**7. (Electromagnetism)**

A linearly polarized plane wave of wavenumber $k$ and frequency $\omega$ is propagating in the $\hat{z}$ direction. The electric field is in the $\hat{y}$ direction. The wave is scattered by two small dielectric spheres of radius $a$ separated by a distance $b$ with $b \gg a$. The first sphere is centered on the origin while the second sphere is located on the $z$-axis with $z = b$. The two spheres have dielectric constant $\epsilon/\epsilon_0 = 1 + \chi$ with $\chi \ll 1$.

(a) Consider the case $kb \ll 1$. Determine the differential and total cross section for the two spheres to leading order in $kb$.

(b) Consider the case $kb \sim 1$, but still $ka \ll 1$. Determine the differential cross section of the two spheres for the light scattered at an angle $\theta$ in the $xz$-plane, measured with respect to the $z$-axis.

(c) Can you find values of $kb$ for which the total radiated power from part (a) and (b) are the same?

**Solution:** *Solution by Jonah Hyman (jthyman@g.ucla.edu)*

This problem lies at the intersection of two approximations. Since each of the spheres is small compared to the wavelength of the incident plane wave ($ka \ll 1$), we can approximate each of the spheres as an ideal dipole (Rayleigh scattering). Since the dielectric response of each sphere is small ($\chi \ll 1$), we can use the Born approximation. There are two approaches, depending on which approximation we apply first.

Both approaches yield the same answer. Starting with the dipole approximation is probably more familiar to you, and you are more likely to know the relevant formulas (especially on a closed-book exam). On the other hand, this method is more open-ended and requires more tricks to implement the approximation; it demands a keen intuition of the physical setup. Staring with the Born approximation reduces the problem to algebra and integration. However, you are less likely to have the Born approximation formula memorized. We present both approaches below.

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***Dipole approximation first:***

In both parts (a) and (b), each of the spheres is small compared to the wavelength of the incident wave, $ka \ll 1$. Therefore, we may approximate each of the spheres by an ideal electric dipole, and we may approximate the incident electromagnetic field as constant over the area of each sphere.

Before discussing the scattering problem in (a) and (b), we must answer the question: What is the electric dipole moment of a dielectric sphere in a constant electric field? This is a classic potential theory problem. But wait—before you start writing general solutions to Laplace's equation, remember that we are allowed to assume a weak dielectric response ($\chi \ll 1$). For that reason, the ordinarily unhelpful equation for the polarization density of a linear dielectric

$$\mathbf{P} = \epsilon_0 \chi \mathbf{E} \quad \text{where } \mathbf{E} \text{ is the } \textit{total} \text{ electric field} \tag{235}$$

is replaced by the much more useful equation

$$\mathbf{P} \approx \epsilon_0 \chi \mathbf{E}_0 \quad \text{where } \mathbf{E}_0 \text{ is the } \textit{background} \text{ electric field, assuming } \chi \ll 1 \tag{236}$$

since the dielectric response is weak. The polarization density is the dipole moment per unit volume, so we can find an approximate value for the dipole moment

$$\mathbf{p} = \frac{4}{3}\pi a^3 \mathbf{P} \approx \frac{4}{3}\pi \epsilon_0 a^3 \chi \mathbf{E}_0 \tag{237}$$

(The exact value, calculated using potential theory, is $\mathbf{p} = 4\pi\epsilon_0 a^3 \frac{\chi}{3+\chi}\mathbf{E}_0$. See 2018 Q9. This reduces to the approximate value (237) in the limit $\chi \ll 1$.)

(a) In the limit $kb \ll 1$ (which implies, since $b \gg a$, that $ka \ll 1$ as well), both dielectric spheres are approximately at the origin. Therefore, if we set $t = 0$ at an appropriate point, we can approximate the background electric field at the location of both spheres as

$$\tilde{\mathbf{E}}_0(z = 0) \approx \tilde{\mathbf{E}}_0(z = b) \approx E_0 e^{-i\omega t}\hat{\mathbf{y}} \tag{238}$$

where $E_0$ is the magnitude of the incident electric field. We have added the tilde to emphasize that the electric field is actually the real part of this complex expression. Therefore, by (237), the dipole moment of each sphere is equal to the real part of

$$\tilde{\mathbf{p}} = \frac{4}{3}\pi\epsilon_0 a^3 \chi E_0 e^{-i\omega t}\hat{\mathbf{y}} \tag{239}$$

Now, the formula for the power radiated per solid angle for a single electric dipole is

$$\frac{dP}{d\Omega} = \frac{\mu_0}{(4\pi)^2 c}\left|\hat{\mathbf{r}} \times \ddot{\mathbf{p}}_{\text{ret}}\right|^2 \tag{240}$$

where the subscript "ret" means that we evaluate $\ddot{\mathbf{p}}$ at the retarded time (we will time-average before calculating the differential cross-section, so this is not too important here). You need to memorize this formula (or write it on your formula sheet); its derivation is just too long.

