9. (Electromagnetism)

Consider a dielectric slab waveguide, i.e. a dielectric volume of index of refraction n_1 delimited by two planes $-a \leqslant x \leqslant a$ and infinitely wide in the other two directions, surrounded by dielectric of index of refraction n_2 ($\mu_1 = \mu_2 = \mu_0$). Study the propagation of transverse electric (TE) waves in the z-direction in this system (i.e. assume $\vec{E}(x, z, t) = E(x)e^{i(hz-\omega t)}\hat{y}$).

(a) Write down the wave equation for $E(x)$ in each region of the slab.

(b) Look for even solutions (invariant for $x \to -x$) with fields decaying outside the guide. Apply the boundary conditions at the interfaces to obtain expressions for electric and magnetic fields in the guide.

(c) Calculate the cut-off frequencies in this guide (i.e. the frequencies for which the wave is no longer guided by the dielectric slab). What is the lowest frequency that can propagate in this guide?

Solution: *Solution by Jonah Hyman (jthyman@g.ucla.edu)*

(a) The three-dimensional wave equation for the electric field is

$$\nabla^2 \mathbf{E} - \frac{1}{v^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} = 0 \quad \text{for} \quad v \equiv \frac{c}{n} \tag{26}$$

From the problem statement, we have that

$$\mathbf{E}(x, z, t) = E(x) e^{i(hz-\omega t)}\hat{\mathbf{y}} \tag{27}$$

Then

$$\begin{aligned} \nabla^2 \mathbf{E} &= \frac{\partial^2 \mathbf{E}}{\partial x^2} + \frac{\partial^2 \mathbf{E}}{\partial y^2} + \frac{\partial^2 \mathbf{E}}{\partial z^2} \\ &= \frac{d^2 E}{dx^2} e^{i(hz-\omega t)}\hat{\mathbf{y}} - h^2 E(x) e^{i(hz-\omega t)}\hat{\mathbf{y}} \\ &= \left(\frac{d^2 E}{dx^2} - h^2 E(x)\right) e^{i(hz-\omega t)}\hat{\mathbf{y}} \end{aligned} \tag{28}$$

and

$$\frac{\partial^2 \mathbf{E}}{\partial t^2} = -\omega^2 E(x) e^{i(hz-\omega t)}\hat{\mathbf{y}}$$

Plugging into the wave equation (26), we get that

$$0 = \nabla^2 \mathbf{E} - \frac{n^2}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} = \left(\frac{d^2 E}{dx^2} - h^2 E(x) + \frac{n^2\omega^2}{c^2} E(x)\right) e^{i(hz-\omega t)}\hat{\mathbf{y}} \tag{29}$$

which implies that

$$\frac{d^2 E}{dx^2} + \left(\frac{n^2\omega^2}{c^2} - h^2\right) E(x) = 0 \tag{30}$$

This equation holds in each region of the waveguide. This implies that

$$\begin{aligned} \frac{d^2 E}{dx^2} + \gamma_1^2 E(x) = 0 \quad \text{for} \quad |x| < a \\ \frac{d^2 E}{dx^2} + \gamma_2^2 E(x) = 0 \quad \text{for} \quad |x| > a \end{aligned} \tag{31}$$

where

$$\gamma_i^2 = \frac{n_i^2\omega^2}{c^2} - h^2 \quad \text{for} \quad i \in \{1, 2\} \tag{32}$$

(b) The ordinary differential equation

$$\frac{d^2 f}{dx^2} + \gamma^2 f(x) = 0 \tag{33}$$

has the general solutions

$$f(x) = A\sin(\gamma x) + B\cos(\gamma x) \qquad \text{if} \quad \gamma^2 > 0 \tag{34}$$

$$f(x) = Ae^{+|\gamma| x} + Be^{-|\gamma| x} \qquad \text{if} \quad \gamma^2 < 0 \tag{35}$$

Let's start outside the waveguide, in the region $x > a$. We want the electric field to decay outside the waveguide, so we want the exponential solution that decays as $x \to \infty$ in this region. In other words, we need to require that

$$\gamma_2^2 = \frac{n_2^2\omega^2}{c^2} - h^2 < 0 \tag{36}$$

and we ought to set as our ansatz

$$E(x) = E_{\text{out}} e^{-|\gamma_2| x} \quad \text{for} \quad x > a \tag{37}$$

where E_{out} is a parameter to be determined by the boundary conditions. Since $E(x)$ is supposed to be even (invariant under $x \to -x$), we can immediately write the solution for $x < -a$:

$$E(x) = E_{\text{out}} e^{+|\gamma_2| x} \quad \text{for} \quad x < -a \tag{38}$$

