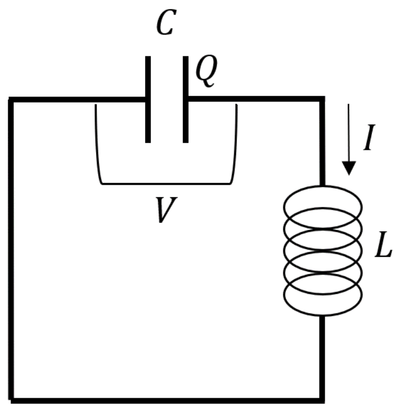
11. (Statistical Mechanics)

Consider a closed LC circuit. It is to be used as a thermometer, by measuring the rms voltage across the capacitance (and inductance). Find an expression for the temperature dependence of the rms voltage, valid for all temperatures. Then find the limits for high and low temperature.

Hint: at low enough temperatures quantum effects may be important.

Solution: *Solution by Jonah Hyman (jthyman@g.ucla.edu)*



To get started on this problem, you need to realize that an LC circuit is a harmonic oscillator. There are three ways to do this, and all are important for this problem:

Differential equation (from first principles):
To find the differential equation, we should use Faraday's law for a clockwise loop around the circuit. The electromotive force (emf) around the circuit is defined by

$$\varepsilon \equiv \int_{\text{loop}} \mathbf{E} \cdot d\boldsymbol{\ell} \tag{90}$$

We can relate the emf to the inductance of the circuit L by using Faraday's law:

$$\begin{aligned}
\varepsilon &\equiv \int_{\text{loop}} \mathbf{E} \cdot d\boldsymbol{\ell} \\
&= \int_{\text{loop interior}} (\nabla \times \mathbf{E}) \cdot d\mathbf{a} \quad \text{by Stokes' theorem} \\
&= -\int_{\text{loop interior}} \frac{d\mathbf{B}}{dt} \cdot d\mathbf{a} \quad \text{by Faraday's law} \\
&= -\frac{d\Phi_{\mathbf{B}}}{dt} \quad \text{where } \Phi_{\mathbf{B}} \equiv \int \mathbf{B} \cdot d\mathbf{a} \text{ is the magnetic flux through the circuit} \\
&= -\frac{d}{dt}(LI) \quad \text{by definition of inductance } \Phi_{\mathbf{B}} = LI \\
&= -L\frac{dI}{dt}
\end{aligned} \tag{91}$$

The emf through the circuit is also related to the capacitance of the circuit via the definition of capacitance:

$$\varepsilon = -\frac{Q}{C} \tag{92}$$

(The sign of the right-hand side is negative since the electric field in the the capacitor points counterclockwise if Q is positive. Therefore, $\int \mathbf{E} \cdot d\boldsymbol{\ell}$ is negative for a clockwise loop element $d\boldsymbol{\ell}$.)

Setting (91) equal to (92), we get

$$\begin{aligned} -L\frac{dI}{dt} &= -\frac{Q}{C} \\ -\frac{dI}{dt} + \frac{Q}{LC} &= 0 \end{aligned} \tag{93}$$

Using the sign convention from the diagram, if I is positive, $\frac{dQ}{dt}$ is negative (since charge flows away from the right plate of the capacitor). Therefore, the continuity equation for electric charge in this setup is

$$I = -\frac{dQ}{dt} \tag{94}$$

Plugging this into (93), we get that

$$\frac{d^2Q}{dt^2} + \frac{Q}{LC} = 0 \tag{95}$$

This is the differential equation for a simple harmonic oscillator with natural frequency $\omega \equiv \frac{1}{\sqrt{LC}}$.