In the approximation in part (a), we have two electric dipoles of the same strength near the origin, so we can simply double the dipole moment in this formula:

$$\frac{dP}{d\Omega} = \frac{\mu_0}{(4\pi)^2 c} \left|\hat{\mathbf{r}} \times 2\ddot{\mathbf{p}}_{\text{ret}}\right|^2 = \frac{4\mu_0}{(4\pi)^2 c} \left|\hat{\mathbf{r}} \times \ddot{\mathbf{p}}_{\text{ret}}\right|^2 \tag{241}$$

Now, we can calculate the cross product in this formula. From (239), we get

$$\ddot{\tilde{\mathbf{p}}}_{\text{ret}} = -\frac{4}{3}\pi\epsilon_0 a^3 \omega^2 \chi E_0 e^{-i\omega t_{\text{ret}}} \hat{\mathbf{y}} \tag{242}$$

and therefore, taking the real part, we get

$$\ddot{\mathbf{p}}_{\text{ret}} = -\frac{4}{3}\pi\epsilon_0 a^3 \omega^2 \chi E_0 \cos\left(\omega t_{\text{ret}}\right) \hat{\mathbf{y}} \tag{243}$$

Note that if we forget to take the real part here, we are likely to lose a factor of 2 later on when we time-average, since $|\operatorname{Re}(A)^2| \neq \operatorname{Re}(|A|^2)$.

Recall that

$$\begin{aligned}
\hat{\mathbf{r}} \times \hat{\mathbf{y}} &= (\sin\theta\cos\varphi\,\hat{\mathbf{x}} + \sin\theta\sin\varphi\,\hat{\mathbf{y}} + \cos\theta\,\hat{\mathbf{z}}) \times \hat{\mathbf{y}} \\
&= \sin\theta\cos\varphi\,\hat{\mathbf{z}} - \cos\theta\,\hat{\mathbf{x}} \quad \text{since } \hat{\mathbf{x}} \times \hat{\mathbf{y}} = \hat{\mathbf{z}} \text{ and } \hat{\mathbf{z}} \times \hat{\mathbf{y}} = -\hat{\mathbf{x}}
\end{aligned} \tag{244}$$

so

$$\left|\hat{\mathbf{r}} \times \hat{\mathbf{y}}\right|^2 = \sin^2\theta\cos^2\varphi + \cos^2\theta \tag{245}$$

and therefore

$$\frac{dP}{d\Omega} = \frac{4\mu_0}{(4\pi)^2 c}\left(\frac{4}{3}\pi\epsilon_0 a^3 \omega^2 \chi E_0\right)^2 \cos^2\left(\omega t_{\text{ret}}\right)\left(\sin^2\theta\cos^2\varphi + \cos^2\theta\right) \tag{246}$$

Time-averaging, we use the fact that $\left\langle\cos^2(\omega t_{\text{ret}})\right\rangle = 1/2$ to get

$$\left\langle\frac{dP}{d\Omega}\right\rangle = \frac{4\mu_0}{2(4\pi)^2 c}\left(\frac{4}{3}\pi\epsilon_0 a^3 \omega^2 \chi E_0\right)^2 \left(\sin^2\theta\cos^2\varphi + \cos^2\theta\right) \tag{247}$$

To get from this to the differential cross-section, we need to divide by the intensity of the incident plane wave, which is

$$I_{\text{inc}} = |\langle\mathbf{S}_{\text{inc}}\rangle| = \frac{1}{2}\epsilon_0 c E_0^2 \tag{248}$$

This equation should probably be memorized (or written on your formula sheet), although it can be quickly derived. Getting to the differential cross-section from the radiated power per solid angle is a matter of dividing by this intensity:

$$\frac{d\sigma}{d\Omega} = \frac{\langle dP/d\Omega\rangle}{I_{\text{inc}}} = \underbrace{\frac{\frac{4\mu_0}{2(4\pi)^2 c}\left(\frac{4}{3}\pi\epsilon_0 a^3\omega^2\chi E_0\right)^2}{\frac{1}{2}\epsilon_0 c E_0^2}}_{\text{prefactor}} \left(\sin^2\theta\cos^2\varphi + \cos^2\theta\right) \tag{249}$$

We can simplify the big prefactor by using the identities $c^2 = \frac{1}{\mu_0\epsilon_0}$ and $\omega = ck$. After some simplification, this gives us

$$\text{Prefactor } = \frac{4}{9}a^6 k^4 \chi^2 \quad \text{where } k \equiv \frac{\omega}{c}$$

