9. (Electromagnetism)

Consider a closed circuit formed into a circular coil of N turns with radius a, resistance R. You can neglect the self-inductance of the circuit. The coil rotates around the $\hat{\mathbf{z}}$-axis in a uniform magnetic field $\mathbf{B}$ directed along the $\hat{\mathbf{x}}$-axis (see below).

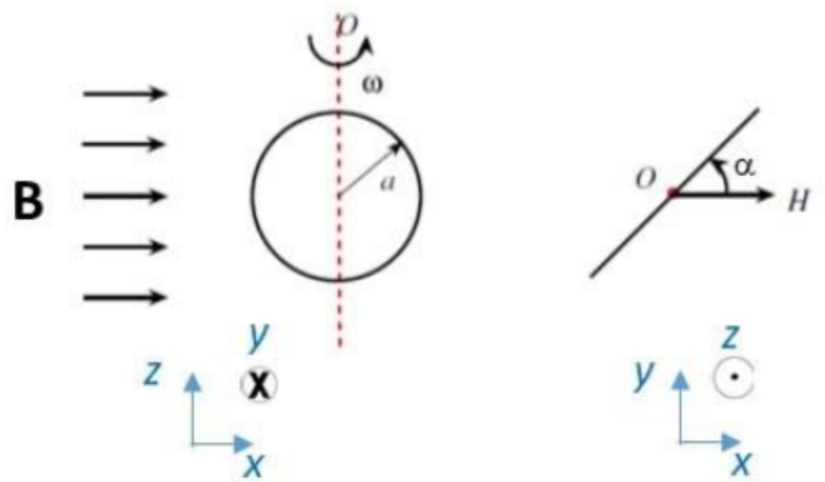
(a) Find the current in the coil as a function of α for rotation at a constant angular velocity ω. Here $\alpha(t) = \omega t$ is the angle between the plane of the coil and $\mathbf{B}$ (the $\hat{\mathbf{x}}$-axis).

(b) Find the externally applied torque that is needed to maintain the coil's uniform rotation.

(c) Due to the time-dependent currents induced in the coil, electromagnetic waves are radiated. What is the frequency of the radiation?

(d) What is the polarization of the radiated waves propagating along the positive $\hat{\mathbf{z}}$-axis?

(e) Compute the total power radiated by the rotating coil of the wire.

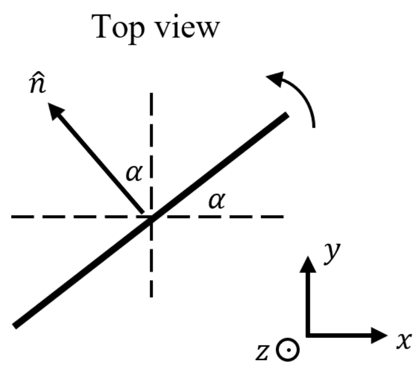


Solution: *Solution by Jonah Hyman (jthyman@g.ucla.edu)*

(a) The source of the current flowing in the wire is the electromagnetic force (emf) ε induced by the changing magnetic flux through the wire. The magnetic flux through a single turn of wire is defined as

$$\Phi_{\mathbf{B},1} \equiv \int_{\text{turn of wire}} \mathbf{B} \cdot d\mathbf{a} \tag{144}$$

Recall that $d\mathbf{a} = \hat{\mathbf{n}}\, dA$, where $\hat{\mathbf{n}}$ is normal to the surface. We can draw a diagram that shows $\hat{\mathbf{n}}$:



From this diagram, we can write

$$\hat{\mathbf{n}} = -\hat{\mathbf{x}} \sin\alpha + \hat{\mathbf{y}} \cos\alpha \tag{145}$$

Since $\mathbf{B} = B\hat{\mathbf{x}}$ in this problem, we have

$$\begin{aligned} \mathbf{B} \cdot d\mathbf{a} &= (B\hat{\mathbf{x}}) \cdot (-\hat{\mathbf{x}} \sin\alpha + \hat{\mathbf{y}} \cos\alpha)\, dA \\ &= -B \sin\alpha\, dA \end{aligned} \tag{146}$$