Differential equation (from Kirchoff's loop rule):
When finding the differential equation for a circuit, one way to bypass some of the reasoning above is to use Kirchoff's loop rule. Using the rules on the next page and taking a clockwise loop around the circuit in this problem, we can write the differential equation directly:

$$0 = \underbrace{+\frac{Q}{C}}_{\text{capacitor}} \quad \underbrace{-L\frac{dI}{dt}}_{\text{inductor}} \tag{96}$$

Using the sign convention from the diagram, if I is positive, $\frac{dQ}{dt}$ is negative (since charge flows away from the right plate of the capacitor). Therefore, the continuity equation for electric charge in this setup is

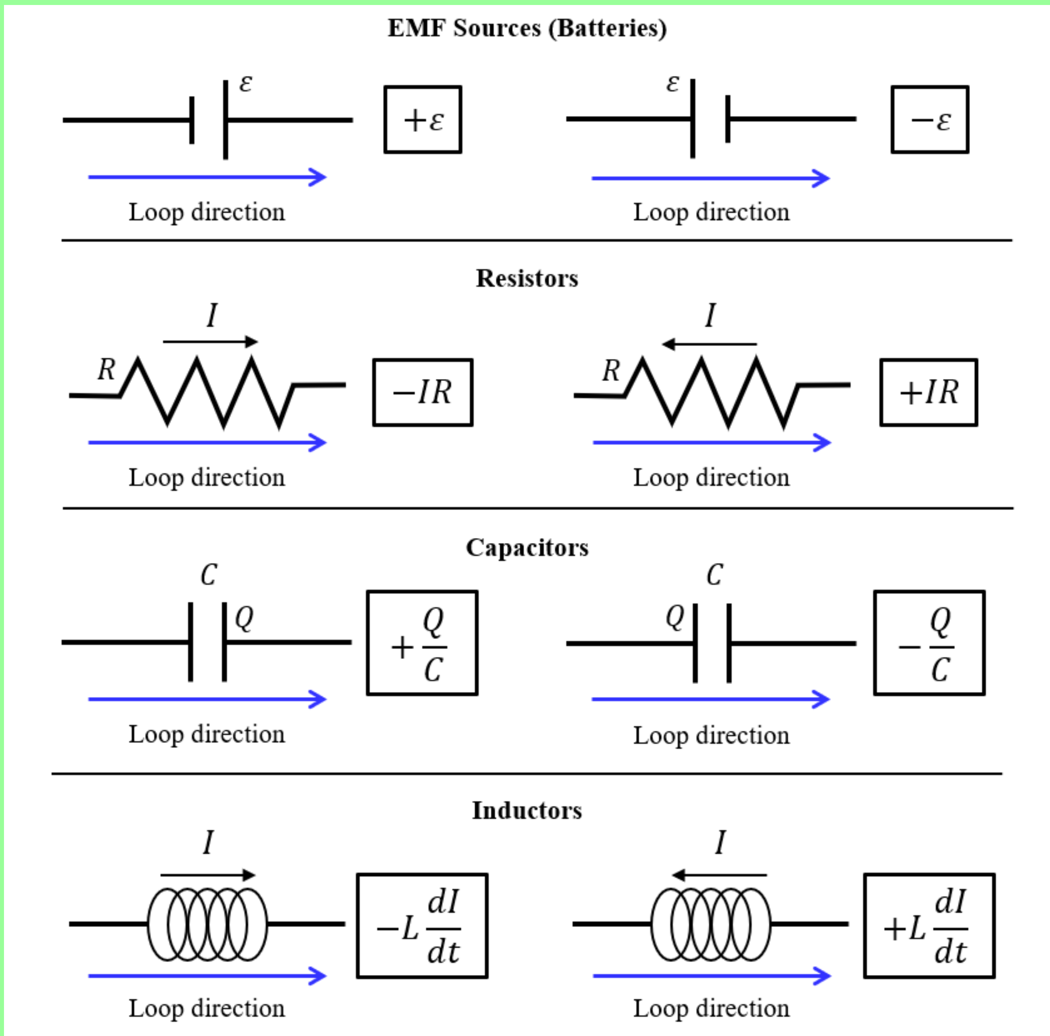
$$I = -\frac{dQ}{dt} \tag{97}$$

Plugging this into (96), we get that

$$\begin{aligned} 0 &= +\frac{Q}{c} + L\frac{d^2Q}{dt^2} \\ \implies 0 &= \frac{d^2Q}{dt^2} + \frac{Q}{LC} \end{aligned} \tag{98}$$

This is the differential equation for a simple harmonic oscillator with natural frequency $\omega \equiv \frac{1}{\sqrt{LC}}$.