Note that the prefactor has units of area, as a cross-section should. Putting this prefactor back in, we recover the differential cross-section:

$$\boxed{\frac{d\sigma}{d\Omega} = \frac{4}{9}a^6k^4\chi^2\left(\sin^2\theta\cos^2\varphi + \cos^2\theta\right) \quad \text{where } k \equiv \frac{\omega}{c}\text{, assuming } kb \ll 1} \tag{250}$$

The total cross-section is defined by

$$\sigma \equiv \int \frac{d\sigma}{d\Omega}\,d\Omega = \int_{-1}^{1} d(\cos\theta)\int_0^{2\pi} d\varphi\,\frac{d\sigma}{d\Omega} \tag{251}$$

Bearing in mind that $\int_0^{2\pi} d\varphi\cos^2\varphi = \int_0^{2\pi} d\varphi\,\frac{1}{2}(1+\cos\varphi) = \pi$, we can write

$$\begin{aligned}
\int d\Omega\,\left(\sin^2\theta\cos^2\varphi + \cos^2\theta\right) &= \int_{-1}^{1} d(\cos\theta)\int_0^{2\pi} d\varphi\left(\sin^2\theta\cos^2\varphi + \cos^2\theta\right)\\
&= \int_{-1}^{1} d(\cos\theta)\,\left(\pi\sin^2\theta + 2\pi\cos^2\theta\right)\\
&= \int_{-1}^{1} d(\cos\theta)\,\left(\pi\left(1-\cos^2\theta\right) + 2\pi\cos^2\theta\right)\\
&= \int_{-1}^{1} d(\cos\theta)\,\left(\pi + \pi\cos^2\theta\right)\\
&= \left[\pi\cos\theta + \frac{\pi}{3}\cos^3\theta\right]_{\cos\theta=-1}^{\cos\theta=1}\\
&= \frac{8\pi}{3}
\end{aligned} \tag{252}$$

Therefore,

$$\begin{aligned}
\sigma &= \frac{4}{9}a^6k^4\chi^2\int_{-1}^{1} d(\cos\theta)\int_0^{2\pi} d\varphi\,\left(\sin^2\theta\cos^2\varphi + \cos^2\theta\right)\\
&= \frac{4}{9}a^6k^4\chi^2\left(\frac{8\pi}{3}\right)
\end{aligned}$$

$$\boxed{\sigma = \frac{32}{27}\pi a^6k^4\chi^2 \quad \text{where } k \equiv \frac{\omega}{c}\text{, assuming } kb \ll 1} \tag{253}$$

(b) In the limit of $kb \sim 1$, we can no longer assume that both dielectric spheres are approximately at the origin. But since $ka \ll 1$, we can still assume that each of the spheres is an ideal electric dipole, and we may still approximate the incident electromagnetic field as constant over the area of each sphere. The differences from part (a) are the following:

1. The incident electric field is different at the location of each sphere.
2. One sphere is not at the origin.

Let's start with the first difference. If we set $t = 0$ at an appropriate point, the approximate electric field at the location of the first dielectric sphere is

$$\tilde{\mathbf{E}}_0(z=0) \approx E_0e^{-i\omega t}\hat{\mathbf{y}} \tag{254}$$

and the approximate electric field at the location of the second dielectric sphere is

$$\tilde{\mathbf{E}}_0(z=b) \approx E_0e^{i(kb-\omega t)}\hat{\mathbf{y}} \tag{255}$$

Note the phase difference between the two background electric fields. Letting the first dipole be the one at $z = 0$ and using (237), we get that the dipole moments of the two spheres are

$$\tilde{\mathbf{p}}^{(1)} = \frac{4}{3}\pi\epsilon_0 a^3 \chi E_0 e^{-i\omega t}\hat{\mathbf{y}} \quad \text{and} \quad \tilde{\mathbf{p}}^{(2)} = \frac{4}{3}\pi\epsilon_0 a^3 \chi E_0 e^{i(kb-\omega t)}\hat{\mathbf{y}} \tag{256}$$

As in part (a), we use the formula for the power per solid angle radiated by a single dipole. As part of the derivation of this formula, we take the radiated electromagnetic fields and square them:

