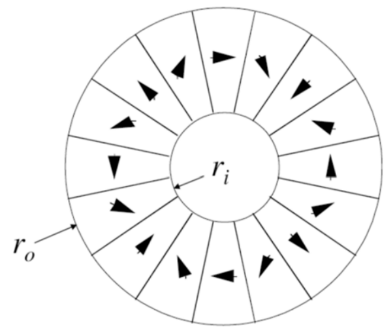
10. (Electromagnetism)

A *Halbach quadrupole* is made by assembling segmented permanent magnet pieces where the magnetization vector is rotated through 6π as one travels around the azimuth, as shown below. In order



to calculate the magnetic field of this magnet, approximate the magnetization vector as a continuous function of φ as follows

$$\vec{M} = M_0 \left(-\hat{\rho}\sin(2\varphi) + \hat{\varphi}\cos(2\varphi)\right)$$

for $r_i < \rho < r_o$ and zero elsewhere. Assume the magnet to be infinitely long in the z-direction.

(a) Calculate the magnetization currents.

(b) Calculate the magnetic field in the vicinity of the axis (i.e. for $\rho < r_i$).

(c) Calculate the magnetic field outside the quadrupole (i.e. for $\rho > r_o$).

Hint: In order to solve this problem, calculate first the magnetic field due to an azimuthal current sheet $\vec{K} = K_0 \sin 2\varphi\, \hat{\varphi}$ located at $\rho = a$ and recall the general solution of the Laplace equation for the magnetic scalar potential in 2D polar coordinates.

The hint should have given the equation of the azimuthal current sheet as $\mathbf{K} = K_0 \cos(2\varphi)\hat{\mathbf{z}}$.

Solution: *Solution by Jonah Hyman (jthyman@g.ucla.edu)*

The key equations to memorize or write on your formula sheet for this problem are the equations for the bound currents in a magnetic material with magnetization (magnetic dipole moment per unit volume) $\mathbf{M}$:

$$\text{Bound volume current:} \quad \mathbf{J}_b = \nabla \times \mathbf{M} \tag{70}$$

$$\text{Bound surface current:} \quad \mathbf{K}_b = \mathbf{M} \times \hat{\mathbf{n}} \tag{71}$$

(a) To calculate the magnetization currents, apply equations (70) and (71) to the given expression for magnetization:

$$\mathbf{M} = M_0(-\hat{\rho}\sin(2\varphi) + \hat{\varphi}\cos(2\varphi)) \tag{72}$$

The bound surface currents occur at radii $\rho = r_i$ and $\rho = r_0$. The unit vector $\hat{\mathbf{n}}$ points away from the magnetic material. Therefore, at $\rho = r_i$, $\hat{\mathbf{n}} = -\hat{\rho}$. At $\rho = r_o$, $\hat{\mathbf{n}} = +\hat{\rho}$. Keeping in mind the unit-vector cross products in cylindrical coordinates, $\hat{\rho} \times \hat{\varphi} = \hat{\mathbf{z}}$ and cyclic permutations thereof, we get the bound surface current density

$$\mathbf{K}_b(\rho = r_i) = M_0\cos(2\varphi)\,(\hat{\varphi} \times -\hat{\rho}) = M_0\cos(2\varphi)\hat{\mathbf{z}} \tag{73}$$

$$\mathbf{K}_b(\rho = r_o) = M_0\cos(2\varphi)\,(\hat{\varphi} \times \hat{\rho}) = -M_0\cos(2\varphi)\hat{\mathbf{z}} \tag{74}$$

To get the bound volume current density, use the formula for curl in cylindrical coordinates:

$$\nabla \times \mathbf{A} = \left[\frac{1}{s}\frac{\partial A_z}{\partial \varphi} - \frac{\partial A_\varphi}{\partial z}\right]\hat{\rho} + \left[\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial \rho}\right]\hat{\varphi} + \frac{1}{\rho}\left[\frac{\partial(\rho A_\varphi)}{\partial \rho} - \frac{\partial A_\rho}{\partial \varphi}\right]\hat{\mathbf{z}} \tag{75}$$