Kirchoff's loop rule:
For any closed, oriented loop in a circuit, the emf change over that loop must be zero.
The emf change a loop is the sum of the emf change for all the circuit elements in that loop.
The rules for em changes for given circuit elements are shown in the diagrams below (note the importance of the placement of charges, currents, and the direction of the loop in determining the signs).



Hamiltonian:
A different way to see that an LC circuit is a harmonic oscillator is to write its Hamiltonian. The energy stored in a capacitor with charge Q and capacitance C is

$$U_C = \frac{Q^2}{2C} \tag{99}$$

The energy stored in an inductor with current I and inductance L is

$$U_L = \frac{1}{2}LI^2 \tag{100}$$

Adding the two energies together, we get the Hamiltonian for this circuit in terms of the charge of the capacitor Q, the current through the inductor I, the capacitance of the circuit C, and the inductance of the circuit L:

$$H = \frac{Q^2}{2C} + \frac{1}{2}LI^2 \tag{101}$$

Using the sign convention from the diagram, if I is positive, $\frac{dQ}{dt}$ is negative (since charge flows away from the right plate of the capacitor). Therefore, the continuity equation for electric charge in this setup is

$$I = -\frac{dQ}{dt} \tag{102}$$

Plugging this into (101), we get the Hamiltonian

$$H = \frac{Q^2}{2C} + \frac{1}{2}L\left(\frac{dQ}{dt}\right)^2 \tag{103}$$

We can rearrange this Hamiltonian and relabel some of the variables to make it look more like the Hamiltonian for a harmonic oscillator:

$$\begin{aligned} H &= \frac{1}{2}L\left(\frac{dQ}{dt}\right)^2 + \frac{1}{2}L\left(\frac{1}{LC}\right)Q^2 \\ H &= \frac{1}{2}\overline{m}\left(\frac{dQ}{dt}\right)^2 + \frac{1}{2}\overline{m}\omega^2 Q^2 \quad \text{for} \quad \overline{m} \equiv L \quad \text{and} \quad \omega \equiv \frac{1}{\sqrt{LC}} \end{aligned} \tag{104}$$

This is the Hamiltonian for a harmonic oscillator in the variable Q with natural frequency $\omega \equiv \frac{1}{\sqrt{LC}}$.

By any of the three methods, we now know that an LC circuit is classically a harmonic oscillator with natural frequency of $\omega \equiv \frac{1}{\sqrt{LC}}$. Therefore, the LC circuit is quantized just like a harmonic oscillator. The energy levels for a quantum harmonic oscillator are

$$E_n = \hbar\omega\left(n + \frac{1}{2}\right) \quad \text{for} \quad n = 0, 1, 2, 3, \ldots \tag{105}$$

To calculate any thermodynamic quantity at equilibrium in the canonical equilibrium, we should calculate the partition function. In this case, the microstates are identified by the label n, so the partition function is

$$\begin{aligned} Z &= \sum_{n=0}^{\infty} e^{-\beta E_n} \quad \text{for} \quad \beta \equiv \frac{1}{kT} \\ &= \sum_{n=0}^{\infty} e^{-\beta\hbar\omega\left(n+\frac{1}{2}\right)} \quad \text{by (105)} \\ &= e^{-\beta\hbar\omega/2}\sum_{n=0}^{\infty}\left(e^{-\beta\hbar\omega}\right)^n \\ Z &= e^{-\beta\hbar\omega/2}\left(\frac{1}{1-e^{-\beta\hbar\omega}}\right) \quad \text{using the geometric series} \sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \text{ for } |x| < 1 \end{aligned} \tag{106}$$