Carrying out the integral (144) over one circular turn of wire, we get

$$\begin{aligned} \Phi_{\mathbf{B},1} &= \int_{\text{turn of wire}} (-B \sin\alpha\, dA) \\ &= -B \sin\alpha \left(\pi a^2\right) \quad \text{since the area of the circular turn is } \pi a^2 \end{aligned} \tag{147}$$

This is the magnetic flux through *one* turn of wire. To get the magnetic flux through the entire loop $\Phi_{\mathbf{B}}$, we must multiply this value by the number of turns N:

$$\Phi_{\mathbf{B}} = N\Phi_{\mathbf{B},1} = -NB\pi a^2 \sin\alpha \tag{148}$$

A version of Faraday's law gives us the induced emf in terms of the time derivative of $\Phi_{\mathbf{B}}$:

$$\varepsilon = -\frac{d\Phi_{\mathbf{B}}}{dt} \tag{149}$$

Plugging in our value for $\Phi_{\mathbf{B}}$, we get

$$\begin{aligned} \varepsilon &= -\frac{d}{dt}\left(-NB\pi a^2 \sin\alpha\right) \\ &= NB\pi a^2 \frac{d\alpha}{dt} \cos\alpha \quad \text{by the chain rule} \\ &= NB\pi a^2 \omega \cos\alpha \quad \text{since } \alpha = \omega t \end{aligned} \tag{150}$$

By Ohm's law, the current is related to the emf:

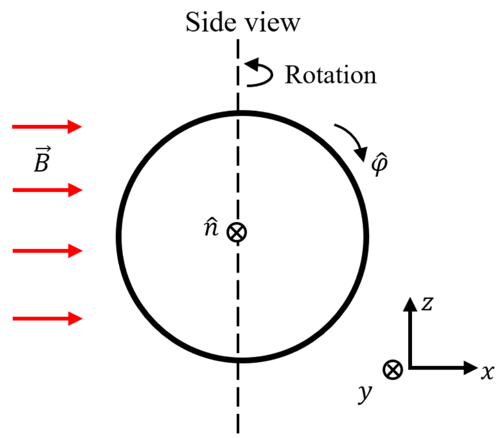
$$\varepsilon = IR \qquad \Longrightarrow \qquad I = \frac{\varepsilon}{R} \tag{151}$$

Plugging in (150), we get

$$I = \frac{NB\pi a^2 \omega \cos\alpha}{R} \tag{152}$$

All that remains is to determine the direction of the current using Lenz's law. Let $+\hat{\mathbf{n}}$ be considered to be the positive direction. At $t = 0$, the magnetic flux through the loop is increasing in the negative direction, since shortly after $t = 0$, $\hat{\mathbf{n}}$ points in the opposite direction as $\mathbf{B}$. Therefore, at $t = 0$, the induced current produces a magnetic field that increases the magnetic flux through the loop in the positive direction. By looking at the diagram and using the right-hand rule, you can see that such a current would point counterclockwise with respect to $\hat{\mathbf{n}}$. Therefore, since $\cos\alpha > 0$ at $t = 0$, the induced current in the loop is given by

$$\boxed{I = \frac{NB\pi a^2 \omega \cos\alpha}{R}\hat{\varphi} \quad \text{where } \hat{\varphi} \text{ points counterclockwise with respect to } \hat{\mathbf{n}}} \tag{153}$$



(b) For this part of the problem, there are two possible methods:

Magnetic dipole formula:
For a current loop in a uniform magnetic field, the torque on the current loop is

$$\boldsymbol{\tau} = \mathbf{m} \times \mathbf{B} \quad \text{where } \mathbf{m} \text{ is the magnetic dipole moment of the loop} \tag{154}$$

Note that this formula holds regardless of the size of the loop. Roughly speaking, that is because the loop can be broken into many magnetic dipoles (infinitesimally small current loops) with overlapping sides. Since the torques on the overlapping sides cancel out, the torque on the collection of magnetic dipoles is equal to the torque on the entire loop.

