1. (Quantum Mechanics)

A particle of charge q is subjected to a magnetic field $\vec{B} = B\hat{z}$.

(a) Consider the symmetric gauge for the vector potential

$$\vec{A} = \frac{B}{2}(-y\hat{x} + x\hat{y})$$

and show that is correctly gives the magnetic field. Write down the Hamiltonian in the symmetric gauge and define

$$Q = \frac{1}{qB}\left(cp_x + \frac{qB}{2}y\right), \quad P = \left(p_y - \frac{qB}{2c}x\right)$$

Show that the commutator $[Q, P] = i\hbar$. (c is the velocity of light)

(b) Show that H in terms of P and Q becomes a one-dimensional harmonic oscillator problem, where $\omega = qB/mc$. Find the energy eigenvalues.

(c) Write down the harmonic oscillator annihilation operator a in terms of the complex coordinates $z = x + iy$ and $z^* = x - iy$ and show that the ground state wave function is given by

$$\psi_0(z, z^*) = u(z, z^*)\, e^{-\frac{qB}{4\hbar c}\, zz^*}$$

and u is an arbitrary analytic function $\frac{\partial}{\partial z^*}u(z, z^*) = 0$, for example, $u(z, z^*) = z^n$ (n is an arbitrary positive integer).

Hint: The Cauchy-Riemann conditions for the analyticity of a function $f(x, y) = U(x, y) + V(x, y)$ are $\frac{\partial U}{\partial x} = \frac{\partial V}{\partial y}$ and $\frac{\partial V}{\partial x} = -\frac{\partial U}{\partial y}$.

Solution: *Solution by Jonah Hyman (jthyman@g.ucla.edu)*

This problem is in Gaussian units for electromagnetism. While the problem doesn't directly state this, you can figure it out by noticing that there are extra factors of c, the speed of light, in some of the answers. For example, from the definition of ω in part (b), $\omega = (qB)/(mc)$, we get that in SI units, the units of the magnetic field B are

$$[B] = \frac{[m]\,[c]\,[\omega]}{[q]} = \frac{\mathrm{kg}\cdot(\mathrm{m/s})\cdot(1/\mathrm{s})}{\mathrm{C}} = \frac{\mathrm{N}}{\mathrm{C}} \tag{1}$$

Therefore, in this problem, **the magnetic field has the same units as the electric field**. This is the hallmark of Gaussian units. We will make a few special remarks about the differences between Gaussian and SI formulas in the context of this problem. Many electrodymanics textbooks include an appendix about Gaussian units, which should be consulted for a more complete overview of the subject (e.g. Griffiths, *Introduction to Electrodynamics*, Appendix C; Zangwill, *Modern Electrodynamics*, Appendix B).

(a) The magnetic field is defined in terms of the vector potential by

$$\mathbf{B} = \nabla \times \mathbf{A} \quad \text{(same in SI and Gaussian units)} \tag{2}$$

Applying this to the given vector potential, $\mathbf{A} = \frac{B}{2}\left(-y\hat{\mathbf{x}} + x\hat{\mathbf{y}}\right)$, and using the fact that $\hat{\mathbf{x}}\times\hat{\mathbf{y}} = -\hat{\mathbf{y}}\times\hat{\mathbf{x}} = \hat{\mathbf{z}}$, we get that

$$\begin{aligned}\mathbf{B} &= \left(\hat{\mathbf{x}}\frac{\partial}{\partial x} + \hat{\mathbf{y}}\frac{\partial}{\partial y} + \hat{\mathbf{z}}\frac{\partial}{\partial z}\right) \times \frac{B}{2}\left(-y\hat{\mathbf{x}} + x\hat{\mathbf{y}}\right) \\ &= \frac{B}{2}\left(\frac{\partial}{\partial y}(-y)\left(\hat{\mathbf{y}}\times\hat{\mathbf{x}}\right) + \frac{\partial}{\partial x}(x)\left(\hat{\mathbf{x}}\times\hat{\mathbf{y}}\right)\right) \\ &= \frac{B}{2}\left(\hat{\mathbf{z}} + \hat{\mathbf{z}}\right) \\ &= B\hat{\mathbf{z}}\end{aligned} \tag{3}$$

which is the given magnetic field.