Inside the waveguide, we need to guess that the solution is a sine or cosine, and not an exponential. This is based on our understanding that guided waves should be based on superpositions of propagating plane waves, not evanescent waves. (If you are skeptical of this logic, please see the note at the end of the problem.) Therefore, we need to require that

$$\gamma_1^2 = \frac{n_1^2 \omega^2}{c^2} - h^2 > 0 \tag{39}$$

We are interested in even solutions (invariant for $x \to -x$), so we should select the cosine solution for our ansatz:

$$E(x) = E_{\text{in}} \cos(\gamma_1 x) \quad \text{for} \quad -a < x < a \tag{40}$$

where E_{in} is a parameter to be determined by the boundary conditions. Therefore, we have a complete ansatz for $\mathbf{E}$:

$$\mathbf{E}(x,z,t) = \begin{cases} E_{\text{out}} e^{-|\gamma_2| x} \hat{\mathbf{y}}\, e^{i(hz-\omega t)} & \text{for} \quad x > a \\ E_{\text{in}} \cos(\gamma_1 x)\, \hat{\mathbf{y}} e^{i(hz-\omega t)} & \text{for} \quad -a < x < a \\ E_{\text{out}} e^{+|\gamma_2| x}\, \hat{\mathbf{y}} e^{i(hz-\omega t)} & \text{for} \quad x < -a \end{cases} \tag{41}$$

We can find the magnetic field using Faraday's law:

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{42}$$

Here,

$$\begin{aligned} \nabla \times \mathbf{E} &= \left(\hat{\mathbf{x}} \frac{\partial}{\partial x} + \hat{\mathbf{y}} \frac{\partial}{\partial y} + \hat{\mathbf{z}} \frac{\partial}{\partial z} \right) \times E(x) e^{i(hz-\omega t)} \hat{\mathbf{y}} \\ &= \left(\frac{dE}{dx} \hat{\mathbf{z}} - ihE(x)\hat{\mathbf{x}} \right) e^{i(hz-\omega t)} \\ &= \begin{cases} E_{\text{out}} \left(-|\gamma_2| \hat{\mathbf{z}} - ih\hat{\mathbf{x}}\right) e^{-|\gamma_2| x}\, e^{i(hz-\omega t)} & \text{for} \quad x > a \\ E_{\text{in}} \left(-\gamma_1 \sin(\gamma_1 x) \hat{\mathbf{z}} - ih \cos(\gamma_1 x)\hat{\mathbf{x}}\right)\, e^{i(hz-\omega t)} & \text{for} \quad -a < x < a \\ E_{\text{out}} \left(|\gamma_2| \hat{\mathbf{z}} - ih\hat{\mathbf{x}}\right) e^{+|\gamma_2| x}\, e^{i(hz-\omega t)} & \text{for} \quad x < -a \end{cases} \end{aligned} \tag{43}$$

We may now use Faraday's law, undoing the time derivative (and ignoring any static fields) to get

$$\mathbf{B}(x,z,t) = \begin{cases} \frac{E_{\text{out}}}{i\omega} \left(-|\gamma_2| \hat{\mathbf{z}} - ih\hat{\mathbf{x}}\right) e^{-|\gamma_2| x}\, e^{i(hz-\omega t)} & \text{for} \quad x > a \\ \frac{E_{\text{in}}}{i\omega} \left(-\gamma_1 \sin(\gamma_1 x) \hat{\mathbf{z}} - ih \cos(\gamma_1 x)\hat{\mathbf{x}}\right)\, e^{i(hz-\omega t)} & \text{for} \quad -a < x < a \\ \frac{E_{\text{out}}}{i\omega} \left(|\gamma_2| \hat{\mathbf{z}} - ih\hat{\mathbf{x}}\right) e^{+|\gamma_2| x}\, e^{i(hz-\omega t)} & \text{for} \quad x < -a \end{cases} \tag{44}$$

This tells us that inside the slab ($-a \leqslant x \leqslant a$), the electric and magnetic fields are given by

$$\mathbf{E}(x,z,t) = E_{\text{in}} \cos(\gamma_1 x) \hat{\mathbf{y}} e^{i(hz-\omega t)} \tag{45}$$

$$\mathbf{B}(x,z,t) = \frac{E_{\text{in}}}{i\omega} \left(-\gamma_1 \sin(\gamma_1 x) \hat{\mathbf{z}} - ih \cos(\gamma_1 x)\hat{\mathbf{x}}\right)\, e^{i(hz-\omega t)} \tag{46}$$

where E_{in} is a constant and $\gamma_1 = \frac{n_1^2 \omega^2}{c^2} - h^2$.