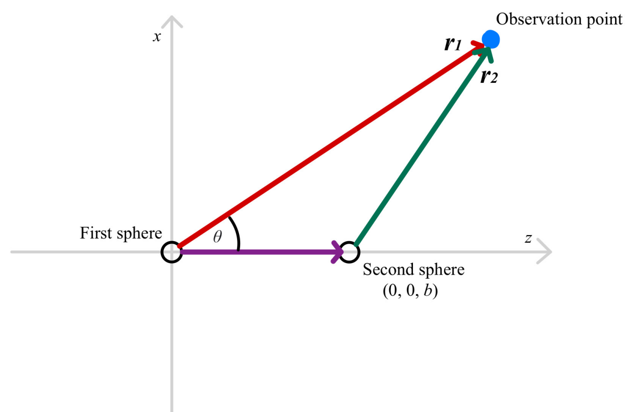
$$\frac{dP}{d\Omega} = \frac{\mu_0}{(4\pi)^2 c}\Big|\underbrace{\hat{\mathbf{r}} \times \ddot{\mathbf{p}}_{\text{ret}}}_{\substack{\mathbf{E}_{\text{rad}},\, \mathbf{B}_{\text{rad}} \\ \text{live here}}}\Big|^2 \tag{257}$$

To take into account the effect of the second dipole, we need to remember the following: For two different sources of radiation, the radiated electromagnetic *fields* add together, not the radiated *power*. Radiated power is proportional to the square of the electromagnetic fields, so we need to add the contributions of the two dipoles together *within* the absolute value sign.

Note that this formula assumes the dipole is at the origin. We now must take into account the second difference from part (a), namely that the second dipole is not at the origin. The radiation field from a dipole is a spherical wave

$$\mathbf{E}_{\text{rad}} \propto \frac{e^{i(kr-\omega t)}}{r} \quad \text{where } \mathbf{r} \text{ points from the dipole to the observation point} \tag{258}$$

Here, we have dipoles in two different places, so we have two different $\mathbf{r}$ vectors, $\mathbf{r}_1$ and $\mathbf{r}_2$.




In terms of these vectors, the total radiated electric field from both dipoles is

$$\mathbf{E}_{\text{rad}} = (\text{prefactor 1})\,\frac{e^{i(kr_1-\omega t)}}{r_1} + (\text{prefactor 2})\,\frac{e^{i(kr_2-\omega t)}}{r_2} \tag{259}$$

We calculate the radiated power very far away from the source, so $r_1, r_2 \gg b$. So we don't need to worry about the difference between $r_1$ and $r_2$ in the denominator of (259), and we can set $r_1 \approx r_2$ there. However, since $kb \sim 1$, we do need to worry about the difference between $r_1$ and $r_2$ in the exponent of (259) (i.e. the interference effects between the dipole sources).

Since the second dipole is at $z = b$,

$$\mathbf{r}_1 = b\hat{\mathbf{z}} + \mathbf{r}_2 \tag{260}$$

We want to calculate $r_2$ in terms of $r_1$ and $b$. To do this, solve for $\mathbf{r}_2$ and square:

$$r_2^2 = (\mathbf{r}_1 - b\hat{\mathbf{z}})^2 = r_1^2 - 2b\mathbf{r}_1 \cdot \hat{\mathbf{z}} + b^2$$

The scattering angle $\theta$ is defined as the angle between $\mathbf{r}_1$ and $\hat{\mathbf{z}}$. We can use this information to simplify the dot product:

$$r_2^2 = r_1^2 - 2br_1\cos\theta + b^2 \tag{261}$$

We now take the square root to get $r_2$ and Taylor expand, dropping all terms subleading in the small parameter $b/r_1$.

$$\begin{aligned} r_2 &= \sqrt{r_1^2 - 2br_1\cos\theta + b^2} \\ &= r_1\sqrt{1 - \frac{2b}{r_1}\cos\theta + \frac{b^2}{r_1^2}} \\ &= r_1\left(1 - \frac{b}{r_1}\cos\theta + \mathcal{O}\left(\frac{b^2}{r_1^2}\right)\right) \\ r_2 &\approx r_1 - b\cos\theta \quad \text{as } b \ll r_1 \end{aligned} \tag{262}$$

Using this information to simplify (259), we get

$$\begin{aligned} \mathbf{E}_{\text{rad}} &\approx (\text{prefactor 1})\,\frac{e^{i(kr_1-\omega t)}}{r_1} + (\text{prefactor 2})\,\frac{e^{i(k(r_1-b\cos\theta)-\omega t)}}{r_1 - b\cos\theta} \\ &\approx \frac{e^{ikr_1}}{r_1}\left((\text{prefactor 1}) + (\text{prefactor 2})e^{-ib\cos\theta}\right) \quad \text{since } b \ll r_1 \end{aligned} \tag{263}$$

The formula for the power radiated by a single dipole ignores the effect of the second dipole:

$$\frac{dP}{d\Omega} = \frac{\mu_0}{(4\pi)^2 c}\left|\hat{\mathbf{r}} \times \ddot{\mathbf{p}}_{\text{ret}}\right|^2 = \frac{\mu_0}{(4\pi)^2 c}\left|\hat{\mathbf{r}} \times \text{Re}\left(\ddot{\tilde{\mathbf{p}}}_{\text{ret}}\right)\right|^2 \quad \text{if (prefactor 2)} = 0$$