The Hamiltonian for a free particle of charge q in a magnetic field with vector potential $\mathbf{A}$ is

$$H = \frac{1}{2m}\left(\mathbf{p} - \frac{q\mathbf{A}}{c}\right)^2 \quad \text{(Gaussian);} \qquad H = \frac{1}{2m}\left(\mathbf{p} - q\mathbf{A}\right)^2 \quad \text{(SI)} \tag{4}$$

Note that $\mathbf{A}_{\text{Gaussian}} = c\mathbf{A}_{\text{SI}}$, so the two equations are equivalent. It is probably best to memorize (or write on your formula sheet) the prescriptions for Lagrangians and Hamiltonians with a vector potential:

Incorporating a vector potential into the Lagrangian:
SI units: Add $+q\dot{\mathbf{r}}\cdot\mathbf{A}$.
Gaussian units: Add $+\frac{q\dot{\mathbf{r}}\cdot\mathbf{A}}{c}$.
Incorporating a vector potential into the Hamiltonian:
SI units: Replace $\mathbf{p}$ with $\mathbf{p} - q\mathbf{A}$.
Gaussian units: Replace $\mathbf{p}$ with $\mathbf{p} - \frac{q\mathbf{A}}{c}$.

Using the Gaussian version of equation (4) and applying the given vector potential, we get

$$\begin{aligned} H &= \frac{1}{2m}\left(\mathbf{p} - \frac{q\mathbf{A}}{c}\right)^2 \\ &= \frac{1}{2m}\left(\left(p_x - \frac{qA_x}{c}\right)^2 + \left(p_y - \frac{qA_y}{c}\right)^2 + \left(p_z - \frac{qA_z}{c}\right)^2\right) \\ H &= \frac{1}{2m}\left(\left(p_x + \frac{qB}{2c}y\right)^2 + \left(p_y - \frac{qB}{2c}x\right)^2 + p_z^2\right) \quad \text{since } \mathbf{A} = \frac{B}{2}\left(-y\hat{\mathbf{x}} + x\hat{\mathbf{y}}\right) \end{aligned} \tag{5}$$

We will write H in a more convenient form in part (b).

The problem defines

$$Q \equiv \frac{1}{qB}\left(cp_x + \frac{qB}{2}y\right) \quad \text{and} \quad P \equiv p_y - \frac{qB}{2c}x \tag{6}$$

Applying the linearity of the commutator

$$[aA + bB, cC + dD] = ac\,[A, C] + ad\,[A, D] + bc\,[B, C] + bd\,[B, D] \tag{7}$$

and the canonical commutation relations for x and p

$$[x, p_x] = [y, p_y] = i\hbar \quad \text{and} \quad [x, p_y] = [y, p_x] = [x, y] = [p_x, p_y] = 0 \tag{8}$$

we get that

$$\begin{aligned} [Q, P] &= \frac{1}{qB}\left(c\,[p_x, p_y] - \frac{qB}{2}\,[p_x, x] + \frac{qB}{2}\,[y, p_y] - \frac{q^2B^2}{4c}\,[y, x]\right) \\ &= \frac{1}{qB}\left(-\frac{qB}{2}\,[p_x, x] + \frac{qB}{2}\,[y, p_y]\right) \\ &= \frac{1}{qB}\left(-\frac{qB}{2}(-i\hbar) + \frac{qB}{2}(i\hbar)\right) \\ &= \frac{1}{qB}\left(qBi\hbar\right) \\ [Q, P] &= i\hbar \end{aligned} \tag{9}$$

This implies that Q and P have the same canonical quantization relations as x and p_x.

