Disc 210A, Q1:

a) $\frac{1}{2} m v^2(x) = q\phi(x)$

$\Rightarrow v(x) = \sqrt{\frac{2q\phi(x)}{m}}$

b) $I = \rho(x) A v(x) = \text{const.}$, where $\rho(x)$ = charge density

$\Rightarrow \rho(x) = \frac{I}{Av(x)} = \frac{I}{A\sqrt{\frac{2q}{m}}} \phi^{-\frac{1}{2}}(x)$

c) Need one more equation$(\rho(x), \phi(x))$, so take $-\partial_x^2 \phi(x) = \rho/\varepsilon_0$.

$$\partial_x^2 \phi(x) = \underbrace{\frac{-I}{A\sqrt{\frac{2q}{m}}\,\varepsilon_0}}_{-k = Ik_0} \phi^{-1/2}(x)$$

d) $\exists$ two methods:

(I) Guess:

$\phi(x) = c x^n$ (Power Law)

Find n by "shaking" $\phi(x)$ with double derivative until x-dependence falls out:

$$\partial_x^2 \phi(x) = c\, n(n-1)\, x^{n-2} = n(n-1)\frac{\phi(x)}{x^2} = -k\, \phi^{-1/2}(x) \Rightarrow \phi^{3/2}(x) = -\frac{k}{n(n-1)} x^2$$

$\Rightarrow \phi(x) \propto x^{4/3}$, $n = 4/3$

Find c by second B.C. (first B.C. already satisfied by guess, $\phi(0) = 0$):

$\phi(d) = \phi_0$, so $c = \frac{\phi_0}{d^{4/3}}$

$$\Rightarrow \boxed{\phi(x) = \phi_0 \left(\frac{x}{d}\right)^{4/3}}$$

(II) Use Identity:

$$\underbrace{\partial_x\phi\, \partial_x^2\phi = \frac{1}{2}\partial_x(\partial_x\phi)^2 = -k\,\phi^{-1/2}\,\partial_x\phi}_{\text{Integrate } \int dx}$$

$$\Rightarrow \quad \frac{1}{2}\int d(\partial_x\phi)^2 = \frac{1}{2}(\partial_x\phi)^2 = -2k\,\phi^{1/2}$$

Take sqrt and integrate after rearranging

$$\Rightarrow \quad \partial_x\phi = 2\sqrt{-k}\,\phi^{1/4}$$

$$\Rightarrow \quad \int \phi^{-1/4}\,d\phi = 2\sqrt{-k}\int dx$$

$$\Rightarrow \quad \frac{4}{3}\phi^{3/4} = 2\sqrt{-k}\,x$$

$$\Rightarrow \quad \phi(x)^{3/2} = \frac{-k}{4/9}x^2$$ (squared to check with Method (I))

$$\Rightarrow \quad \phi(x) \propto x^{4/3}, \quad n = 4/3$$

2^{nd} B.C. $\phi(d) = \phi_0$ gives $\boxed{\phi(x) = \phi_0\left(\frac{x}{d}\right)^{4/3}}$

e) Show $I = k_2 V_0^{3/2}$, find k_2.

$\Rightarrow$ plug in $\phi(x)$ to result from c) to give

$$\phi(x)^{3/2} = -\frac{k_0 I}{4/9}x^2 = \phi_0^{3/2}\frac{x^2}{d^2}$$

$$\Rightarrow \quad \boxed{I = \underbrace{-\frac{9A\sqrt{\frac{2q}{m}}\,\varepsilon_0}{4d^2}}_{k_2}\,\phi_0^{3/2}}$$