Adding in the effect of the second dipole inside the absolute value signs, we get

$$\frac{dP}{d\Omega} = \frac{\mu_0}{(4\pi)^2 c}\left|\hat{\mathbf{r}} \times \text{Re}\left(\ddot{\tilde{\mathbf{p}}}^{(1)}_{\text{ret}} + \ddot{\tilde{\mathbf{p}}}^{(2)}_{\text{ret}}e^{-ib\cos\theta}\right)\right|^2 \tag{264}$$

where $\tilde{\mathbf{p}}^{(1)}$ and $\tilde{\mathbf{p}}^{(2)}$ are the complex versions of the dipole moments of the first and second dipole, respectively. As in part (a), we must remember to take the real part of the quantity inside the absolute value *before* squaring, or else we will get the time-averaging wrong because $|\text{Re}(A)^2| \neq \text{Re}(|A|^2)$. Differentiating (256) twice and adding, we get

$$\ddot{\tilde{\mathbf{p}}}^{(1)}_{\text{ret}} + \ddot{\tilde{\mathbf{p}}}^{(2)}_{\text{ret}}e^{-ib\cos\theta} = \frac{4}{3}\pi\epsilon_0 a^3\omega^2\chi E_0 e^{-i\omega t_{\text{ret}}}\left(1 + e^{ikb(1-\cos\theta)}\right)\hat{\mathbf{y}} \tag{265}$$

This means that

$$\left\langle\frac{dP}{d\Omega}\right\rangle = \frac{\mu_0}{(4\pi)^2 c}\left(\frac{4}{3}\pi\epsilon_0 a^3\omega^2\chi E_0\right)^2\left\langle\text{Re}\left[e^{-i\omega t_{\text{ret}}}\left(1 + e^{ib(1-\cos\theta)}\right)\right]^2\right\rangle|\hat{\mathbf{r}} \times \hat{\mathbf{y}}|^2$$

We now need to take the real part. Setting $\alpha \equiv kb(1-\cos\theta)$, we get

$$\begin{aligned} \text{Re}\left(e^{-i\omega t}\left(1 + e^{i\alpha}\right)\right) &= \text{Re}\left[(\cos(\omega t) - i\sin(\omega t))\,(1 + \cos\alpha + i\sin\alpha)\right] \\ &= \cos(\omega t)\,(1 + \cos\alpha) + \sin(\omega t)\sin\alpha \end{aligned}$$

Since $\left\langle\sin^2 x\right\rangle = \left\langle\cos^2 x\right\rangle = \frac{1}{2}$ and $\langle\sin x\cos x\rangle = 0$, we can square and time-average this quantity:

$$\begin{aligned}
\left\langle \mathrm{Re}\left(e^{-i\omega t}\left(1+e^{i\alpha}\right)\right)^2\right\rangle &= \left\langle \left(\cos(\omega t)\left(1+\cos\alpha\right)+\sin(\omega t)\sin\alpha\right)^2\right\rangle \\
&= (1+\cos\alpha)^2\left\langle \cos(\omega t)^2\right\rangle + \sin^2\alpha\left\langle \sin(\omega t)^2\right\rangle \\
&= \frac{1}{2}\left[(1+\cos\alpha)^2+\sin^2\alpha\right] \\
&= \frac{1}{2}\left[1+2\cos\alpha+\cos^2\alpha+\sin^2\alpha\right] \\
&= \frac{1}{2}\left[2+2\cos\alpha\right] \\
&= 1+\cos\alpha
\end{aligned} \tag{266}$$

Therefore, plugging (265) into (264) and recalling that $|\hat{\mathbf{r}}\times\hat{\mathbf{y}}|^2 = \sin^2\theta\cos^2\varphi+\cos^2\theta$ from part (a), we get the time-averaged power radiated per unit area:

$$\left\langle \frac{dP}{d\Omega}\right\rangle = \underbrace{\frac{\mu_0}{(4\pi)^2 c}}_{\substack{\text{prefactor}\\ \text{from formula}}} \underbrace{\left(\frac{4}{3}\pi\epsilon_0 a^3\omega^2\chi E_0\right)^2}_{\substack{\text{square of real prefactor}\\ \text{in (265)}}} \underbrace{\left[1+\cos\left(kb\left(1-\cos\theta\right)\right)\right]}_{\substack{\text{time-averaged square}\\ \text{of real part of}\\ \text{complex prefactor in (265)}\\ \text{(see (266))}}} \underbrace{\left(\sin^2\theta\cos^2\varphi+\cos^2\theta\right)}_{\substack{|\hat{\mathbf{r}}\times\hat{\mathbf{y}}|^2\\ \text{(calculated in (a))}}} \tag{267}$$