(b) We start by finding H in terms of P and Q:

$$\begin{aligned} H &= \frac{1}{2m}\left(\left(p_x + \frac{qB}{2c}y\right)^2 + \left(p_y - \frac{qB}{2c}x\right)^2 + p_z^2\right) \quad \text{from (5)} \\ &= \frac{1}{2m}\left(\frac{1}{c^2}\left(cp_x + \frac{qB}{2}y\right)^2 + \left(p_y - \frac{qB}{2c}x\right)^2 + p_z^2\right) \\ &= \frac{1}{2m}\left(\frac{1}{c^2}\left(qB\,Q\right)^2 + P^2 + p_z^2\right) \\ H &= \frac{p_z^2}{2m} + \frac{P^2}{2m} + \frac{1}{2}m\left(\frac{qB}{mc}\right)^2 Q^2 \end{aligned} \tag{10}$$

This Hamiltonian is in two parts: The z-dependent part is

$$H_z = \frac{p_z^2}{2m} \tag{11}$$

which is just the Hamiltonian for a free particle in the z-direction. Since P and Q depend on (x, y, p_x, p_y), the (x, y)-dependent part of the Hamiltonian is

$$H_{(x,y)} = \frac{P^2}{2m} + \frac{1}{2}m\left(\frac{qB}{mc}\right)^2 Q^2 = \frac{P^2}{2m} + \frac{1}{2}m\omega^2 Q^2 \quad \text{with } \omega \equiv \frac{qB}{mc} \text{ and } [Q, P] = i\hbar \tag{12}$$

which is the Hamiltonian for a one-dimensional harmonic oscillator.

$H_{(x,y)}$ and H_z commute with another (since one depends on (x, y) and their momenta and the other depends on z and its momentum). Therefore, we can simultaneously diagonalize $H_{(x,y)}$ and H_z, so we can add the energies of each of the two.

The energy spectrum of H_z is the continuous spectrum of energy eigenvalues for a free particle:

$$E_z(p_z) = \frac{p_z^2}{2m} \quad \text{for } -\infty < p_z < \infty \tag{13}$$

The energy spectrum of $H_{(x,y)}$ is the discrete spectrum of energy eigenvalues for a one-dimensional harmonic oscillator. (Note that it is necessary that Q and P satisfy the canonical quantization relation, $[Q, P] = i\hbar$, for this to be correct.)

$$E_{(x,y)}(n) = \hbar\omega\left(n + \frac{1}{2}\right) = \frac{\hbar qB}{mc}\left(n + \frac{1}{2}\right) \quad \text{for} \quad n = 0, 1, 2, \ldots \tag{14}$$

To get the total energy, add the two energies together:

$$\boxed{E_{(n,p_z)} = \hbar\omega\left(n + \frac{1}{2}\right) = \frac{\hbar qB}{mc}\left(n + \frac{1}{2}\right) + \frac{p_z^2}{2m} \quad \text{for} \quad n = 0, 1, 2, \ldots, \ -\infty < p_z < \infty} \tag{15}$$

It is also acceptable to write $p_z = \hbar k_z$ and express the energy in terms of (n, k_z).

(c) The harmonic oscillator annihilation operator in terms of the operators Q and P is

$$a = \frac{1}{\sqrt{2}}\left(\left(\frac{m\omega}{\hbar}\right)^{1/2} Q + \frac{i}{(m\omega\hbar)^{1/2}} P\right) \tag{16}$$

You need to memorize this equation (or learn how to derive it, or write it on your formula sheet). Note that a and $a^\dagger$ only have the correct commutation relations if Q and P have the canonical commutation relation $[Q, P] = i\hbar$, which we verified in part (a). We can simplify this equation to get