We can further simplify this into a hyperbolic sine function:

$$\begin{aligned} Z &= e^{-\beta\hbar\omega/2}\left(\frac{1}{1-e^{-\beta\hbar\omega}}\right) \\ &= \frac{1}{e^{\beta\hbar\omega/2} - e^{-\beta\hbar\omega/2}} \\ Z &= \frac{1}{2\sinh\left(\beta\hbar\omega/2\right)} \quad \text{using the definition } \sinh x = \frac{e^x - e^{-x}}{2} \end{aligned} \tag{107}$$

We can now extract the energy of the harmonic oscillator using the formula

$$\langle E\rangle = -\frac{\partial(\ln Z)}{\partial\beta} = -\frac{1}{Z}\frac{\partial Z}{\partial\beta} \tag{108}$$

Using the fact that $(\sinh x)' = \cosh x$, we get in this case

$$\begin{aligned}\langle E\rangle &= -2\sinh(\beta\hbar\omega/2)\frac{\partial}{\partial\beta}\left(\frac{1}{2\sinh\left(\beta\hbar\omega/2\right)}\right)\\ &= -2\sinh(\beta\hbar\omega/2)\left(-\frac{1}{2\sinh^2(\beta\hbar\omega/2)}\cosh(\beta\hbar\omega/2)\cdot\frac{\hbar\omega}{2}\right)\\ \langle E\rangle &= \frac{\hbar\omega}{2}\coth(\beta\hbar\omega/2) \quad \text{for} \quad \beta\equiv\frac{1}{kT} \end{aligned}\tag{109}$$

From this information, we need to extract the root mean square (rms) voltage across the capacitor, which is the square root of the expectation value of the voltage squared:

$$V_{\text{rms}} \equiv \sqrt{\langle V^2\rangle} \tag{110}$$

To find $\langle V^2\rangle$, recall by the definition of capacitance that $V = \frac{Q}{C}$, so

$$\left\langle V^2\right\rangle = \frac{\left\langle Q^2\right\rangle}{C^2} \quad \text{since } C \text{ is a constant} \tag{111}$$

To find $\langle Q^2\rangle$, note that since H contains a factor of Q^2, we can just take the expectation value of H to get the expectation value of Q^2:

$$\langle E\rangle = \langle H\rangle = \frac{\left\langle Q^2\right\rangle}{2C} + \frac{1}{2}L\left\langle\frac{dQ}{dt}\right\rangle^2 \tag{112}$$

If we work in Fourier components of Q, we are allowed to take $\frac{dQ}{dt} = \omega Q = \frac{1}{\sqrt{LC}}Q$. Plugging this in, we get a direct relation between $\langle Q^2\rangle$ and $\langle E\rangle$:

$$\begin{aligned}\langle E\rangle &= \frac{\left\langle Q^2\right\rangle}{2C} + \frac{1}{2}L\omega^2\langle Q\rangle^2\\ &= \frac{\left\langle Q^2\right\rangle}{2C} + \frac{1}{2C}\langle Q\rangle^2 \quad \text{since } \omega = \frac{1}{\sqrt{LC}}\\ \langle E\rangle &= \frac{\left\langle Q^2\right\rangle}{C}\end{aligned} \tag{113}$$

Using our expression for the expectation value of the energy (109), we can find $\langle Q^2\rangle$:

$$\begin{aligned}\left\langle Q^2\right\rangle &= C\langle E\rangle\\ \left\langle Q^2\right\rangle &= C\frac{\hbar\omega}{2}\coth(\beta\hbar\omega/2) \quad \text{for} \quad \omega\equiv\frac{1}{\sqrt{LC}} \quad \text{and} \quad \beta\equiv\frac{1}{kT}\end{aligned} \tag{114}$$