In this case, we have a flat loop with constant current, so the magnetic dipole moment of a single turn of the loop is given by the formula

$$\mathbf{m}_1 = I\mathbf{A} = IA\,\hat{\mathbf{n}} \quad \text{where } A \text{ is the area of the loop} \tag{155}$$

This is the magnetic dipole moment through *one* turn of wire. To get the magnetic dipole moment for the entire loop $\mathbf{m}$, we must multiply this value by the number of turns N:

$$\mathbf{m} = NIA\,\hat{\mathbf{n}} \tag{156}$$

Here, $A = \pi a^2$, and we have the value of $\hat{\mathbf{n}}$ from (145), so we get

$$\mathbf{m} = NI\pi a^2 \left(-\hat{\mathbf{x}} \sin\alpha + \hat{\mathbf{y}} \cos\alpha\right) \quad \text{where } I \text{ is the current from part (a)} \tag{157}$$

Applying the formula (154), we get the torque on the current loop

$$\begin{aligned} \boldsymbol{\tau} &= NI\pi a^2 \left(-\hat{\mathbf{x}} \sin\alpha + \hat{\mathbf{y}} \cos\alpha\right) \times B\,\hat{\mathbf{x}} \\ &= -NI\pi a^2 B \cos\alpha\,\hat{\mathbf{z}} \quad \text{since } \hat{\mathbf{y}} \times \hat{\mathbf{x}} = -\hat{\mathbf{z}} \end{aligned}$$

The torque we need to *apply* to the current loop to keep it rotating is the opposite of the torque *on* the current loop:

$$\boldsymbol{\tau}_{\text{applied}} = -\boldsymbol{\tau} = NI\pi a^2 B \cos\alpha\,\hat{\mathbf{z}}$$

Plugging in the value of I from part (a) (153), we get

$$\boxed{\boldsymbol{\tau}_{\text{applied}} = (NB\pi a^2)^2 \,\frac{\omega}{R}\, \cos^2\alpha\,\hat{\mathbf{z}}} \tag{158}$$

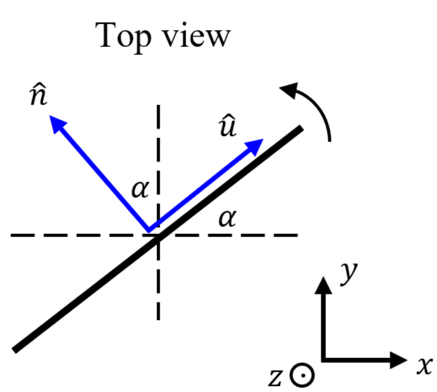
Direct calculation:
To directly calculate the torque on the wire due to the magnetic field, we start by calculating the force on an section of the wire with oriented line element $d\boldsymbol{\ell}$. From the Lorentz force law, we get that the force on a section of the wire with oriented line element $d\ell$ is

$$\begin{aligned} d\mathbf{F} &= dq\,\mathbf{v} \times \mathbf{B} \\ &= (\lambda d\ell)\;\mathbf{v} \times \mathbf{B} \quad \text{where } \lambda \text{ is the linear charge density in the wire} \\ &= \mathbf{I}\,d\ell\, \times \mathbf{B} \quad \text{since } \mathbf{I} = \lambda\mathbf{v} \\ &= I\,d\boldsymbol{\ell}\, \times \mathbf{B} \quad \text{since } d\boldsymbol{\ell} \propto \mathbf{I} \end{aligned} \tag{159}$$

From the definition of torque, $\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}$, we can find the torque on this section of the wire about the center of the loop:

$$\begin{aligned} d\boldsymbol{\tau} &= \mathbf{r} \times d\mathbf{F} \\ &= I\mathbf{r} \times (d\boldsymbol{\ell} \times \mathbf{B}) \end{aligned} \tag{160}$$

