8. (Electromagnetism)

In this problem we consider charged objects in nonrelativistic circular motion in the plane transverse to a uniform magnetic field $\mathbf{B}$.

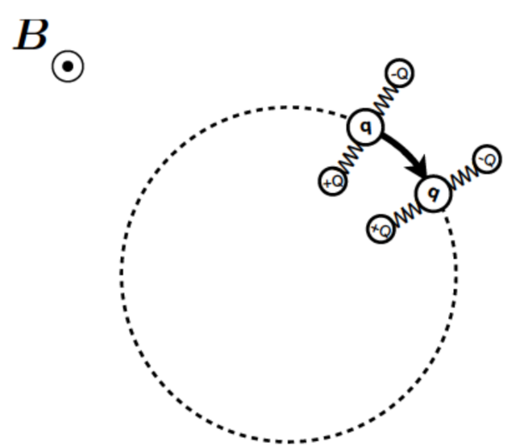
(a) For a point particle with mass m and charge q, what is the angular velocity ω?

We now consider the case that the point particle is replaced by a finite size polarizable molecular ion. As a simple model for this, along with the point charge q we introduce two more point charges $+Q$ and $-Q$, of negligible masses, which are attracted to q by a harmonic potential, $V = \frac{1}{2}k\left[|\mathbf{x}_Q - \mathbf{x}_q|^2 + |\mathbf{x}_{-Q} - \mathbf{x}_q|^2\right]$. Since this is meant to capture all the interactions between the particles you should not include any additional Coulomb interactions between the particles.

(b) We now look for a solution in which the molecule undergoes circular motion at some angular velocity ω'. Take the orbital radius of charge q to be R, and assume that the spring constant k is large so that the molecular size is small compared to R, i.e., $|\mathbf{x}_{\pm Q} - \mathbf{x}_q| \ll R$. Compute that angular velocity ω' to lowest nontrivial order in the small parameter $1/k$. As shown in the figure, you can assume that the distance between the charges is constant and that the axis of the molecule always points toward the center of the orbit. You can also set the masses of charges Q and $-Q$ to zero.

(c) We want to express the result for ω' in terms of the electric polarizability of the molecule, α, which you now compute. Place the molecule in a constant electric field $\mathbf{E}$ (assume the charge q is held in place by some other force). Compute the induced electric dipole moment $\mathbf{d}$, taking the location of charge q to be the origin of coordinates. Compute the polarizability α via $\mathbf{d} = \alpha\mathbf{E}$.

(d) Using your results above, express the fractional angular velocity shift, $\Delta\omega/\omega = (\omega' - \omega)/\omega$ in terms of α, working to first nontrivial order in small α. The resulting expression for the angular velocity shift holds for more general molecules, and has been observed experimentally.



Solution: *Solution by Jonah Hyman (jthyman@g.ucla.edu)*

This problem contains a lot of information, so we'll take it step-by-step. We work in SI units throughout.

(a) This problem concerns the motion of a charged particle in a uniform magnetic field, which is called "cyclotron motion." Take the magnetic field to be in the z-direction, $\mathbf{B} = B\hat{\mathbf{z}}$. We are considering circular motion in the plane transverse to $\hat{\mathbf{z}}$, namely the xy-plane. Therefore, $\mathbf{B}$ is perpendicular to the velocity of the particle $\mathbf{v}$. The magnitude of the Lorentz force on the particle is therefore

$$\begin{aligned} F &= |q\mathbf{v} \times \mathbf{B}| \\ &= qvB \quad \text{since } \mathbf{B} \perp \mathbf{v} \end{aligned} \tag{114}$$