(c) Given that there is neither free charge or free current at the interface, there are four boundary conditions at the interface $x = a$ (for which the vector normal to the interface is $\hat{\mathbf{x}}$):

$$\mathbf{D}^{\perp}_{x\to a^-} = \mathbf{D}^{\perp}_{x\to a^+} \Longrightarrow \epsilon_1 \mathbf{E}^{\perp}_{x\to a^-} = \epsilon_2 \mathbf{E}^{\perp}_{x\to a^+} \tag{47}$$

$$\mathbf{E}^{\parallel}_{x\to a^-} = \mathbf{E}^{\parallel}_{x\to a^+} \tag{48}$$

$$\mathbf{B}^{\perp}_{x\to a^-} = \mathbf{B}^{\perp}_{x\to a^+} \tag{49}$$

$$\mathbf{H}^{\parallel}_{x\to a^-} = \mathbf{H}^{\parallel}_{x\to a^+} \Longrightarrow \frac{1}{\mu_1}\mathbf{B}^{\parallel}_{x\to a^-} = \frac{1}{\mu_2}\mathbf{B}^{\parallel}_{x\to a^+} \tag{50}$$

Equation (47) is automatically satisfied here because $\mathbf{E}$ is in the $\hat{\mathbf{y}}$ direction, which is parallel to the interface. Plugging equations (43) and (44) into boundary conditions (48), (49), and (50), and using the fact (given in the problem) that $\mu_1 = \mu_2 = \mu_0$, we get the following:

$$E_{\text{in}} \cos(\gamma_1 a) = E_{\text{out}} e^{-|\gamma_2| a} \tag{51}$$

$$\frac{E_{\text{in}}}{i\omega}(-ih)\cos(\gamma_1 a) = \frac{E_{\text{out}}}{i\omega}(-ih)e^{-|\gamma_2| a} \tag{52}$$

$$\frac{E_{\text{in}}}{i\omega}(-\gamma_1 \sin(\gamma_1 a)) = \frac{E_{\text{out}}}{i\omega}(-|\gamma_2|)e^{-|\gamma_2| a} \tag{53}$$

The first two of these equations are redundant. We can simplify the non-redundant equations to get

$$E_{\text{in}} \cos(\gamma_1 a) = E_{\text{out}} e^{-|\gamma_2| a} \quad \text{and} \quad E_{\text{in}}\gamma_1 \sin(\gamma_1 a) = E_{\text{out}}|\gamma_2| e^{-|\gamma_2| a} \tag{54}$$

These two equations impose two constraints. One constraint relates E_{in} to E_{out}. The other constraint gives a relationship between γ_1 and γ_2. γ_1 and γ_2 give us the wave number h in terms of n_i, ω, and c. Therefore, the latter constraint tells us, given a frequency, which wave numbers (modes) can propagate in the waveguide. The highest frequency for which *no* wave numbers can propagate in the waveguide for a given mode is the cutoff frequency that we are looking for in this problem.

The wave is only guided by the dielectric slab when $\gamma_2^2 < 0$ (when γ_2^2 reaches zero, the electric field outside switches from an exponentially decaying evanescent wave to a sinusoidal wave that carries energy away from the waveguide). Setting $|\gamma_2| = 0$ in the second equation of (54), we get

$$\sin(\gamma_1 a)|_{\omega=\omega_{c,m}} = 0 \qquad \Longrightarrow \qquad \gamma_1 = \frac{m\pi}{a} \quad \text{for} \quad m = 0, 1, 2, 3, \ldots \tag{55}$$

where $\omega_{c,m}$ is the cutoff frequency for a particular mode m. We may now use the structural dispersion relations (32) to get that

$$\frac{m^2\pi^2}{a^2} = \gamma_1^2 = \frac{n_1^2\omega_{c,m}^2}{c^2} - h^2 \tag{56}$$

$$0 = \gamma_2^2 = \frac{n_2^2\omega_{c,m}^2}{c^2} - h^2 \tag{57}$$

We now want to solve for the cutoff frequency $\omega_{c,m}$. Subtracting equation (57) from equation (56) to eliminate the wave number h^2, we get that

$$\frac{m^2\pi^2}{a^2} = \frac{(n_1^2 - n_2^2)\omega_{c,m}^2}{c^2} \qquad \Longrightarrow \qquad \boxed{\omega_{c,m} = \frac{m\pi c}{a}\frac{1}{\sqrt{n_1^2 - n_2^2}} \quad \text{for} \quad m = 0, 1, 2, 3, \ldots} \tag{58}$$

The lowest ($m = 0$) mode can propagate at arbitrarily low frequency.