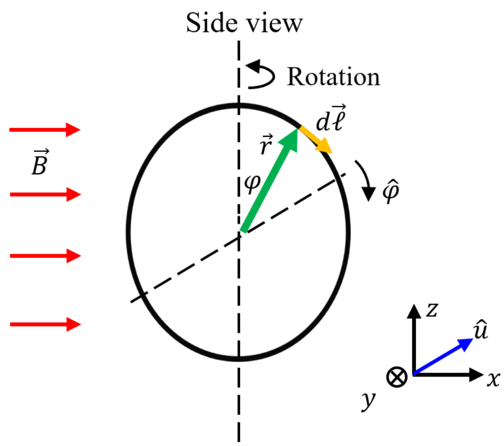
We now need to find $\mathbf{r}$ and $d\boldsymbol{\ell}$ for the problem setup. To do so, introduce the rotating coordinate system $(\hat{\mathbf{u}}, \hat{\mathbf{n}})$ shown in the diagram below:



From this diagram, we can write that

$$\hat{\mathbf{u}} = \hat{\mathbf{x}} \cos\alpha + \hat{\mathbf{y}} \sin\alpha \tag{161}$$

The diagram below shows $\mathbf{r}$ and $d\boldsymbol{\ell}$ for a given value of φ:



From this diagram, recalling that a is the radius of the circle, we can write

$$\mathbf{r} = a\,(\hat{\mathbf{u}}\,\sin\varphi + \hat{\mathbf{z}}\cos\varphi) \tag{162}$$

$$d\boldsymbol{\ell} = d\ell\,(\hat{\mathbf{u}}\,\cos\varphi - \hat{\mathbf{z}}\,\sin\varphi) \tag{163}$$

Using this information, we can build up to (160):

$$\begin{aligned} d\boldsymbol{\ell}\times\mathbf{B} &= d\ell\,(\hat{\mathbf{u}}\,\cos\varphi - \hat{\mathbf{z}}\,\sin\varphi)\times B\,\hat{\mathbf{x}} \\ &= B\,d\ell\,[(\hat{\mathbf{u}}\times\hat{\mathbf{x}})\,\cos\varphi - (\hat{\mathbf{z}}\times\hat{\mathbf{x}})\,\sin\varphi] \\ &= B\,d\ell\,[(\hat{\mathbf{u}}\times\hat{\mathbf{x}})\,\cos\varphi - \hat{\mathbf{y}}\,\sin\varphi] \quad \text{since } \hat{\mathbf{z}}\times\hat{\mathbf{x}} = \hat{\mathbf{y}} \end{aligned}$$

By (161), we can write

$$\begin{aligned} \hat{\mathbf{u}}\times\hat{\mathbf{x}} &= (\hat{\mathbf{x}}\,\cos\alpha + \hat{\mathbf{y}}\,\sin\alpha)\times\hat{\mathbf{x}} \\ &= -\hat{\mathbf{z}}\,\sin\alpha \quad \text{since } \hat{\mathbf{y}}\times\hat{\mathbf{x}} = -\hat{\mathbf{z}} \end{aligned}$$

Therefore,

$$d\boldsymbol{\ell}\times\mathbf{B} = B\,d\ell\,(-\hat{\mathbf{z}}\,\sin\alpha\,\cos\varphi - \hat{\mathbf{y}}\,\sin\varphi) \tag{164}$$

Using (162) and noting that $\hat{\mathbf{z}}\times\hat{\mathbf{z}} = 0$ and $\hat{\mathbf{z}}\times\hat{\mathbf{y}} = -\hat{\mathbf{x}}$, we get

$$\begin{aligned} \mathbf{r}\times(d\boldsymbol{\ell}\times\mathbf{B}) &= a\,(\hat{\mathbf{u}}\,\sin\varphi + \hat{\mathbf{z}}\cos\varphi)\times B\,d\ell\,(-\hat{\mathbf{z}}\,\sin\alpha\,\cos\varphi - \hat{\mathbf{y}}\,\sin\varphi) \\ &= Ba\,d\ell\left[-(\hat{\mathbf{u}}\times\hat{\mathbf{z}})\sin\alpha\,\sin\varphi\,\cos\varphi - (\hat{\mathbf{u}}\times\hat{\mathbf{y}})\sin^2\varphi - (\hat{\mathbf{z}}\times\hat{\mathbf{y}})\sin\varphi\,\cos\varphi\right] \\ &= Ba\,d\ell\left[-(\hat{\mathbf{u}}\times\hat{\mathbf{z}})\sin\alpha\,\sin\varphi\,\cos\varphi - (\hat{\mathbf{u}}\times\hat{\mathbf{y}})\sin^2\varphi + \hat{\mathbf{x}}\,\sin\varphi\,\cos\varphi\right] \end{aligned}$$