In this case, there is no z-component of $\mathbf{M}$, and $\mathbf{M}$ does not depend on z, so $\nabla \times \mathbf{M}$ has only a z-component, which we now calculate:

$$\begin{aligned}
\mathbf{J}_b = \nabla \times \mathbf{M} &= \frac{1}{\rho}\left[\frac{\partial(\rho M_\varphi)}{\partial \rho} - \frac{\partial M_\rho}{\partial \varphi}\right]\hat{\mathbf{z}} \\
&= \frac{M_0}{\rho}\left[\frac{\partial}{\partial \rho}(\rho\cos(2\varphi)) - \frac{\partial}{\partial \varphi}(-\sin(2\varphi))\right]\hat{\mathbf{z}} \\
&= \frac{M_0}{\rho}\left[\cos(2\varphi) + 2\cos(2\varphi)\right]\hat{\mathbf{z}} \\
&= \frac{3M_0}{\rho}\cos(2\varphi)\hat{\mathbf{z}}
\end{aligned} \tag{76}$$

Collecting our answers, we get that

$$\boxed{\mathbf{K}_b = M_0\cos(2\varphi)\hat{\mathbf{z}} \quad \text{for } \rho = r_i} \tag{77}$$

$$\boxed{\mathbf{K}_b = -M_0\cos(2\varphi)\hat{\mathbf{z}} \quad \text{for } \rho = r_o} \tag{78}$$

$$\boxed{\mathbf{J}_b = \frac{3M_0}{\rho}\cos(2\varphi)\hat{\mathbf{z}} \quad \text{for } r_i < \rho < r_o} \tag{79}$$

To solve parts (b) and (c), we start by solving the problem given in the hint. Consider an azimuthal current sheet $\mathbf{K} = K_0\cos(2\varphi)\hat{\mathbf{z}}$ located at $\rho = a$. Inside and outside the current sheet, there is no free current, so $\nabla \times \mathbf{H} = \mathbf{J}_f = 0$. Therefore, we may write $\mathbf{H}$ as the gradient of a magnetic scalar potential (introducing a minus sign solely to strengthen the analogy with the electric potential):

$$\mathbf{H} = -\nabla\psi \tag{80}$$

In this case, we are working with $\mathbf{H}$ just for convention's sake. The problem from the hint has no magnetic materials in it, so $\mathbf{B} = \mu_0 \mathbf{H}$ in this case. (This is not true for the original problem involving the quadrupole, however.) The no-magnetic-monopoles law tells us that

$$\nabla^2 \psi = -\nabla \cdot \mathbf{H} = -\frac{1}{\mu_0} \nabla \cdot \mathbf{B} = 0 \tag{81}$$

so ψ satisfies Laplace's equation inside and outside the sheet. The general solution to Laplace's equation in polar coordinates is

$$\psi(\rho, \varphi) = (A_0 + B_0 \ln \rho)(C_0 + D_0 \varphi) + \sum_{k=1}^{\infty} \left(A_k \rho^k + \frac{B_k}{\rho^k} \right) (C_k \sin(k\varphi) + D_k \cos(k\varphi)) \tag{82}$$

Recall that

In potential theory problems, only the multipole moments in the setup will be present in the solution.

In this case, the surface current is proportional to $\cos(2\varphi)$. The surface current is proportional to the difference in $\mathbf{H}$, which is the gradient of ψ, so we expect ψ to be proportional to $\sin(2\varphi)$. Our ansatz for ψ is therefore

$$\psi_{\text{in/out}}(\rho, \varphi) = \left(A' \rho^2 + \frac{B'}{\rho^2} \right) \sin(2\varphi) \tag{83}$$

A' and B' do not have the same units, and this problem has only one length scale (a), so let's redefine the constants so that they do have the same units:

$$\psi_{\text{in/out}}(\rho, \varphi) = \left(A \frac{\rho^2}{a^2} + B \frac{a^2}{\rho^2} \right) \sin(2\varphi) \tag{84}$$

The magnetic scalar potential ought to be finite as $\rho \to 0$ and $\rho \to \infty$, so we can refine the ansatz to meet these boundary conditions:

$$\psi(\rho, \varphi) = \begin{cases} A \frac{\rho^2}{a^2} \sin(2\varphi) & \text{for } \rho < a \\ B \frac{a^2}{\rho^2} \sin(2\varphi) & \text{for } \rho > a \end{cases} \tag{85}$$