To get to the differential cross-section, divide by the intensity of the incident plane wave (as in part (a)):

$$\begin{aligned}
\frac{d\sigma}{d\Omega} &= \frac{\langle dP/d\Omega\rangle}{I_{\text{inc}}} \\
&= \underbrace{\frac{\frac{\mu_0}{(4\pi)^2 c}\left(\frac{4}{3}\pi\epsilon_0 a^3\omega^2\chi E_0\right)^2}{\frac{1}{2}\epsilon_0 c E_0^2}}_{\text{prefactor}}\left[1+\cos\left(kb\left(1-\cos\theta\right)\right)\right]\left(\sin^2\theta\cos^2\varphi+\cos^2\theta\right)
\end{aligned} \tag{268}$$

This prefactor is similar to the one in part (a); it is just missing an overall factor of 2. Simplifying it yields

$$\text{Prefactor } = \frac{2}{9}a^6k^4\chi^2 \quad \text{where } k\equiv\frac{\omega}{c}$$

Therefore, we get

$$\frac{d\sigma}{d\Omega} = \frac{2}{9}a^6k^4\chi^2\left[1+\cos\left(kb\left(1-\cos\theta\right)\right)\right]\left(\sin^2\theta\cos^2\varphi+\cos^2\theta\right) \quad \text{where } k\equiv\frac{\omega}{c} \tag{269}$$

Note that if $b=0$, this answer reduces to the one from part (a).

The problem asks for the differential cross-section in the $xz$-plane. In that plane, $\varphi = 0$, so $\sin^2\theta\cos^2\varphi+\cos^2\theta = \sin^2\theta+\cos^2\theta = 1$ and our answer simplifies to

$$\boxed{\frac{d\sigma}{d\Omega} = \frac{2}{9}a^6k^4\chi^2\left[1+\cos\left(kb\left(1-\cos\theta\right)\right)\right] \quad \text{in the } xz\text{-plane, where } k\equiv\frac{\omega}{c}} \tag{270}$$

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***Born approximation first:***
The dielectric response of each sphere is small ($\chi \ll 1$), so we can use the Born approximation. The formula for the differential cross section in this approximation is

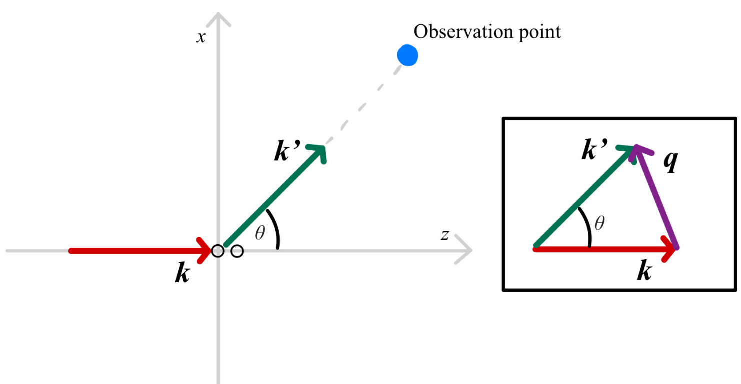
$$\frac{d\sigma}{d\Omega} = \left(\frac{k^2}{4\pi}\right)^2|\hat{\mathbf{r}}\times\hat{\mathbf{e}}_0|^2\left|\int d^3r'\,\chi(\mathbf{r}',\omega)e^{i\mathbf{q}\cdot\mathbf{r}'}\right|^2 \tag{271}$$

Note that the integral here is just the Fourier transform of the electric susceptibility $\chi$. Here, $\hat{\mathbf{e}}_0$ is the polarization vector for the incoming electric field, $k$ is equal to $\omega/c$, and $\mathbf{q}$ is the momentum transfer, the difference between the incoming and outgoing momentum:

$$\mathbf{q} \equiv \mathbf{k}' - \mathbf{k} \tag{272}$$

This is elastic scattering, so the incoming and outgoing wave vectors $\mathbf{k}$ and $\mathbf{k}'$ have the same magnitude. The incoming wave vector points in the $\hat{\mathbf{z}}$ direction, while the outgoing wave vector points in the $\hat{\mathbf{r}}$ direction. This means that

$$\begin{aligned}
\mathbf{k} &= k\hat{\mathbf{z}} \\
\mathbf{k}' &= k\hat{\mathbf{r}} = k\sin\theta\cos\varphi\,\hat{\mathbf{x}} + k\sin\theta\sin\varphi\,\hat{\mathbf{y}} + k\cos\theta\,\hat{\mathbf{z}} \\
\mathbf{q} = \mathbf{k}' - \mathbf{k} &= k\sin\theta\cos\varphi\,\hat{\mathbf{x}} + k\sin\theta\sin\varphi\,\hat{\mathbf{y}} + k\left(\cos\theta - 1\right)\hat{\mathbf{z}}
\end{aligned} \tag{273}$$