By (161), we can write

$$\begin{aligned} \hat{\mathbf{u}}\times\hat{\mathbf{y}} &= (\hat{\mathbf{x}}\,\cos\alpha + \hat{\mathbf{y}}\,\sin\alpha)\times\hat{\mathbf{y}} \\ &= \hat{\mathbf{z}}\,\cos\alpha \quad \text{since } \hat{\mathbf{x}}\times\hat{\mathbf{y}} = \hat{\mathbf{z}} \end{aligned}$$

$$\begin{aligned} \hat{\mathbf{u}}\times\hat{\mathbf{z}} &= (\hat{\mathbf{x}}\,\cos\alpha + \hat{\mathbf{y}}\,\sin\alpha)\times\hat{\mathbf{z}} \\ &= -\hat{\mathbf{y}}\,\cos\alpha + \hat{\mathbf{x}}\,\sin\alpha \quad \text{since } \hat{\mathbf{x}}\times\hat{\mathbf{z}} = -\hat{\mathbf{y}} \text{ and } \hat{\mathbf{y}}\times\hat{\mathbf{z}} = \hat{\mathbf{x}} \end{aligned}$$

Therefore,

$$\begin{aligned} \mathbf{r}\times(d\boldsymbol{\ell}\times\mathbf{B}) &= Ba\,d\ell\big[-(-\hat{\mathbf{y}}\,\cos\alpha + \hat{\mathbf{x}}\,\sin\alpha)\sin\alpha\,\sin\varphi\,\cos\varphi \\ &\qquad - (\hat{\mathbf{z}}\,\cos\alpha)\sin^2\varphi + \hat{\mathbf{x}}\,\sin\varphi\,\cos\varphi\big] \\ &= Ba\,d\ell\big[\hat{\mathbf{x}}\,\left(-\sin^2\alpha\,\sin\varphi\,\cos\varphi + \sin\varphi\,\cos\varphi\right) \\ &\qquad + \hat{\mathbf{y}}\,(\sin\alpha\,\cos\alpha\,\sin\varphi\,\cos\varphi) - \hat{\mathbf{z}}\cos\alpha\,\sin^2\varphi\big] \end{aligned} \tag{165}$$

By (160), the torque on a section of wire with oriented line element $d\boldsymbol{\ell}$ is $\mathbf{r} \times (d\boldsymbol{\ell} \times \mathbf{B})$. Since the circle is of raidus a, we have $d\ell = a d\varphi$. This gives us

$$d\boldsymbol{\tau} = IBa^2\, d\varphi \big[\hat{\mathbf{x}}\, \big(-\sin^2\alpha\, \sin\varphi\, \cos\varphi + \sin\varphi\, \cos\varphi\big) + \hat{\mathbf{y}}\, (\sin\alpha\, \cos\alpha\, \sin\varphi\, \cos\varphi) - \hat{\mathbf{z}} \cos\alpha\, \sin^2\varphi\big] \tag{166}$$

where I is the current calculated in part (a). The net torque on one circular turn of wire is the integral of this differential torque over the circular loop:

$$\begin{aligned} \boldsymbol{\tau}_1 &= \int_{\varphi=0}^{\varphi=2\pi} d\boldsymbol{\tau} \\ &= IBa^2 \int_0^{2\pi} d\varphi\, \big[\hat{\mathbf{x}}\, \big(-\sin^2\alpha\, \sin\varphi\, \cos\varphi + \sin\varphi\, \cos\varphi\big) \\ &\qquad + \hat{\mathbf{y}}\, (\sin\alpha\, \cos\alpha\, \sin\varphi\, \cos\varphi) - \hat{\mathbf{z}} \cos\alpha\, \sin^2\varphi\big] \end{aligned} \tag{167}$$

