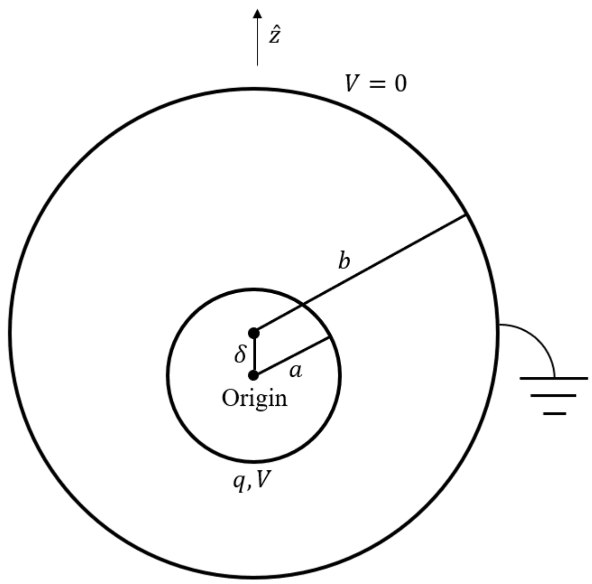
13. (Electromagnetism)

An isolated conducting sphere of radius a is placed inside a thin conducting spherical shell of radius b. The centers of the two spheres are not coincident, but are instead displaced from each other by a small distance δ, with $\delta \ll a, b$. The total charge of the inner sphere is q, and the outer sphere is grounded. Find the distribution of surface charge σ on the inner sphere and the force F acting on it, to first order in δ.

Solution: *Solution by Jonah Hyman (jthyman@g.ucla.edu)*

Here is a diagram of the setup.



These two spheres are conducting, so the surface charge on each sphere redistributes itself so that the sphere is an equipotential. Let V be the electric potential of the inner sphere. The outer sphere is grounded, so its electric potential is zero.

In order to find the surface charge distribution on the inner sphere, we need to find the electric field between the spheres first. The discontinuity in the electric field at the inner sphere will tell us about the surface charge density on the inner sphere, since by Gauss's law,

$$E_{2,\perp} - E_{1,\perp} = \frac{\sigma}{\epsilon_0} \quad \text{at the interface between 1 and 2} \tag{42}$$

To find the electric field between the spheres, use potential theory. Assuming that the volume charge density between the spheres is zero, Laplace's equation is satisfied between the spheres:

$$\nabla^2 V(\mathbf{r}) = 0 \quad \text{for} \quad a < r < b \tag{43}$$

We also know the boundary conditions: $V(\mathbf{r}) = V$ on the inner sphere and $V(\mathbf{r}) = 0$ on the outer sphere. Describing the inner sphere mathematically is straightforward: If the origin is placed at the center of the inner sphere, then the surface of the inner sphere is described by $r = a$. However, since the spheres are misaligned, we must do a calculation to describe the surface of the outer sphere.

Setting $\hat{\mathbf{z}}$ to point from the center of the inner sphere to the center of the outer sphere, we have that the center of the outer sphere is located at $\delta\,\hat{\mathbf{z}}$. Therefore, if the vector $\mathbf{r}$ describes a point P on the surface of the outer sphere, the vector from the center of the outer sphere to the point P is given by $\mathbf{r} - \delta\,\hat{\mathbf{z}}$. Point P is on the surface of the outer sphere if and only if the distance from the center of the outer sphere to point P is equal to b. In other words, $\mathbf{r}$ describes a point P on the surface of the outer sphere if and only if

$$b = |\mathbf{r} - \delta\,\hat{\mathbf{z}}| \tag{44}$$

Using some vector algebra, we can derive a relation between b and $\mathbf{r}$. Assuming $\mathbf{r}$ lies on the surface of the outer sphere,

$$\begin{aligned} b^2 &= |\mathbf{r} - \delta\,\hat{\mathbf{z}}|^2 \\ &= (\mathbf{r} - \delta\,\hat{\mathbf{z}}) \cdot (\mathbf{r} - \delta\,\hat{\mathbf{z}}) \\ &= r^2 - 2\delta\,\mathbf{r}\cdot\hat{\mathbf{z}} + \delta^2 \\ b^2 &= r^2 - 2\delta\, r\cos\theta + \delta^2 \quad \text{since } \mathbf{r}\cdot\hat{\mathbf{z}} = r\cos\theta \end{aligned} \tag{45}$$

