\right)' = \frac{1}{\sqrt{x^2 + c^2}}; \qquad \left(\operatorname{arctanh}\left(\frac{x}{c}\right)\right)' = \frac{c}{c^2 - x^2}$$

Calculus

$$\int_{-\infty}^{\infty} dx\, e^{-\alpha x^2} = \frac{\pi^{1/2}}{\alpha^{1/2}}; \qquad \int_{-\infty}^{\infty} dx\, x^2 e^{-\alpha x^2} = \frac{1}{2}\frac{\pi^{1/2}}{\alpha^{3/2}}; \qquad \int_{-\infty}^{\infty} dx\, x^4 e^{-\alpha x^2} = \frac{3}{4}\frac{\pi^{1/2}}{\alpha^{5/2}}$$

$$\int_{-\infty}^{\infty} dx\, e^{-\alpha x^2 + \beta x} = \frac{\pi^{1/2}}{\alpha^{1/2}}\, e^{\beta^2/(4\alpha)}$$

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk\, \tilde{f}(k) e^{ikx}; \quad \tilde{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx\, f(x) e^{-ikx}; \quad \int_{-\infty}^{\infty} dx\, |f(x)|^2 = \int_{-\infty}^{\infty} dk\, |\tilde{f}(k)|^2$$

$$\delta(x - x_0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk\, e^{ik(x - x_0)}; \qquad \delta(ax) = \frac{\delta(x)}{|a|}; \qquad \delta(g(x)) = \sum_{\substack{i \\ g(x_i) = 0}} \frac{\delta(x - x_i)}{|g'(x_i)|}$$

$$\delta(\mathbf{r}) = \frac{\delta(r)\delta(\theta)\delta(\varphi)}{r^2 \sin\theta}$$

$$\nabla^2 \frac{1}{|\mathbf{r} - \mathbf{r}'|} = -4\pi\delta(\mathbf{r} - \mathbf{r}')$$

Unit vectors and vector algebra

$$\hat{\mathbf{s}} = \hat{\mathbf{x}}\cos\varphi + \hat{\mathbf{y}}\sin\varphi$$
$$\hat{\varphi} = -\hat{\mathbf{x}}\sin\varphi + \hat{\mathbf{y}}\cos\varphi$$

$$\hat{\mathbf{x}} = \hat{\mathbf{s}}\cos\varphi - \hat{\varphi}\sin\varphi$$
$$\hat{\mathbf{y}} = \hat{\mathbf{s}}\sin\varphi + \hat{\varphi}\cos\varphi$$

$$\hat{\mathbf{r}} = \hat{\mathbf{x}}\sin\theta\cos\varphi + \hat{\mathbf{y}}\sin\theta\sin\varphi + \hat{\mathbf{z}}\cos\theta$$
$$\hat{\theta} = \hat{\mathbf{x}}\cos\theta\cos\varphi + \hat{\mathbf{y}}\cos\theta\sin\varphi - \hat{\mathbf{z}}\sin\theta$$
$$\hat{\varphi} = -\hat{\mathbf{x}}\sin\varphi + \hat{\mathbf{y}}\cos\varphi$$

$$\hat{\mathbf{x}} = \hat{\mathbf{r}}\sin\theta\cos\varphi + \hat{\theta}\cos\theta\cos\varphi - \hat{\varphi}\sin\varphi$$
$$\hat{\mathbf{y}} = \hat{\mathbf{r}}\sin\theta\sin\varphi + \hat{\theta}\cos\theta\sin\varphi + \hat{\varphi}\cos\varphi$$
$$\hat{\mathbf{z}} = \hat{\mathbf{r}}\cos\theta - \hat{\theta}\sin\theta$$

$$\epsilon_{ijk}\epsilon_{i\ell m} = \delta_{j\ell}\delta_{km} - \delta_{jm}\delta_{k\ell}$$
$$(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d}) = (\mathbf{a}\cdot\mathbf{c})(\mathbf{b}\cdot\mathbf{d}) - (\mathbf{a}\cdot\mathbf{d})(\mathbf{b}\cdot\mathbf{c})$$

VECTOR DERIVATIVES

Cartesian. $d\mathbf{l} = dx\,\hat{\mathbf{x}} + dy\,\hat{\mathbf{y}} + dz\,\hat{\mathbf{z}}; \quad d\tau = dx\,dy\,dz$

Gradient : $$\nabla t = \frac{\partial t}{\partial x}\hat{\mathbf{x}} + \frac{\partial t}{\partial y}\hat{\mathbf{y}} + \frac{\partial t}{\partial z}\hat{\mathbf{z}}$$

Divergence : $$\nabla \cdot \mathbf{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$

Curl : $$\nabla \times \mathbf{v} = \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right)\hat{\mathbf{x}} + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right)\hat{\mathbf{y}} + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\hat{\mathbf{z}}$$

Laplacian : $$\nabla^2 t = \frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} + \frac{\partial^2 t}{\partial z^2}$$