The integral of $\sin\varphi\cos\varphi$ over a full circle ($0 \le \varphi \le 2\pi$) is zero, so the x- and y-components of $\boldsymbol{\tau}_1$ cancel out. All that remains is the z-component, which we can write:

$$\boldsymbol{\tau}_1 = -\hat{\mathbf{z}}\, IBa^2 \cos\alpha \int_0^{2\pi} d\varphi\, \sin^2\varphi \tag{168}$$

Using a power-reducing formula, we can take the integral:

$$\begin{aligned} \int_0^{2\pi} d\varphi\, \sin^2\varphi &= \int_0^{2\pi} d\varphi\, \frac{1-\cos(2\varphi)}{2} \\ &= \left[\frac{\varphi - \frac{1}{2}\cos(2\varphi)}{2}\right]_0^{2\pi} \\ &= \pi \end{aligned}$$

This gets us

$$\boldsymbol{\tau}_1 = -\hat{\mathbf{z}}\, IB\, \pi a^2\, \cos\alpha \tag{169}$$

This is the torque on *one* turn of wire. To get the torque on the entire loop, we must multiply this value by the number of turns N:

$$\boldsymbol{\tau} = N\boldsymbol{\tau}_1 = -\hat{\mathbf{z}}\, NIB\, \pi a^2\, \cos\alpha \tag{170}$$

The torque we need to *apply* to the current loop to keep it rotating is the opposite of the torque *on* the current loop:

$$\boldsymbol{\tau}_{\text{applied}} = -\boldsymbol{\tau} = \hat{\mathbf{z}}\, NIB\, \pi a^2\, \cos\alpha \tag{171}$$

Plugging in the value of I from part (a) (153), we get

$$\boxed{\boldsymbol{\tau}_{\text{applied}} = (NB\pi a^2)^2\, \frac{\omega}{R}\, \cos^2\alpha\, \hat{\mathbf{z}}} \tag{172}$$

(c) In this setup, there is only one length scale a, the radius of the circular loop. Therefore, the characteristics of the radiation field should be the same no matter how small the circular loop is. (Remember that the radiation field is measured infinitely far away from the source, so the location where we measure it does not provide a length scale.) Therefore, we can take the circular loop to be infinitesimally small, meaning that we can take it to be a magnetic dipole.

Here's another way to think about this: In a scattering problem, the dipole approximation is valid when $a \ll \lambda$, where a is the characteristic length of the charge/current distribution and λ is the wavelength of the incident electromagnetic wave. Here, the **B** field is static, and a static field is the limit of an electromagnetic wave with infinite wavelength. Therefore, a is automatically much less than λ, so we can take the current loop to be a magnetic dipole.

To determine the frequency of the radiation, recall the following:

The allowed frequencies for the radiation field for a dipole are the same as the frequencies of the time dependence of the dipole moment.

The magnetic dipole moment for this current loop was calculated in (157) to be

$$\mathbf{m} = NI\pi a^2 \left(-\hat{\mathbf{x}} \sin\alpha + \hat{\mathbf{y}} \cos\alpha\right) \quad \text{where } I \text{ is the current from part (a)}$$

Plugging in the value for the current I from part (a) (153), we get

$$\begin{aligned} \mathbf{m} &= \frac{N^2 B \left(\pi a^2\right)^2 \omega}{R} \cos\alpha \left(-\hat{\mathbf{x}} \sin\alpha + \hat{\mathbf{y}} \cos\alpha\right) \\ &= \frac{N^2 B \left(\pi a^2\right)^2 \omega}{R} \cos(\omega t) \left(-\hat{\mathbf{x}} \sin(\omega t) + \hat{\mathbf{y}} \cos(\omega t)\right) \quad \text{since } \alpha = \omega t \end{aligned} \tag{173}$$