Using the gradient operator in polar coordinates, $\nabla = \hat{\rho} \frac{\partial}{\partial \rho} + \hat{\varphi} \frac{1}{\rho} \frac{\partial}{\partial \varphi}$, we get that

$$\mathbf{H}(\rho, \varphi) = -\nabla \psi(\rho, \varphi) = \begin{cases} \frac{2A\rho}{a^2} \left[-\sin(2\varphi) \hat{\rho} - \cos(2\varphi) \hat{\varphi} \right] & \text{for } \rho < a \\ \frac{2Ba^2}{\rho^3} \left[\sin(2\varphi) \hat{\rho} - \cos(2\varphi) \hat{\varphi} \right] & \text{for } \rho > a \end{cases} \tag{86}$$

Now to apply the boundary conditions at $\rho = a$. The normal component of $\mathbf{B}$ is always continuous, and $\mathbf{B} = \mu_0 \mathbf{H}$ on both sides of the boundary in this setup. Therefore, the normal component of $\mathbf{H}$, which is the $\hat{\rho}$ component, must be continuous at $\rho = a$. This implies that

$$-\frac{2A\rho}{a^2} = \frac{2Ba^2}{\rho^3} \bigg|_{\rho = a} \implies A = -B \tag{87}$$

The boundary condition for the tangential component of the $\mathbf{H}$ field is

$$\hat{\mathbf{n}} \times (\mathbf{H}_2 - \mathbf{H}_1) = \mathbf{K}_f \quad \text{where } \hat{\mathbf{n}} \text{ points from medium 1 to medium 2} \tag{88}$$

This implies that

$$\begin{aligned} -\frac{2Ba^2}{\rho^3} + \frac{2A\rho}{a^2}\bigg|_{\rho=a} &= K_0 \\ -B + A &= \frac{aK_0}{2} \\ B &= -\frac{aK_0}{4} \quad \text{since } B = -A \end{aligned} \tag{89}$$

Since $A = -B$, this means that

$$A = \frac{K_0 a}{4} \quad \text{and} \quad B = -\frac{K_0 a}{4} \tag{90}$$

Plugging into equation (86), we get the following:

$$\mathbf{H}(\rho,\varphi) = \begin{cases} \frac{K_0\rho}{2a}\left[-\sin(2\varphi)\hat{\rho} - \cos(2\varphi)\hat{\varphi}\right] & \text{for } \rho < a \\ -\frac{K_0 a^3}{2\rho^3}\left[\sin(2\varphi)\hat{\rho} - \cos(2\varphi)\hat{\varphi}\right] & \text{for } \rho > a \end{cases} \tag{91}$$

Simplifying further and using the fact that $\mathbf{B} = \mu_0\mathbf{H}$ on both sides of the boundary in this setup, we get that

$$\mathbf{B}(\rho,\varphi;K_0,a) = \begin{cases} -\frac{\mu_0 K_0\rho}{2a}\left[\sin(2\varphi)\hat{\rho} + \cos(2\varphi)\hat{\varphi}\right] & \text{for } \rho < a \\ -\frac{\mu_0 K_0 a^3}{2\rho^3}\left[\sin(2\varphi)\hat{\rho} - \cos(2\varphi)\hat{\varphi}\right] & \text{for } \rho > a \end{cases} \tag{92}$$

We have added the explicit dependence (K_0, a) to clarify that the value of $\mathbf{B}$ depends on the strength and location of the surface current.

Now we are ready to discuss parts (b) and (c). Using our results for the bound currents in part (a), the quadrupole problem is equivalent to a superposition of azimuthal current sheets of infinitesimal thickness located at $r_i \leqslant \rho \leqslant r_0$, with the following magnitudes:

$$\mathbf{K}(\rho) = \begin{cases} M_0\cos(2\varphi)\hat{\mathbf{z}} & \text{for } \rho = r_i \\ \frac{3M_0}{\rho}d\rho\,\cos(2\varphi)\hat{\mathbf{z}} & \text{for } r_i < \rho < r_o \\ -M_0\cos(2\varphi)\hat{\mathbf{z}} & \text{for } \rho = r_o \end{cases} \tag{93}$$

Notice the use of the differential $d\rho$ to turn the volume current density inside the bulk of the permanent magnet into a superposition of surface current densities.