For an alternate, very instructive derivation of $\langle Q^2\rangle$, see Richard Meyers' notes for this question at `https://physwiki.com/w/images/a/ac/2017.pdf` (page 28). Using (111) to relate this to $\langle Q^2\rangle$ to $\langle V^2\rangle$, we get

$$\begin{aligned}\left\langle V^2\right\rangle &= \frac{\left\langle Q^2\right\rangle}{C^2}\\ &= \frac{\hbar\omega}{2C}\coth(\beta\hbar\omega/2) \quad \text{for} \quad \omega\equiv\frac{1}{\sqrt{LC}} \quad \text{and} \quad \beta\equiv\frac{1}{kT}\\ \left\langle V^2\right\rangle &= \frac{\hbar}{2C\sqrt{LC}}\coth\left(\frac{\hbar}{2kT\sqrt{LC}}\right)\end{aligned} \tag{115}$$

To find the rms voltage, take the square root:

$$\boxed{V_{\text{rms}} = \left[\frac{\hbar}{2C\sqrt{LC}}\coth\left(\frac{\hbar}{2kT\sqrt{LC}}\right)\right]^{1/2}} \tag{116}$$

This expression is valid for all temperatures. The problem also wants us to find the limits for high and low temperature. To do this, we need to exam the limiting behavior of $\coth x$ for small and large x. By the definitions of $\sinh x \equiv \frac{e^x - e^{-x}}{2}$ and $\cosh x \equiv \frac{e^x + e^{-x}}{2}$, we can write

$$\coth x = \frac{\sinh x}{\cosh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}} \tag{117}$$

For small x, we can take a Taylor series for the exponentials:

$$\begin{aligned}
\coth x &= \frac{\left(1 + x + \frac{x^2}{2} + \mathcal{O}(x^3)\right) + \left(1 - x + \frac{x^2}{2} + \ldots \mathcal{O}(x^3)\right)}{\left(1 + x + \frac{x^2}{2} + \mathcal{O}(x^3)\right) - \left(1 - x + \frac{x^2}{2} + \ldots \mathcal{O}(x^3)\right)} \\
&= \frac{2 + x^2 + \mathcal{O}(x^3)}{2x + \mathcal{O}(x^3)} \\
\coth x &= \frac{1}{x} + \mathcal{O}(x) \quad \text{for small } x
\end{aligned} \tag{118}$$

For large x, $e^x \gg e^{-x}$, so we can write

$$\coth x \approx \frac{e^x}{e^x} = 1 \quad \text{for large } x \tag{119}$$

For $x \equiv \frac{\hbar}{2kT\sqrt{LC}}$, high temperature corresponds to low x. Therefore, for high temperature, using (118), we get

$$\begin{aligned}
V_{\text{rms}} &= \left[\frac{\hbar}{2C\sqrt{LC}}\coth\left(\frac{\hbar}{2kT\sqrt{LC}}\right)\right]^{1/2} \\
&\approx \left[\frac{\hbar}{2C\sqrt{LC}}\frac{2kT\sqrt{LC}}{\hbar}\right]^{1/2}
\end{aligned}$$

$$\boxed{V_{\text{rms}} \approx \left[\frac{kT}{C}\right]^{1/2} \qquad \text{for high temperature}} \tag{120}$$

For $x \equiv \frac{\hbar}{2kT\sqrt{LC}}$, low temperature corresponds to low x. Therefore, for low temperature, using (119), we get

$$V_{\text{rms}} = \left[\frac{\hbar}{2C\sqrt{LC}}\coth\left(\frac{\hbar}{2kT\sqrt{LC}}\right)\right]^{1/2}$$

$$\boxed{V_{\text{rms}} \approx \left[\frac{\hbar}{2C\sqrt{LC}}\right]^{1/2} \qquad \text{for low temperature}} \tag{121}$$

Note that for low temperature, the rms voltage is temperature-independent.