We need to examine what the frequencies of this dipole moment are. Note that the frequency of a product of two sine functions is not the same as the frequency of a single sine function. Instead, we can use the double-angle formulas

$$\sin(2x) = 2\sin x \cos x \quad \text{and} \quad \cos(2x) = 2\cos^2 x - 1$$

to write

$$\begin{aligned} \mathbf{m} &= \frac{N^2 B \left(\pi a^2\right)^2 \omega}{R} \left(-\hat{\mathbf{x}} \sin(\omega t) \cos\omega t + \hat{\mathbf{y}} \cos^2(\omega t)\right) \\ &= \frac{N^2 B \left(\pi a^2\right)^2 \omega}{R} \left(-\hat{\mathbf{x}} \left(\frac{\sin(2\omega t)}{2}\right) + \hat{\mathbf{y}} \left(\frac{\cos(2\omega t) + 1}{2}\right)\right) \\ &= \frac{N^2 B \left(\pi a^2\right)^2 \omega}{2R} \left[-\hat{\mathbf{x}} \sin(2\omega t) + \hat{\mathbf{y}} \left(\cos(2\omega t) + 1\right)\right] \end{aligned} \tag{174}$$

The time-dependent part of the magnetic dipole moment has an angular frequency 2ω. (There is also a time-independent part of the magnetic dipole moment, but since the radiation field depends on the second time derivative of the magnetic dipole moment, this part does not contribute to the radiation field.) Therefore, the angular frequency of the radiation is also 2ω, and the frequency of the radiation is $(2\omega)/(2\pi) = \omega/\pi$.

(d) This part requires you to know the following:

For a radiation field observed at a point in the $\hat{\mathbf{r}}$ direction from the origin, the radiated magnetic field is perpendicular to both $\hat{\mathbf{r}}$ and the "radiation vector" $\boldsymbol{\alpha}$. For dipole radiation, $\boldsymbol{\alpha}$ is equal to $\ddot{\mathbf{p}}_{\text{ret}}$ (electric dipole radiation) or $\ddot{\mathbf{m}}_{\text{ret}}/c$ (magnetic dipole radiation). The subscript "ret" indicates that the dipole moment is evaluated at the retarded time. The radiated electric field is perpendicular to both $\hat{\mathbf{r}}$ and the radiated magnetic field. Just like a plane wave, we have $\mathbf{E}_{\text{rad}} \times \mathbf{B}_{\text{rad}} \propto \hat{\mathbf{r}}$.

From (174), we can write the second time derivative of the magnetic dipole moment as

$$\ddot{\mathbf{m}}_{\text{ret}} = \frac{2N^2 B \left(\pi a^2\right)^2 \omega^3}{R} \left[\hat{\mathbf{x}} \sin(2\omega t) - \hat{\mathbf{y}} \cos(2\omega t)\right]_{\text{ret}} \tag{175}$$

The electromagnetic waves propagating along the z-axis must be perpendicular to $\hat{\mathbf{z}}$. The radiated magnetic field is also perpendicular to the direction of $\ddot{\mathbf{m}}_{\text{ret}}$, which is

$$\ddot{\mathbf{m}}_{\text{ret}} \propto \left[\hat{\mathbf{x}}\,\sin(2\omega t) - \hat{\mathbf{y}}\,\cos(2\omega t)\right]_{\text{ret}}$$

Therefore, the radiated magnetic field must be in the direction perpendicular to $\hat{\mathbf{z}}$ and $\ddot{\mathbf{m}}_{\text{ret}}$

$$\mathbf{B}_{\text{rad}} \propto \hat{\mathbf{z}} \times \left[\hat{\mathbf{x}}\,\sin(2\omega t) - \hat{\mathbf{y}}\,\cos(2\omega t)\right]_{\text{ret}} = \left[\hat{\mathbf{x}}\,\cos(2\omega t) + \hat{\mathbf{y}}\,\sin(2\omega t)\right]_{\text{ret}} \tag{176}$$

This is the definition of a circularly polarized electromagnetic field: The direction of $\mathbf{B}$ spins around in a circle over time. Therefore, the radiated electromagnetic waves propagating along the positive z-axis are circularly polarized, with polarization vector (176).