All of these surface current densities are of the form $K_0\cos(2\varphi)\hat{\mathbf{z}}$, and we calculated the magnetic field due to such a current in the hint! We may now forget all about the permanent magnet and simply treat this as a superposition problem, integrating up the solutions in equation (92) to get the magnetic field wherever we please:

$$\mathbf{B}(\rho,\varphi) = \mathbf{B}(\rho,\varphi;M_0,r_i) + \mathbf{B}(\rho,\varphi;-M_0,r_o) + \int_{r_i}^{r_o}\mathbf{B}\left(\rho,\varphi;\ \frac{3M_0}{\rho'}d\rho',\rho'\right) \tag{94}$$

In the integral, note the difference between the radius of the observation point ρ and the location of the current sheet ρ'.

(b) For $\rho < r_i$, we are inside all the azimuthal current sheets, so we consistently use the top line in equation (92). Therefore, the magnetic field points in the $\sin(2\varphi)\hat{\rho} + \cos(2\varphi)\hat{\varphi}$ direction, and its (signed) magnitude is given by

$$\begin{aligned} B(\rho, \varphi) &= -\frac{\mu_0 M_0 \rho}{2r_i} - \frac{\mu_0(-M_0)\rho}{2r_0} + \int_{r_i}^{r_o} \left(-\frac{\mu_0 \rho}{2\rho'} \frac{3M_0}{\rho'} d\rho' \right) \\ &= -\frac{\mu_0 M_0 \rho}{2r_i} + \frac{\mu_0 M_0 \rho}{2r_0} - \frac{3\mu_0 M_0 \rho}{2} \int_{r_i}^{r_o} d\rho' \frac{1}{(\rho')^2} \\ &= -\frac{\mu_0 M_0 \rho}{2r_i} + \frac{\mu_0 M_0 \rho}{2r_0} - \frac{3\mu_0 M_0 \rho}{2} \left(\frac{1}{r_i} - \frac{1}{r_o} \right) \\ &= -2\mu_0 M_0 \rho \left(\frac{1}{r_i} - \frac{1}{r_o} \right) \end{aligned} \tag{95}$$

Putting everything together, we get that

$$\boxed{\mathbf{B}(\rho, \varphi) = -2\mu_0 M_0 \rho \left(\frac{1}{r_i} - \frac{1}{r_o} \right) [\sin(2\varphi)\hat{\rho} + \cos(2\varphi)\hat{\varphi}] \quad \text{for } \rho < r_i} \tag{96}$$

(c) For $\rho > r_o$, we are outside all the azimuthal current sheets, so we consistently use the bottom line in equation (92). Therefore, the magnetic field points in the $\sin(2\varphi)\hat{\rho} - \cos(2\varphi)\hat{\varphi}$ direction, and its (signed) magnitude is given by

$$\begin{aligned} B(\rho, \varphi) &= -\frac{\mu_0 M_0 r_i^3}{2\rho^3} - \frac{\mu_0(-M_0) r_o^3}{2\rho^3} + \int_{r_i}^{r_o} \left(-\frac{\mu_0 (\rho')^3}{2\rho^3} \frac{3M_0}{\rho'} d\rho' \right) \\ &= -\frac{\mu_0 M_0 r_i^3}{2\rho^3} + \frac{\mu_0 M_0 r_o^3}{2\rho} - \frac{3\mu_0 M_0}{2\rho^3} \int_{r_i}^{r_o} (\rho')^2 d\rho' \\ &= -\frac{\mu_0 M_0 r_i^3}{2\rho^3} + \frac{\mu_0 M_0 r_o^3}{2\rho} - \frac{3\mu_0 M_0}{2\rho^3} \left(\frac{r_o^3}{3} - \frac{r_i^3}{3} \right) \\ &= -\frac{\mu_0 M_0 r_i^3}{2\rho^3} + \frac{\mu_0 M_0 r_o^3}{2\rho} - \frac{\mu_0 M_0 r_o^3}{2\rho^3} + \frac{\mu_0 M_0 r_i^3}{2\rho^3} \\ &= 0 \end{aligned} \tag{97}$$

Putting everything together, we get that

$$\boxed{\mathbf{B}(\rho, \varphi) = 0 \quad \text{for } \rho > r_o} \tag{98}$$