Two more notes about this problem:

Why sinusoidal waves?
Earlier, we presented a semi-intuitive argument for why we ought to set

$$0 < \gamma_1^2 \equiv \frac{n_1^2\omega^2}{c^2} - h^2 \tag{59}$$

to get sinusoidal solutions inside the waveguide. If you want a more rigorous argument, assume that $\gamma_1^2 < 0$. Then, the even solution that decays outside the waveguide would be given by the ansatz

$$\mathbf{E}(x,z,t) = \begin{cases} E_{\text{out}} e^{-|\gamma_2|x} \hat{\mathbf{y}}\, e^{i(hz-\omega t)} & \text{for} \quad x > a \\ E_{\text{in}} \cosh(|\gamma_1| x)\, \hat{\mathbf{y}} e^{i(hz-\omega t)} & \text{for} \quad -a < x < a \\ E_{\text{out}} e^{+|\gamma_2|x}\, \hat{\mathbf{y}} e^{i(hz-\omega t)} & \text{for} \quad x < -a \end{cases} \tag{60}$$

where

$$|\gamma_1| \equiv \sqrt{h^2 - \frac{n_1^2\omega^2}{c^2}} > 0 \quad \text{and} \quad |\gamma_2| \equiv \sqrt{h^2 - \frac{n_2^2\omega^2}{c^2}} > 0 \tag{61}$$

As an exercise, go through equations (44) through (54) with this ansatz. What you should get is

$$E_{\text{in}} \cosh(|\gamma_1|a) = E_{\text{out}} e^{-|\gamma_2|a} \quad \text{and} \quad E_{\text{in}} |\gamma_1| \sinh(|\gamma_1|a) = -E_{\text{out}} |\gamma_2| e^{-|\gamma_2|a} \tag{62}$$

Divide these equations by one another and simplify to get

$$\tanh(|\gamma_1|a) = -\frac{|\gamma_1|}{|\gamma_2|} \tag{63}$$

The left-hand side of this equation is positive, while the right-hand side of this equation is negative. Therefore, this solution is impossible, and so we must have $\gamma_1^2 > 0$.

Total internal reflection
We can think of the wave inside a dielectric slab waveguide as being an incident plane wave propagating at an angle of incidence θ and reflecting off the slabs:

$$\mathbf{E}_I = E_I e^{i\mathbf{k}\cdot\mathbf{r} - \omega t} \quad \text{for} \quad \mathbf{k} = \hat{\mathbf{z}}\, k \sin\theta_1 + \hat{\mathbf{x}}\, k \cos\theta_1 \tag{64}$$

By inspection, the parameter we called h is the z-component of the incident wave vector:

$$h = k \sin\theta_1 = \frac{n_1\omega}{c} \sin\theta_1 \tag{65}$$

where we used the dispersion relation for a plane wave, $\omega = \frac{ck}{n}$. Equation (57) tells us that at the cutoff frequency,

$$\frac{n_2^2\omega_{c,m}^2}{c^2} = h^2 = \frac{n_1^2\omega_{c,m}^2}{c^2} \sin^2\theta_1 \tag{66}$$

Simplifying, we get that

$$n_2^2 = n_1^2 \sin^2\theta_1 \Longrightarrow \sin\theta_1 = \frac{n_2}{n_1} \tag{67}$$

But this is just the condition for *total internal reflection*: By Snell's law

$$n_1 \sin\theta_1 = n_2 \sin\theta_2 \tag{68}$$

where θ_2 is the angle of the transmitted wave. But for waves confined to the waveguide, there ought not to be any transmitted wave. This is accomplished if so $\sin\theta_2 > 1$, since in that case we cannot find a real angle of transmission θ_2. The condition $\sin\theta_2 > 1$ gives us

$$n_1 \sin\theta_1 > n_2 \Longrightarrow \sin\theta_1 > \frac{n_2}{n_1} \tag{69}$$