We want to find an equation of the form $r = f(\theta)$ that describes the surface of the outer sphere. To find this equation, use the quadratic formula and expand to first order in δ:

$$\begin{aligned} 0 &= r^2 - 2\delta\, r\cos\theta + \delta^2 + (\delta^2 - b^2) \\ \implies r &= \frac{1}{2}\left(2\delta\cos\theta \pm \sqrt{(2\delta\cos\theta)^2 - 4(\delta^2 - b^2)}\right) \\ &= \delta\cos\theta \pm \sqrt{\delta^2\cos^2\theta - \delta^2 + b^2} \\ &= \delta\cos\theta \pm \sqrt{b^2 - \delta^2\sin^2\theta} \quad \text{using the Pythagorean identity } \sin^2 x + \cos^2 x = 1 \\ &= \delta\cos\theta \pm \left(b - \frac{1}{2}\frac{\delta^2}{b}\sin^2\theta\right) \quad \text{expanding the square root using } \sqrt{1+x} = 1 - \frac{1}{2}x + \dots \\ r &= \pm b + \delta\cos\theta + \mathcal{O}(\delta^2) \end{aligned} \tag{46}$$

We should choose the positive answer since $b \gg \delta$ and we want to have $r > 0$. Therefore, the condition for the surface of the outer sphere is

$$r = b + \delta\cos\theta + \mathcal{O}(\delta^2) \tag{47}$$

Thus, we have a full mathematical setup for a boundary value problem for the electric potential $V(\mathbf{r})$:

$$\begin{aligned} \nabla^2 V(\mathbf{r}) &= 0 \quad \text{for} \quad a < r < b && \text{(Laplace's equation)} \\ V(\mathbf{r}) &= V \quad \text{for} \quad r = a && \text{(boundary condition on inner sphere)} \\ V(\mathbf{r}) &= 0 \quad \text{for} \quad r = b + \delta\cos\theta + \mathcal{O}(\delta^2) && \text{(boundary condition on outer sphere)} \end{aligned} \tag{48}$$

To solve this problem, use the general solution to Laplace's equation in spherical coordinates with azimuthal symmetry:

$$V(\mathbf{r}) = \sum_{\ell=0}^{\infty}\left(A_\ell\, r^\ell + \frac{B_\ell}{r^{\ell+1}}\right) P_\ell(\cos\theta) \tag{49}$$

To simplify the algebra, note that

In potential theory problems, only the multipole moments in the setup will be present in the solution.

In this setup, the only multipole moments that are present are $\ell = 0$ (the spherically symmetric moment) and $\ell = 1$ (the moment that goes like $\cos\theta$, which comes from the expression for the surface of the outer sphere (47)). For that reason, we will take only the $\ell = 0$ and $\ell = 1$ terms of this general setup as our ansatz:

$$V(\mathbf{r}) = \left(A_0 + \frac{B_0}{r}\right) + \left(A_1 r + \frac{B_1}{r^2}\right)\cos\theta \quad \text{to order } \delta \tag{50}$$

By the uniqueness theorem, if we can find a solution for $V(\mathbf{r})$ that satisfies the boundary condition using this ansatz, that solution will be the only possible solution for the electric potential between the spheres.

To work to first order in δ consistently, we should try to figure out what order the A's and B's are in terms of δ. If $\delta = 0$, then the spheres are completely concentric, so the setup is spherically symmetric. This means that only the zeroeth multipole moment is present, so if $\delta = 0$, A_0 and B_0 are the only nonzero coefficients. We can use this information to write the order of A_0, B_0, A_1, and B_1:

$$\begin{aligned} &A_0 \text{ and } B_0 \text{ are of order } \mathcal{O}(1) \text{ since they are nonzero even when } \delta = 0 \\ &A_1 \text{ and } B_1 \text{ are of order at least } \mathcal{O}(\delta) \text{ since they are zero when } \delta = 0 \end{aligned} \tag{51}$$