$$\begin{aligned}
a &= \left(\frac{m\omega}{2\hbar}\right)^{1/2}\left(Q + \frac{i}{m\omega}P\right) \\
&= \left(\frac{qB}{2\hbar c}\right)^{1/2}\left(Q + \frac{ic}{qB}P\right) \quad \text{since } \omega = \frac{qB}{mc}, \text{ so } m\omega = \frac{qB}{c} \\
&= \left(\frac{qB}{2\hbar c}\right)^{1/2}\left(\frac{1}{qB}\left(cp_x + \frac{qB}{2}y\right) + \frac{ic}{qB}\left(p_y - \frac{qB}{2c}x\right)\right) \quad \text{by (6)} \\
&= \left(\frac{qB}{2\hbar c}\right)^{1/2}\left(\frac{c}{qB}p_x + \frac{1}{2}y + \frac{ic}{qB}p_y - \frac{i}{2}x\right) \\
&= \left(\frac{qB}{2\hbar c}\right)^{1/2}\left(\frac{c}{qB}\left(p_x + ip_y\right) - \frac{i}{2}\left(x + iy\right)\right)
\end{aligned} \tag{17}$$

We now need to get the differential version of a, so we must apply the differential operators p_x and p_y as applied to a wavefunction $\psi(x,y)$:

$$p_x = \frac{\hbar}{i}\frac{\partial}{\partial x} \quad \text{and} \quad p_y = \frac{\hbar}{i}\frac{\partial}{\partial y} \tag{18}$$

This gets us to

$$\begin{aligned} a &= \left(\frac{qB}{2\hbar c}\right)^{1/2}\left(\frac{c}{qB}\left(\frac{\hbar}{i}\frac{\partial}{\partial x} + i\frac{\hbar}{i}\frac{\partial}{\partial y}\right) - \frac{i}{2}(x+iy)\right) \\ &= \left(\frac{qB}{2\hbar c}\right)^{1/2}\left(-i\frac{\hbar c}{qB}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) - \frac{i}{2}(x+iy)\right) \end{aligned} \tag{19}$$

As instructed by the problem, we must now convert from the coordinates (x,y) to the independent coordinates (z,z^*), where

$$z \equiv x+iy \quad \text{and} \quad z^* \equiv x-iy \tag{20}$$

(Note that this complex coordinate z is unrelated to the real spatial coordinate z.) Using the chain rule, we can write the partial derivatives of x and y in terms of partial derivatives of z:

$$\frac{\partial}{\partial x} = \frac{\partial z}{\partial x}\frac{\partial}{\partial z} + \frac{\partial z^*}{\partial x}\frac{\partial}{\partial z^*} = \frac{\partial}{\partial z} + \frac{\partial}{\partial z^*} \tag{21}$$

$$\frac{\partial}{\partial y} = \frac{\partial z}{\partial y}\frac{\partial}{\partial z} + \frac{\partial z^*}{\partial y}\frac{\partial}{\partial z^*} = i\frac{\partial}{\partial z} - i\frac{\partial}{\partial z^*} \tag{22}$$

This implies that

$$\begin{aligned} \frac{\partial}{\partial x} + i\frac{\partial}{\partial y} &= \left(\frac{\partial}{\partial z} + \frac{\partial}{\partial z^*}\right) + i\left(i\frac{\partial}{\partial z} - i\frac{\partial}{\partial z^*}\right) \\ &= 2\frac{\partial}{\partial z^*} \end{aligned} \tag{23}$$

Substituting equations (20) and (23) into equation (19), we get that

$$a = \left(\frac{qB}{2\hbar c}\right)^{1/2}\left(-i\frac{\hbar c}{qB}\left(2\frac{\partial}{\partial z^*}\right) - \frac{i}{2}z\right)$$

$$\boxed{a = -i\left(\frac{qB}{2\hbar c}\right)^{1/2}\left(\frac{2\hbar c}{qB}\frac{\partial}{\partial z^*} + \frac{1}{2}z\right)} \tag{24}$$

The ground state wave function for the harmonic oscillator is the wave function that, when lowered with the annihilation operator, yields zero:

$$a\,\psi_0(z,z^*) = 0 \tag{25}$$

Using the differential operator in equation (24), this implies that

$$\begin{aligned} \frac{2\hbar c}{qB}\frac{\partial\psi_0}{\partial z^*} + \frac{1}{2}z\psi_0 &= 0 \\ \frac{\partial\psi_0}{\partial z^*} + \frac{1}{4}\frac{qB}{\hbar c}z\psi_0 &= 0 \end{aligned} \tag{26}$$