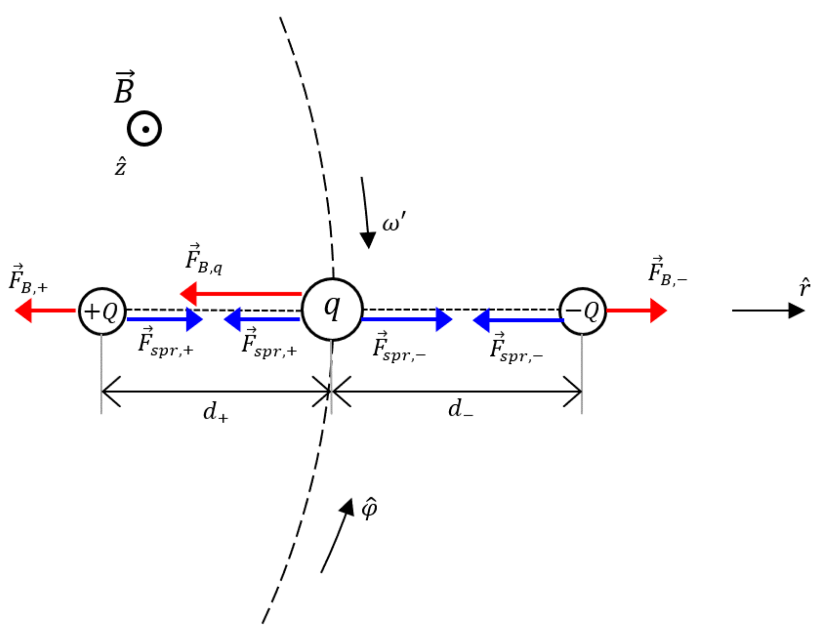
This is the only force acting on the particle. Since the particle is in circular motion, the net force on the particle must equal the mass times the centripetal acceleration. We can then solve for the speed of the particle v:

$$\begin{aligned} F &= m\, a_{\text{centrip}} \\ qvB &= m\frac{v^2}{R} \\ v &= \frac{qBR}{m} \end{aligned} \tag{115}$$

For circular motion, the speed is related to the angular speed by $v = \omega R$. From this, we can derive the "cyclotron frequency"

$$\omega = \frac{v}{R} = \boxed{\frac{qB}{m}} \tag{116}$$

(b) For this part, it is helpful to first draw a force diagram including the Lorentz force on each charge due to the external magnetic field $\mathbf{F}_B$ (direction calculated by the right-hand rule, assuming q and Q are positive) as well as the spring forces $\mathbf{F}_{\text{spr}}$ between the charges, which come from the harmonic potential mentioned by the problem. (As specified in the problem, we ignore the Coulomb forces between the charges.)



We can write the Newton's second law equations $\mathbf{F} = m\mathbf{a}$ for each of the three charges. Since charges $\pm Q$ are assumed to be massless, we can set the net force on each charge to be zero:

$$\begin{aligned}
\text{For charge } q\text{:} \quad & \left(-F_{B,q} - F_{\text{spr},+} + F_{\text{spr},-}\right)\hat{\mathbf{r}} = m\mathbf{a}_q && (117)\\
\text{For charge } +Q\text{:} \quad & \left(-F_{B,+} + F_{\text{spr},+}\right)\hat{\mathbf{r}} = 0 && (118)\\
\text{For charge } -Q\text{:} \quad & \left(-F_{\text{spr},-} + F_{B,-}\right)\hat{\mathbf{r}} = 0 && (119)
\end{aligned}$$

We can calculate the magnitude of the Lorentz forces $F_{B,q}$ and $F_{B,\pm}$ by the Lorentz force law. Let $\mathbf{v}_q$, $\mathbf{v}_+$, and $\mathbf{v}_-$ be the velocities of the charges q, $+Q$, and $-Q$, respectively. Since $\mathbf{B}$ is perpendicular to all the velocities, the magnitudes of the Lorentz forces are

$$F_{B,q} = qvB \quad \text{and} \quad F_{B,\pm} = Qv_{\pm}B \tag{120}$$

Using the relation $v = \omega r$ for circular motion, and taking into account the fact that the charges $\pm Q$ are orbiting at radii $R_+ \equiv R - d_+$ and $R_- \equiv R + d_-$, respectively, we get

$$F_{B,q} = q\omega' RB; \qquad F_{B,+} = Q\omega'(R - d_+)B; \qquad F_{B,-} = Q\omega'(R + d_-)B \tag{121}$$

The spring forces are of magnitude $k\left|\mathbf{x}_{\pm Q} - \mathbf{x}_q\right|$, which means that we can write their magnitudes

$$F_{\text{spr},+} = kd_+ \quad \text{and} \quad F_{\text{spr},-} = kd_- \tag{122}$$