We can now impose the boundary condition on the inner sphere, $V(\mathbf{r}) = V$ for $r = a$. Since V is constant and the Legendre polynomials are orthogonal, the only nonzero contribution to the electric potential at $r = a$ comes from the zeroeth Legendre polynomial, which is the constant one: $P_0(\cos\theta) = 1$. This implies that

$$V = A_0 + \frac{B_0}{a} \tag{52}$$

$$0 = A_1\, a + \frac{B_1}{a^2} \tag{53}$$

This gets us a relation between A_ℓ and B_ℓ for each value of ℓ. To get a second relation, we need to impose the boundary condition on the inner sphere, $V(\mathbf{r}) = 0$ for $r = b + \delta\cos\theta + \mathcal{O}(\delta^2)$:

$$\begin{aligned} 0 &= V(\mathbf{r})\big|_{r=b+\delta\cos\theta+\mathcal{O}(\delta^2)} \\ 0 &= \left(A_0 + \frac{B_0}{b+\delta\cos\theta+\mathcal{O}(\delta^2)}\right) + \left(A_1\left(b+\delta\cos\theta+\mathcal{O}(\delta^2)\right) + \frac{B_1}{\left(b+\delta\cos\theta+\mathcal{O}(\delta^2)\right)^2}\right)\cos\theta \end{aligned} \tag{54}$$

Expanding each term to first order in δ, using the Taylor expansion $(1+x)^n = 1 + nx + \mathcal{O}(x^2)$, we get

$$\begin{aligned} \frac{B_0}{b+\delta\cos\theta+\mathcal{O}(\delta^2)} &= \frac{B_0}{b}\left(\frac{1}{1+\frac{\delta}{b}\cos\theta+\mathcal{O}(\delta^2)}\right) \\ &= \frac{B_0}{b}\left(1 - \frac{\delta}{b}\cos\theta + \mathcal{O}(\delta^2)\right) \quad \text{since } B_0 \text{ is of order } \mathcal{O}(1) \end{aligned} \tag{55}$$

$$A_1\left(b+\delta\cos\theta+\mathcal{O}(\delta^2)\right) = A_1 b + \mathcal{O}(\delta^2) \quad \text{since } A_1 \text{ is of order } \mathcal{O}(\delta) \tag{56}$$

$$\begin{aligned} \frac{B_1}{\left(b+\delta\cos\theta+\mathcal{O}(\delta^2)\right)^2} &= \frac{B_1}{b^2+2b\delta\cos\theta+\mathcal{O}(\delta^2)} \\ &= \frac{B_1}{b^2} \quad \text{since } B_1 \text{ is of order } \mathcal{O}(\delta) \end{aligned} \tag{57}$$

Plugging back into (54), we get

$$\begin{aligned} 0 &= \left(A_0 + \frac{B_0}{b}\left(1 - \frac{\delta}{b}\cos\theta\right)\right) + \left(A_1 b + \frac{B_1}{b^2}\right)\cos\theta + \mathcal{O}(\delta^2) \\ 0 &= \left(A_0 + \frac{B_0}{b}\right) + \left(-B_0\frac{\delta}{b^2} + A_1 b + \frac{B_1}{b^2}\right)\cos\theta + \mathcal{O}(\delta^2) \end{aligned} \tag{58}$$

This is true for all θ, so the constant term and the θ-dependent term must both identically vanish. This gives us

$$A_0 + \frac{B_0}{b} = 0 \tag{59}$$

$$-B_0\frac{\delta}{b^2} + A_1 b + \frac{B_1}{b^2} = 0 \tag{60}$$

Equations (52), (53), (59), and (60) are four linear equations for the four unknowns A_0, B_0, A_1, and B_1:

$$V = A_0 + \frac{B_0}{a} \qquad A_0 + \frac{B_0}{b} = 0$$

$$0 = A_1\, a + \frac{B_1}{a^2} \qquad -B_0 \frac{\delta}{b^2} + A_1 b + \frac{B_1}{b^2} = 0$$