The problem gives us the ansatz for the ground state

$$\psi_0(z,z^*) = u(z,z^*)\exp\left(-\frac{qB}{4\hbar c}zz^*\right) \tag{27}$$

The first step is to show which functions of the form (27) satisfies the differential equation (26). We can check this directly by noting that

$$\frac{\partial \psi_0}{\partial z^*} = \left(\frac{\partial u}{\partial z^*} - \frac{1}{4} \frac{qB}{\hbar c} z \right) \exp\left(-\frac{qB}{4\hbar c} z z^* \right) \tag{28}$$

Therefore,

$$\frac{\partial \psi_0}{\partial z^*} + \frac{1}{4} \frac{qB}{\hbar c} z \psi_0 = \frac{\partial u}{\partial z^*} \tag{29}$$

We want this to equal zero, so we must set $\frac{\partial}{\partial z^*} u(z, z^*) = 0$ for the differential equation (26) to be satisfied.

So far, we have shown that the solution set for the differential equation (26) is

$$\psi_0(z, z^*) = u(z, z^*) \exp\left(-\frac{qB}{4\hbar c} z z^* \right) \quad \text{for } u(z, z^*) \text{ with } \frac{\partial}{\partial z^*} u(z, z^*) = 0 \tag{30}$$

There are no other solutions because if we consider z to be fixed, differential equation (26) is a first-order ordinary differential equation in z^*. Thus, up to a (possibly z-dependent) prefactor, there is only one solution to this differential equation.

All that remains is to show that $u(z, z^*)$ is analytic. We do so by showing that the condition $\frac{\partial}{\partial z^*} u(z, z^*) = 0$ is equivalent to the condition that $u(z, z^*)$ is analytic. First, divide $u(z, z^*)$ into its real and imaginary components:

$$u(z, z^*) = U(z, z^*) + iV(z, z^*) \quad \text{for } U(z, z^*) \text{ and } V(z, z^*) \text{ real-valued} \tag{31}$$

Then, take the partial derivative with respect to z^*, using equation (22) to write the partial derivatives with respect to x and y:

$$\begin{aligned} \frac{\partial u}{\partial z^*} &= \frac{\partial U}{\partial z^*} + i \frac{\partial V}{\partial z^*} \\ &= \frac{1}{2} \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right) U + \frac{i}{2} \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right) V \\ &= \frac{1}{2} \left(\frac{\partial U}{\partial x} - \frac{\partial V}{\partial y} \right) + \frac{i}{2} \left(\frac{\partial U}{\partial y} + \frac{\partial V}{\partial x} \right) \end{aligned} \tag{32}$$

Since U and V are both real, this equation tells us that

$$\frac{\partial u}{\partial z^*} = 0 \quad \text{if and only if} \quad \frac{\partial U}{\partial x} = \frac{\partial V}{\partial y} \quad \text{and} \quad \frac{\partial V}{\partial x} = -\frac{\partial U}{\partial y} \tag{33}$$

The hint in the problem tells us that the right-hand side of this equation is the definition of an analytic function. Therefore,

$$\frac{\partial u}{\partial z^*} = 0 \quad \text{if and only if } u(z, z^*) \text{ is analytic} \tag{34}$$

Putting everything together, we have established that the ground state wave function (solution to differential equation (26) is given by

$$\psi_0(z, z^*) = u(z, z^*) \exp\left(-\frac{qB}{4\hbar c} z z^* \right) \quad \text{for any analytic function } u(z, z^*) \tag{35}$$

The quantum mechanics of a charged spinless particle in a constant magnetic field (in either the symmetric gauge, as in this problem, or in the related Landau gauge) is a fairly common comp problem. See also 2016 Q7, 2014 Q7, and 2012 Q5.