Finally, the acceleration of the middle charge q must be of magnitude equal to the centripetal acceleration: Using $v = \omega r$, we get

$$\mathbf{a}_q = -a_{\text{centrip}}\,\hat{\mathbf{r}} = -\frac{v_q^2}{R}\hat{\mathbf{r}} = -(\omega')^2 R\,\hat{\mathbf{r}} \tag{123}$$

Putting everything together, we can write the Newton's second law equations

$$\begin{aligned}
\text{For charge } q\text{:} \quad & -q\omega' RB - kd_+ + kd_- = -m(\omega')^2 R && (124)\\
\text{For charge } +Q\text{:} \quad & -Q\omega'(R - d_+)B + kd_+ = 0 && (125)\\
\text{For charge } -Q\text{:} \quad & -kd_- + Q\omega'(R + d_-)B = 0 && (126)
\end{aligned}$$

We can use (125) and (126) to solve for the distances d_+ and d_-:

$$\begin{aligned}
Q\omega' Bd_+ + kd_+ &= Q\omega' BR \quad \text{from (125)}\\
d_+ &= \frac{Q\omega' BR}{Q\omega' B + k} && (127)\\
& && (128)
\end{aligned}$$

$$\begin{aligned}
-kd_- + Q\omega' Bd_- &= -Q\omega' BR \quad \text{from (126)}\\
d_- &= -\frac{Q\omega' BR}{Q\omega' B - k} && (129)\\
& && (130)
\end{aligned}$$

Plugging these into (124), we get

$$\begin{aligned}
-q\omega' BR - k\left(d_+ - d_-\right) &= -m\left(\omega'\right)^2 R\\
-q\omega' BR - k\left(\frac{Q\omega' BR}{Q\omega' B + k} + \frac{Q\omega' BR}{Q\omega' B - k}\right) &= -m\left(\omega'\right)^2 R\\
q\omega' BR + k\left(\frac{Q\omega' BR}{Q\omega' B + k} + \frac{Q\omega' BR}{Q\omega' B - k}\right) &= m\left(\omega'\right)^2 R && (131)
\end{aligned}$$

The second term contains factors of ω', so it needs to be simplified. Here, we use the fact that $1/k$ can be considered to be a small parameter, taking a Taylor expansion to lowest nontrivial order:

$$\begin{aligned} k\left(\frac{Q\omega' BR}{Q\omega' B+k}+\frac{Q\omega' BR}{Q\omega' B-k}\right) &= Q\omega' BRk\left(\frac{1}{Q\omega' B+k}+\frac{1}{Q\omega' B-k}\right) \\ &= Q\omega' BR\left(\frac{1}{1+\frac{Q\omega' B}{k}}-\frac{1}{1-\frac{Q\omega' B}{k}}\right) \\ &= Q\omega' BR\left(\left(1-\frac{Q\omega' B}{k}+\dots\right)-\left(1+\frac{Q\omega' B}{k}+\dots\right)\right) \\ &\approx -2\frac{\left(Q\omega' B\right)^2 R}{k} \end{aligned} \tag{132}$$

Plugging this into (131), we get an equation we can solve for ω':

$$\begin{aligned} q\omega' BR - 2\frac{\left(Q\omega' B\right)^2 R}{k} &= m\left(\omega'\right)^2 R \\ qB - 2\frac{Q^2\omega' B^2}{k} &= m\omega' \\ qB &= \omega'\left(m+2\frac{(QB)^2}{k}\right) \\ \omega' &= \frac{qB}{m+2\frac{(QB)^2}{k}} \end{aligned} \tag{133}$$

We can again use the fact that $1/k$ is a small parameter to approximate this result to lowest nontrivial order using a Taylor expansion:

$$\begin{aligned} \omega' &= \frac{qB}{m}\left(\frac{1}{1+2\frac{(QB)^2}{mk}}\right) \\ &= \frac{qB}{m}\left(1-2\frac{(QB)^2}{mk}+\dots\right) \end{aligned}$$