The top two equations can be used to solve for A_0 and B_0:

$$\begin{aligned} V &= A_0 + \frac{B_0}{a} && (61) \\ &= -\frac{B_0}{b} + \frac{B_0}{a} \quad \text{since } A_0 + \frac{B_0}{b} = 0 \\ V &= B_0 \left(\frac{1}{a} - \frac{1}{b}\right) \\ \implies B_0 &= \frac{V}{\frac{1}{a} - \frac{1}{b}} = V \frac{ab}{b-a} && (62) \end{aligned}$$

$$\begin{aligned} A_0 &= -\frac{B_0}{b} \quad \text{since } A_0 + \frac{B_0}{b} = 0 \\ &= -\frac{1}{b}\left(V \frac{ab}{b-a}\right) \quad \text{by (62)} \\ A_0 &= -V \frac{a}{b-a} && (63) \end{aligned}$$

Before moving on to A_1 and B_1, note that the A_0 and B_0 terms in the electric potential give a complete expression for $V(\mathbf{r})$ to zeroeth order in δ:

$$\begin{aligned} V(\mathbf{r}) &= A_0 + \frac{B_0}{r} + \mathcal{O}(\delta) \\ &= -V \frac{a}{b-a} + V \frac{ab}{b-a} \frac{1}{r} + \mathcal{O}(\delta) \\ V(\mathbf{r}) &= -V \frac{a}{b-a} \left(1 - \frac{b}{r}\right) + \mathcal{O}(\delta) && (64) \end{aligned}$$

You can check that this satisfies the order $\mathcal{O}(1)$ version of the boundary conditions: $V(\mathbf{r}) = V$ for $r = a$ and $V(\mathbf{r}) = 0$ for $r = b + \mathcal{O}(\delta)$. In other words, it is the solution to this setup if the spheres were concentric ($\delta = 0$).

We are now ready to use the bottom two equations to solve for A_1 and B_1:

$$\begin{aligned} 0 &= A_1 a + \frac{B_1}{a^2} \\ \implies A_1 &= -\frac{B_1}{a^3} && (65) \end{aligned}$$

$$\begin{aligned} 0 &= -B_0 \frac{\delta}{b^2} + A_1 b + \frac{B_1}{b^2} \\ 0 &= -\left(V \frac{ab}{b-a}\right) \frac{\delta}{b^2} + \left(-\frac{B_1}{a^3}\right) b + \frac{B_1}{b^2} \quad \text{plugging in (62) and (65)} \\ 0 &= -V \frac{a}{b-a} \frac{\delta}{b} + \frac{B_1}{a^3 b^2}\left(a^3 - b^3\right) \end{aligned}$$

$$
\begin{aligned}
B_1 \frac{b^3 - a^3}{a^3 b^2} &= -V \frac{a}{b-a} \frac{\delta}{b} \\
\implies B_1 &= -V \frac{a}{b-a} \frac{\delta}{b} \frac{a^3 b^2}{b^3 - a^3} \\
B_1 &= -\delta\, V \frac{a^4 b}{(b-a)(b^3 - a^3)} \qquad (66)
\end{aligned}
$$

$$
\begin{aligned}
A_1 &= -\frac{B_1}{a^3} \quad \text{by (65)} \\
&= -\frac{1}{a^3} \left(-\delta\, V \frac{a^4 b}{(b-a)(b^3 - a^3)} \right) \\
A_1 &= \delta\, V \frac{ab}{(b-a)(b^3 - a^3)} \qquad (67)
\end{aligned}
$$

Plugging this into (50), we get that the term in $V(\mathbf{r})$ at first order in δ is

$$
\begin{aligned}
\left(A_1 r + \frac{B_1}{r^2} \right) \cos\theta &= \left(\delta\, V \frac{ab}{(b-a)(b^3 - a^3)} r - \delta\, V \frac{a^4 b}{(b-a)(b^3 - a^3)} \frac{1}{r^2} \right) \cos\theta \\
&= \delta\, V \frac{ab}{(b-a)(b^3 - a^3)} \left(r - \frac{a^3}{r^2} \right) \cos\theta \qquad (68)
\end{aligned}
$$

Combining (64) and (68), we get our final answer for $V(\mathbf{r})$ to first order in δ:

$$
V(\mathbf{r}) = \underbrace{-V \frac{a}{b-a}}_{A_0} \left(1 - \frac{b}{r} \right) + \underbrace{\delta\, V \frac{ab}{(b-a)(b^3 - a^3)}}_{A_1} \left(r - \frac{a^3}{r^2} \right) \cos\theta + \mathcal{O}(\delta^2) \quad \text{for} \quad a < r < b \qquad (69)
$$