We will do parts (a) and (b) in one fell swoop, considering part (a) as a special case of part (b). Since $ka \ll 1$, we can approximate each sphere as a point scatterer. Therefore, the electric susceptibility is a sum of two delta functions, one centered at the origin and one centered at $b = z$:

$$\chi(\mathbf{r},\omega) = N\left(\delta^{(3)}(\mathbf{r}) + \delta^{(3)}(\mathbf{r} - b\,\hat{\mathbf{z}})\right) \quad \text{for a constant } N \tag{274}$$

What is the height of the delta function $N$? To determine this, note that if we integrate (274) over one of the spheres, we get $N$. Since the electric susceptibility $\chi$ is constant over each sphere, we also know that the integral of $\chi(\mathbf{r})$ over one of the spheres is just $\chi$ times the volume of the sphere: $\frac{4}{3}\pi a^3\chi$. Comparing these two answers tells us that $N = \frac{4}{3}\pi a^3\chi$. With this approximation, the integral in the Born approximation is trivial:

$$\begin{aligned}
\int d^3r'\,\chi(\mathbf{r}',\omega)e^{i\mathbf{q}\cdot\mathbf{r}'} &= \int d^3r'\,\frac{4}{3}\pi a^3\chi\left(\delta^{(3)}(\mathbf{r}') + \delta^{(3)}(\mathbf{r}' - b\,\hat{\mathbf{z}})\right)e^{i\mathbf{q}\cdot\mathbf{r}'} \\
&= \frac{4}{3}\pi a^3\chi\left(1 + e^{ib\mathbf{q}\cdot\hat{\mathbf{z}}}\right) \\
&= \frac{4}{3}\pi a^3\chi\left(1 + e^{ikb(\cos\theta - 1)}\right) \quad \text{by (273)}
\end{aligned} \tag{275}$$

We can now square this integral:

$$\begin{aligned}
\left|\int d^3r'\,\chi(\mathbf{r}',\omega)e^{i\mathbf{q}\cdot\mathbf{r}'}\right|^2 &= \left(\frac{4}{3}\pi a^3\chi\right)^2\left|1+e^{ikb(\cos\theta-1)}\right|^2 \\
&= \left(\frac{4}{3}\pi a^3\chi\right)^2\left|e^{ikb(\cos\theta-1)/2}\left(e^{ikb(\cos\theta-1)/2}+e^{-ikb(\cos\theta-1)/2}\right)\right|^2 \\
&= \left(\frac{4}{3}\pi a^3\chi\right)^2\left(2\cos\left(\frac{kb\,(\cos\theta-1)}{2}\right)\right)^2 \\
&= \left(\frac{4}{3}\pi a^3\chi\right)^2\left(4\cos^2\left(\frac{kb\,(\cos\theta-1)}{2}\right)\right) \\
&= \left(\frac{4}{3}\pi a^3\chi\right)^2\left(2\left[1+\cos\left(kb\,(1-\cos\theta)\right)\right]\right)
\end{aligned} \tag{276}$$

In the last line, we used the identities $\cos^2 x = \frac{1}{2}(1+\cos(2x))$ and $\cos(-x) = \cos x$. The other parts of the Born approximation formula (271) can now be calculated. In this case, $\hat{\mathbf{e}}_0 = \hat{\mathbf{y}}$, so

$$\begin{aligned}
\hat{\mathbf{r}}\times\hat{\mathbf{y}} &= (\sin\theta\cos\varphi\,\hat{\mathbf{x}}+\sin\theta\sin\varphi\,\hat{\mathbf{y}}+\cos\theta\,\hat{\mathbf{z}})\times\hat{\mathbf{y}} \\
&= \sin\theta\cos\varphi\,\hat{\mathbf{z}}-\cos\theta\,\hat{\mathbf{x}} \quad \text{since } \hat{\mathbf{x}}\times\hat{\mathbf{y}}=\hat{\mathbf{z}} \text{ and } \hat{\mathbf{z}}\times\hat{\mathbf{y}}=-\hat{\mathbf{x}}
\end{aligned} \tag{277}$$

so

$$|\hat{\mathbf{r}}\times\hat{\mathbf{y}}|^2 = \sin^2\theta\cos^2\varphi+\cos^2\theta \tag{278}$$