(e) For this part of the problem, since this was an open-book exam, you could look up the formula for the power radiated by a magnetic dipole

$$P = \frac{\mu_0}{4\pi c^3}\,\frac{2}{3}\,\left|\ddot{\mathbf{m}}_{\text{ret}}\right|^2 \tag{177}$$

and plug in the second time derivative of the magnetic dipole moment (175) to get

$$P = \frac{\mu_0}{4\pi c^3}\,\frac{2}{3}\,\frac{4N^4B^2\left(\pi a^2\right)^4\omega^6}{R^2}\left[\hat{\mathbf{x}}\,\sin(2\omega t) - \hat{\mathbf{y}}\,\cos(2\omega t)\right]^2_{\text{ret}}$$

$$\boxed{P = \frac{2\mu_0\,N^4B^2\left(\pi a^2\right)^4\omega^6}{3\pi c^3R^2}} \tag{178}$$

What if this exam was not open-book? Well, the one formula to memorize is the formula for the power radiated per solid angle from an *electric* dipole:

$$\frac{dP}{d\Omega} = \frac{\mu_0}{(4\pi)^2 c}\,\left|\hat{\mathbf{r}} \times \ddot{\mathbf{p}}_{\text{ret}}\right|^2 \tag{179}$$

The next thing to memorize is the following: To change electric dipole radiation equations to be valid for magnetic dipole radiation, replace $\mathbf{p}$ with $\mathbf{m}/c$. Therefore, without memorizing an additional equation, we can find that for a *magnetic* dipole,

$$\frac{dP}{d\Omega} = \frac{\mu_0}{(4\pi)^2 c^3}\,\left|\hat{\mathbf{r}} \times \ddot{\mathbf{m}}_{\text{ret}}\right|^2 \tag{180}$$

To get from this to the equation for power, we need to integrate over $d\Omega$. Setting $\ddot{\mathbf{m}}_{\text{ret}}$ at a given time to be the in the $\hat{\mathbf{z}}$ direction and using spherical coordinates, we get that

$$\left|\hat{\mathbf{r}} \times \ddot{\mathbf{m}}_{\text{ret}}\right|^2 = \left|\ddot{\mathbf{m}}_{\text{ret}}\right|^2 \sin^2\theta$$

since θ is the angle between $\hat{\mathbf{z}}$ and $\hat{\mathbf{r}}$. Integrating over the solid angle, we get

$$\begin{aligned} P &= \int \frac{dP}{d\Omega}\,d\Omega \\ &= \frac{\mu_0}{(4\pi)^2c^3}\left|\ddot{\mathbf{m}}_{\text{ret}}\right|^2 \int \sin^2\theta\,d\Omega \\ &= \frac{\mu_0}{(4\pi)^2c^3}\left|\ddot{\mathbf{m}}_{\text{ret}}\right|^2 \int_0^{2\pi} d\varphi \int_{-1}^{1} d\left(\cos\theta\right)\,\sin^2\theta \end{aligned}$$

Taking the integral, we get

$$\begin{aligned}
\int_0^{2\pi} d\varphi \int_{-1}^{1} d\left(\cos\theta\right)\,\sin^2\theta &= 2\pi \int_{-1}^{1} d\left(\cos\theta\right)\,\sin^2\theta \\
&= 2\pi \int_{-1}^{1} d\left(\cos\theta\right)\,\big(1-\cos^2\theta\big) \\
&= 2\pi \left[u - \frac{u^3}{3}\right]_{u=-1}^{u=1} \\
&= \frac{8\pi}{3}
\end{aligned}$$

Therefore,

$$\begin{aligned}
`P &= \frac{\mu_0}{(4\pi)^2 c^3}\left|\ddot{\mathbf{m}}_{\mathrm{ret}}\right|^2 \left(\frac{8\pi}{3}\right) \\
&= \frac{\mu_0}{4\pi c^3}\,\frac{2}{3}\left|\ddot{\mathbf{m}}_{\mathrm{ret}}\right|^2
\end{aligned}$$

which is the formula mentioned earlier.