Now, we must find the electric field between the spheres. Actually, we only need the radial component of the electric field, since only the discontinuity of the electric field *perpendicular* to the inner sphere is related to the surface charge density on the inner sphere. Using the gradient operator in polar coordinates

$$
\nabla = \hat{\mathbf{r}} \frac{\partial}{\partial r} + \hat{\theta} \frac{1}{r} \frac{\partial}{\partial \theta} \qquad (70)
$$

and the equation $\mathbf{E} = -\nabla V(\mathbf{r})$, we get

$$
\begin{aligned}
E_r &= -\frac{\partial V}{\partial r} \\
&= -\frac{\partial}{\partial r} \left(A_0 \left(1 - \frac{b}{r} \right) + A_1 \left(r - \frac{a^3}{r^2} \right) \cos\theta + \mathcal{O}(\delta^2) \right) \quad \text{by (69)} \\
&= -\left(A_0 \frac{b}{r^2} + A_1 \left(1 + 2\frac{a^3}{r^3} \right) \cos\theta + \mathcal{O}(\delta^2) \right) \\
E_r &= -A_0 \frac{b}{r^2} - A_1 \left(1 + 2\frac{a^3}{r^3} \right) \cos\theta + \mathcal{O}(\delta^2) \quad \text{for} \quad a < r < b \qquad (71)
\end{aligned}
$$

Taking the limit of this expression as we approach the inner sphere, $r \to a^+$, we get

$$
\begin{aligned}
E_r|_{r \to a^+} &= -A_0 \frac{b}{a^2} - A_1 \left(1 + 2\frac{a^3}{a^3} \right) \cos\theta + \mathcal{O}(\delta^2) \\
E_r|_{r \to a^+} &= -A_0 \frac{b}{a^2} - 3A_1 \cos\theta + \mathcal{O}(\delta^2) \qquad (72)
\end{aligned}
$$

Inside the inner sphere, $V(\mathbf{r}) = V$ everywhere (by the uniqueness theorem), so $\mathbf{E} = 0$ inside the inner sphere. Therefore, the surface charge density on the inner sphere is given by the equation

$$\begin{aligned}\frac{\sigma}{\epsilon_0} &= E_r|_{r\to a^+} - E_r|_{r\to a^-} \\ \frac{\sigma}{\epsilon_0} &= E_r|_{r\to a^+} \quad \text{since } \mathbf{E} = 0 \text{ inside the sphere} \\ \sigma(\theta) &= \epsilon_0 E_r|_{r\to a^+} \end{aligned} \tag{73}$$

Plugging in the value for the radial electric field just outside the inner sphere (72), we get

$$\begin{aligned}\sigma(\theta) &= \epsilon_0 \left(-A_0 \frac{b}{a^2} - 3A_1 \cos\theta + \mathcal{O}(\delta^2)\right) \\ &= \epsilon_0 \left[-\left(-V\frac{a}{b-a}\right)\frac{b}{a^2} - 3\left(\delta\, V \frac{ab}{(b-a)(b^3-a^3)}\cos\theta + \mathcal{O}(\delta^2)\right)\right] \quad \text{plugging in values from (69)} \\ \sigma(\theta) &= \epsilon_0 V \frac{ab}{b-a}\left(\frac{1}{a^2} - 3\delta\frac{1}{b^3-a^3}\cos\theta + \mathcal{O}(\delta^2)\right) \end{aligned} \tag{74}$$

The problem wants σ to be given in terms of the total charge of the inner sphere q. To get this charge, just integrate $\sigma(\theta)$ over the surface of the inner sphere:

$$\begin{aligned} q &= \int \sigma(\theta) dA \\ &= \epsilon_0 V \frac{ab}{b-a}\left(\int \frac{1}{a^2}\, dA - \int 3\delta \frac{1}{b^3-a^3}\cos\theta\, dA + \mathcal{O}(\delta^2)\right) \\ &= \epsilon_0 V \frac{ab}{b-a}\left((4\pi a^2)\frac{1}{a^2} - 3\delta\frac{1}{b^3-a^3}\int_{\cos\theta=-1}^{\cos\theta=+1} d(\cos\theta) 2\pi a^2 \cos\theta + \mathcal{O}(\delta^2)\right)\end{aligned}$$