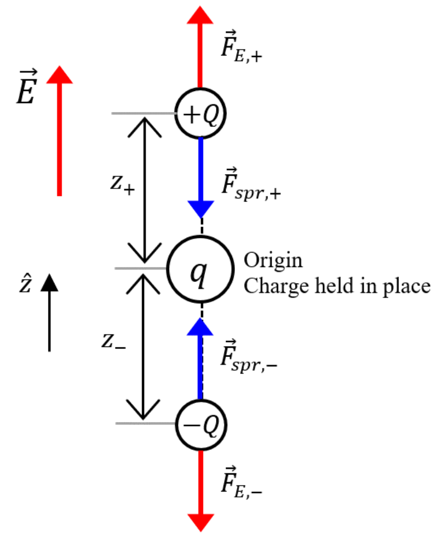
Therefore, recalling that $\omega = qB/m$ was our answer to part (a), we can write

$$\boxed{\omega' = \omega\left(1-2\frac{(QB)^2}{mk}\right) \quad \text{to lowest nontrivial order in } 1/k\text{, where } \omega = \frac{qB}{m}} \tag{134}$$

(c) As stated in the problem, we place the molecule in a constant electric field $\mathbf{E}$, which we set to be in the $\hat{\mathbf{z}}$-direction. We need to use the formula for electric dipole moment for a collection of point charges, which is

$$\mathbf{p} = \sum_i q_i \mathbf{x}_i \quad \text{where } \mathbf{x}_i \text{ is the position of point charge } q_i \tag{135}$$

(We use the standard $\mathbf{p}$ to represent the electric dipole moment, instead of the problem's suggestion of $\mathbf{d}$.) All that remains is to find the position of each of the point charges in the constant electric field $\mathbf{E}$. A diagram of the setup appears on the next page.



We assume that the charges are given enough time to settle into an equilibrium where all three charges are at rest. We can then write Newton's second law for the charges $\pm Q$:

$$F_{E,+} - F_{\text{spr},+} = 0 \quad \text{and} \quad F_{\text{spr},-} - F_{E,-} = 0 \tag{136}$$

The magnitude of the electric force on the charges $\pm Q$ is given by the formula $\mathbf{F}_E = q\mathbf{E}$:

$$F_{E,+} = QE \quad \text{and} \quad F_{E,-} = QE \tag{137}$$

The magnitude of the spring force (from the harmonic potential between q and $\pm Q$) is given by the formula $k\,|\mathbf{x}_{\pm Q} - \mathbf{x}_q|$, which means that we can write

$$F_{\text{spr},+} = kz_+ \quad \text{and} \quad F_{\text{spr},-} = kz_- \tag{138}$$

Plugging this into (136), we can solve for the distance of each charge from the origin $z_\pm$:

$$\begin{gathered} QE - kz_+ = 0 \quad \text{and} \quad kz_- - QE = 0 \\ \Longrightarrow z_+ = z_- = \frac{Q}{k}E \end{gathered} \tag{139}$$

Using formula (135) to calculate the dipole moment about the origin, we get

$$\begin{aligned} \mathbf{p} &= q\,\mathbf{x}_q + Q\,\mathbf{x}_{+Q} - Q\mathbf{x}_{-Q} \\ &= q\,(0) + Q(z_+\hat{\mathbf{z}}) - Q(-z_-\hat{\mathbf{z}}) \\ &= Q\,(z_+ + z_-)\,\hat{\mathbf{z}} \\ &= Q\left(\frac{2Q}{k}E\right)\hat{\mathbf{z}} \\ \mathbf{p} &= \frac{2Q^2}{k}\mathbf{E} \end{aligned} \tag{140}$$

Using the given formula for polarizability $\mathbf{p} = \alpha\mathbf{E}$, this tells us that the polarizability is

$$\boxed{\alpha = \frac{2Q^2}{k}} \tag{141}$$

(d) Our starting point is the approximate value for ω' calculated in part (b) (134):

$$\omega' = \omega\left(1 - 2\frac{(QB)^2}{mk}\right) \quad \text{to lowest nontrivial order in } 1/k$$

From this equation, we can calculate the fractional angular velocity shift

$$\frac{\Delta\omega}{\omega} = \frac{\omega' - \omega}{\omega} = -2\frac{(QB)^2}{mk} \quad \text{to lowest nontrivial order in } 1/k \tag{142}$$

Since $\alpha = 2Q^2/k$ by our answer from part (c) (141), large k corresponds to small α. We can therefore write this answer in terms of the polarizability

$$\boxed{\frac{\Delta\omega}{\omega} = -\frac{\alpha B^2}{m} \quad \text{to lowest nontrivial order in } \alpha} \tag{143}$$