Spherical. $d\mathbf{l} = dr\,\hat{\mathbf{r}} + r\,d\theta\,\hat{\boldsymbol{\theta}} + r\sin\theta\,d\phi\,\hat{\boldsymbol{\phi}}; \quad d\tau = r^2\sin\theta\,dr\,d\theta\,d\phi$

Gradient : $$\nabla t = \frac{\partial t}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial t}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial t}{\partial \phi}\hat{\boldsymbol{\phi}}$$

Divergence : $$\nabla \cdot \mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta\, v_\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi}$$

Curl : $$\nabla \times \mathbf{v} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta\, v_\phi) - \frac{\partial v_\theta}{\partial\phi}\right]\hat{\mathbf{r}} + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial v_r}{\partial\phi} - \frac{\partial}{\partial r}(r v_\phi)\right]\hat{\boldsymbol{\theta}} + \frac{1}{r}\left[\frac{\partial}{\partial r}(r v_\theta) - \frac{\partial v_r}{\partial\theta}\right]\hat{\boldsymbol{\phi}}$$

Laplacian : $$\nabla^2 t = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial t}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial t}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 t}{\partial\phi^2}$$

Cylindrical. $d\mathbf{l} = ds\,\hat{\mathbf{s}} + s\,d\phi\,\hat{\boldsymbol{\phi}} + dz\,\hat{\mathbf{z}}; \quad d\tau = s\,ds\,d\phi\,dz$

Gradient : $$\nabla t = \frac{\partial t}{\partial s}\hat{\mathbf{s}} + \frac{1}{s}\frac{\partial t}{\partial\phi}\hat{\boldsymbol{\phi}} + \frac{\partial t}{\partial z}\hat{\mathbf{z}}$$

Divergence : $$\nabla \cdot \mathbf{v} = \frac{1}{s}\frac{\partial}{\partial s}(s v_s) + \frac{1}{s}\frac{\partial v_\phi}{\partial\phi} + \frac{\partial v_z}{\partial z}$$

Curl : $$\nabla \times \mathbf{v} = \left[\frac{1}{s}\frac{\partial v_z}{\partial\phi} - \frac{\partial v_\phi}{\partial z}\right]\hat{\mathbf{s}} + \left[\frac{\partial v_s}{\partial z} - \frac{\partial v_z}{\partial s}\right]\hat{\boldsymbol{\phi}} + \frac{1}{s}\left[\frac{\partial}{\partial s}(s v_\phi) - \frac{\partial v_s}{\partial\phi}\right]\hat{\mathbf{z}}$$

Laplacian : $$\nabla^2 t = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial t}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 t}{\partial\phi^2} + \frac{\partial^2 t}{\partial z^2}$$

VECTOR IDENTITIES

Triple Products

(1) $\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \mathbf{B}\cdot(\mathbf{C}\times\mathbf{A}) = \mathbf{C}\cdot(\mathbf{A}\times\mathbf{B})$

(2) $\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = \mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}(\mathbf{A}\cdot\mathbf{B})$

Product Rules

(3) $\nabla(fg) = f(\nabla g) + g(\nabla f)$

(4) $\nabla(\mathbf{A}\cdot\mathbf{B}) = \mathbf{A}\times(\nabla\times\mathbf{B}) + \mathbf{B}\times(\nabla\times\mathbf{A}) + (\mathbf{A}\cdot\nabla)\mathbf{B} + (\mathbf{B}\cdot\nabla)\mathbf{A}$

(5) $\nabla\cdot(f\mathbf{A}) = f(\nabla\cdot\mathbf{A}) + \mathbf{A}\cdot(\nabla f)$

(6) $\nabla\cdot(\mathbf{A}\times\mathbf{B}) = \mathbf{B}\cdot(\nabla\times\mathbf{A}) - \mathbf{A}\cdot(\nabla\times\mathbf{B})$

(7) $\nabla\times(f\mathbf{A}) = f(\nabla\times\mathbf{A}) - \mathbf{A}\times(\nabla f)$

(8) $\nabla\times(\mathbf{A}\times\mathbf{B}) = (\mathbf{B}\cdot\nabla)\mathbf{A} - (\mathbf{A}\cdot\nabla)\mathbf{B} + \mathbf{A}(\nabla\cdot\mathbf{B}) - \mathbf{B}(\nabla\cdot\mathbf{A})$

Second Derivatives

(9) $\nabla\cdot(\nabla\times\mathbf{A}) = 0$

(10) $\nabla\times(\nabla f) = 0$

(11) $\nabla\times(\nabla\times\mathbf{A}) = \nabla(\nabla\cdot\mathbf{A}) - \nabla^2\mathbf{A}$

FUNDAMENTAL THEOREMS

Gradient Theorem : $\int_{\mathbf{a}}^{\mathbf{b}}(\nabla f)\cdot d\mathbf{l} = f(\mathbf{b}) - f(\mathbf{a})$

Divergence Theorem : $\int(\nabla\cdot\mathbf{A})\,d\tau = \oint\mathbf{A}\cdot d\mathbf{a}$

Curl Theorem : $\int(\nabla\times\mathbf{A})\cdot d\mathbf{a} = \oint\mathbf{A}\cdot d\mathbf{l}$