In the last step, we used the fact that the surface area of the inner sphere is $4\pi a^2$, and that for the inner sphere, $dA = a^2 d\Omega = 2\pi a^2 d(\cos\theta)$ if the integrand does not depend on φ. Since the only remaining integral is odd in its integration variable ($\cos\theta$), it vanishes, and we get

$$\begin{aligned} q &= \epsilon_0 V \frac{ab}{b-a}\left((4\pi a^2)\frac{1}{a^2}\right) + \mathcal{O}(\delta^2) \\ &= 4\pi\epsilon_0 V \frac{ab}{b-a} + \mathcal{O}(\delta^2)\end{aligned}$$

Therefore, we have a relation between q and V:

$$V = \frac{q}{4\pi\epsilon_0}\frac{b-a}{ab} + \mathcal{O}(\delta^2) \tag{75}$$

(Note that this is the same relation as would exist if the spheres were concentric: $q = CV$ for the capacitance of two spheres $C = \frac{4\pi\epsilon ab}{b-a}$.) Plugging this back into (74), we get

$$\sigma(\theta) = \epsilon_0\left(\frac{q}{4\pi\epsilon_0}\frac{b-a}{ab}\right)\frac{ab}{b-a}\left(\frac{1}{a^2} - 3\delta\frac{1}{b^3-a^3}\cos\theta + \mathcal{O}(\delta^2)\right)$$

$$\boxed{\sigma(\theta) = \frac{q}{4\pi}\left(\frac{1}{a^2} - 3\delta\frac{1}{b^3-a^3}\cos\theta + \mathcal{O}(\delta^2)\right)} \tag{76}$$

Now, we need to find the force on the inner sphere. The force density on a patch of surface with charge dq is given by averaging the electric fields on both sides of the surface, $\mathbf{E}_1$ and $\mathbf{E}_2$, and multiplying by the charge dq:

$$d\mathbf{F} = dq\,\frac{\mathbf{E}_1 + \mathbf{E}_2}{2} = \sigma\, dA\,\frac{\mathbf{E}_1 + \mathbf{E}_2}{2} \tag{77}$$

In this case, the electric field inside the inner sphere is zero, as previously mentioned. The tangential component of the electric field is always continuous across a surface (assuming a static situation), so the electric field just outside the inner sphere is solely in the radial direction:

$$\begin{aligned} \mathbf{E}|_{r\to a^-} = 0 \quad \text{and} \quad \mathbf{E}|_{r\to a^+} &= \hat{\mathbf{r}}\, E_r|_{r\to a^+} \\ \mathbf{E}|_{r\to a^+} &= \hat{\mathbf{r}}\frac{\sigma(\theta)}{\epsilon_0} \quad \text{by (73)} \end{aligned} \tag{78}$$

Therefore, the force on a small portion of the inner sphere is

$$\begin{aligned} d\mathbf{F} &= \sigma(\theta)\, dA\, \frac{\mathbf{E}|_{r\to a^-} + \mathbf{E}_{r\to a^+}}{2} \\ &= \hat{\mathbf{r}}\, dA \frac{(\sigma(\theta))^2}{2\epsilon_0} \\ &= \hat{\mathbf{r}}\, dA\, \frac{1}{2\epsilon_0}\left[\frac{q}{4\pi}\left(\frac{1}{a^2} - 3\delta\frac{1}{b^3-a^3}\cos\theta + \mathcal{O}(\delta^2)\right)\right]^2 \quad \text{plugging in (76)} \\ &= \hat{\mathbf{r}}\, dA\, \frac{1}{2\epsilon_0}\frac{q^2}{(4\pi)^2}\left[\frac{1}{a^4} - 6\delta\frac{1}{a^2(b^3-a^3)}\cos\theta + \mathcal{O}(\delta^2)\right] \\ d\mathbf{F} &= \hat{\mathbf{r}}\, dA \frac{q^2}{32\pi^2\epsilon_0 a^2}\left[\frac{1}{a^2} - 6\delta\frac{1}{b^3-a^3}\cos\theta + \mathcal{O}(\delta^2)\right] \end{aligned} \tag{79}$$

This force consists of two term: a constant term and a θ-dependent term. The integral of the radial unit vector $\hat{\mathbf{r}}$ over a sphere is zero, so the constant term vanishes upon integration:

$$\begin{aligned} \mathbf{F} &= \int \hat{\mathbf{r}}\, dA \frac{q^2}{32\pi^2\epsilon_0 a^2}\left[-6\delta\frac{1}{b^3-a^3}\cos\theta + \mathcal{O}(\delta^2)\right] \quad \text{ignoring the constant term} \\ &= -\frac{3q^2}{16\pi^2\epsilon_0 a^2}\frac{\delta}{(b^3-a^3)}\int \hat{\mathbf{r}}\, dA \cos\theta + \mathcal{O}(\delta^2) \\ &= -\frac{3q^2}{16\pi^2\epsilon_0 a^2}\frac{\delta}{(b^3-a^3)}\int_0^{2\pi} d\varphi \int_{\cos\theta=-1}^{\cos\theta=+1} d(\cos\theta) a^2 \hat{\mathbf{r}}\cos\theta + \mathcal{O}(\delta^2) \\ \mathbf{F} &= -\frac{3q^2}{16\pi^2\epsilon_0}\frac{\delta}{(b^3-a^3)}\int_0^{2\pi} d\varphi \int_{\cos\theta=-1}^{\cos\theta=+1} d(\cos\theta)\hat{\mathbf{r}}\cos\theta + \mathcal{O}(\delta^2) \end{aligned} \tag{80}$$

In the last two lines, we used the area element $dA = a^2 d(\cos\theta)d\varphi$. We can now split the unit vector $\hat{\mathbf{r}}$ up into $\hat{\mathbf{z}}$ and another component, which we will call $\hat{\mathbf{s}}$:

$$\hat{\mathbf{r}} = \underbrace{(\hat{\mathbf{x}}\,\cos\varphi + \hat{\mathbf{y}}\,\sin\varphi)}_{\hat{\mathbf{s}}}\sin\theta + \hat{\mathbf{z}}\cos\theta \tag{81}$$

Thus,

$$\int_0^{2\pi} d\varphi \int_{-1}^{+1} d(\cos\theta)\hat{\mathbf{r}}\cos\theta = \int_0^{2\pi} d\varphi \int_{-1}^{+1} d(\cos\theta)\hat{\mathbf{s}}\sin\theta\cos\theta + \int_0^{2\pi} d\varphi \int_{-1}^{+1} d(\cos\theta)\hat{\mathbf{z}}\cos^2\theta \tag{82}$$

Since $\hat{\mathbf{s}} = \hat{\mathbf{x}}\cos\varphi + \hat{\mathbf{y}}\,\sin\varphi$, the first integral vanishes once we take the integral over φ. All that remains is the second integral:

$$\begin{aligned} \int_0^{2\pi} d\varphi \int_{-1}^{+1} d(\cos\theta)\hat{\mathbf{r}}\cos\theta &= \hat{\mathbf{z}}\int_0^{2\pi} d\varphi \int_{-1}^{+1} d(\cos\theta)\cos^2\theta \\ &= 2\pi\hat{\mathbf{z}}\left[\frac{\cos^3\theta}{3}\right]_{\cos\theta=-1}^{\cos\theta=+1} \\ &= \frac{4\pi}{3}\,\hat{\mathbf{z}} \end{aligned} \tag{83}$$

Plugging this back into (80), we get an expression for the force on the inner sphere

$$\mathbf{F} = -\frac{3q^2}{16\pi^2\epsilon_0}\frac{\delta}{b^3 - a^3}\left(\frac{4\pi}{3}\,\hat{\mathbf{z}}\right) + \mathcal{O}(\delta^2)$$

or

$$\boxed{\mathbf{F} = -\hat{\mathbf{z}}\,\frac{q^2}{4\pi\epsilon_0}\frac{\delta}{b^3 - a^3} + \mathcal{O}(\delta^2)} \tag{84}$$

To this order in δ, the force tends to push the inner capacitor further off-center.