Putting everything together, we get the differential cross-section

$$\begin{aligned}
\frac{d\sigma}{d\Omega} &= \left(\frac{k^2}{4\pi}\right)^2|\hat{\mathbf{r}}\times\hat{\mathbf{y}}|^2\left|\int d^3r'\,\chi(\mathbf{r}',\omega)e^{i\mathbf{q}\cdot\mathbf{r}'}\right|^2 \\
&= \left(\frac{k^2}{4\pi}\right)^2\left(\sin^2\theta\cos^2\varphi+\cos^2\theta\right)\left(\frac{4}{3}\pi a^3\chi\right)^2\left(2\left[1+\cos\left(kb\,(1-\cos\theta)\right)\right]\right) \\
&= \frac{2}{9}a^6k^4\chi^2\left[1+\cos\left(kb\,(1-\cos\theta)\right)\right]\left(\sin^2\theta\cos^2\varphi+\cos^2\theta\right)
\end{aligned} \tag{279}$$

We can now answer parts (a) and (b) directly:

(a) For this part, $kb \ll 1$, so we can set $b$ to zero in (279) to get

$$\boxed{\frac{d\sigma}{d\Omega} = \frac{4}{9}a^6k^4\chi^2\left(\sin^2\theta\cos^2\varphi+\cos^2\theta\right) \quad \text{where } k\equiv\frac{\omega}{c}\text{, assuming } kb\ll 1} \tag{280}$$

The total cross-section is defined by

$$\sigma \equiv \int\frac{d\sigma}{d\Omega}\,d\Omega = \int_{-1}^{1}d(\cos\theta)\int_0^{2\pi}d\varphi\,\frac{d\sigma}{d\Omega} \tag{281}$$

Bearing in mind that $\int_0^{2\pi} d\varphi \cos^2\varphi = \int_0^{2\pi} d\varphi\, \frac{1}{2}(1+\cos\varphi) = \pi$, we can write

$$\begin{aligned}\int d\Omega\, \left(\sin^2\theta\cos^2\varphi + \cos^2\theta\right) &= \int_{-1}^{1} d(\cos\theta) \int_0^{2\pi} d\varphi\, \left(\sin^2\theta\cos^2\varphi + \cos^2\theta\right)\\ &= \int_{-1}^{1} d(\cos\theta)\, \left(\pi\sin^2\theta + 2\pi\cos^2\theta\right)\\ &= \int_{-1}^{1} d(\cos\theta)\, \left(\pi\left(1-\cos^2\theta\right) + 2\pi\cos^2\theta\right)\\ &= \int_{-1}^{1} d(\cos\theta)\, \left(\pi + \pi\cos^2\theta\right)\\ &= \left[\pi\cos\theta + \frac{\pi}{3}\cos^3\theta\right]_{\cos\theta=-1}^{\cos\theta=1}\\ &= \frac{8\pi}{3}\end{aligned} \tag{282}$$

Therefore,

$$\begin{aligned}\sigma &= \frac{4}{9}a^6k^4\chi^2 \int_{-1}^{1} d(\cos\theta) \int_0^{2\pi} d\varphi\, \left(\sin^2\theta\cos^2\varphi + \cos^2\theta\right)\\ &= \frac{4}{9}a^6k^4\chi^2\left(\frac{8\pi}{3}\right)\end{aligned}$$

$$\boxed{\sigma = \frac{32}{27}\pi a^6k^4\chi^2 \quad \text{where } k \equiv \frac{\omega}{c}, \text{ assuming } kb \ll 1} \tag{283}$$

(b) For this part, the problem tells us to assume that $kb \sim 1$ and asks us for the differential cross-section in the $xz$-plane. In that plane, $\varphi = 0$, so $\sin^2\theta\cos^2\varphi + \cos^2\theta = \sin^2\theta + \cos^2\theta = 1$ and (279) simplifies to

$$\boxed{\frac{d\sigma}{d\Omega} = \frac{2}{9}a^6k^4\chi^2\left[1+\cos\left(kb\left(1-\cos\theta\right)\right)\right] \quad \text{in the } xz\text{-plane, where } k \equiv \frac{\omega}{c}} \tag{284}$$

---

(c) No matter how we got to our answers for parts (a) and (b), the approach for part (c) is the same. The first thing to note is that $\frac{d\sigma}{d\Omega} = \frac{\langle dP/d\Omega\rangle}{I_{\text{inc}}}$. Here, the intensity of the incident wave $I_{\text{inc}}$ is independent of $\theta$ and $\varphi$, and $I_{\text{inc}}$ is the same in parts (a) and (b). Therefore, the total cross-section in each part is proportional to the total radiated power, and the proportionality constant is the same in parts (a) and (b):

$$\sigma = \int \frac{d\sigma}{d\Omega} d\Omega = \frac{1}{I_{\text{inc}}} \int \left\langle \frac{dP}{d\Omega} \right\rangle d\Omega = \frac{P_{\text{rad}}}{I_{\text{inc}}} \tag{285}$$

Therefore, finding out when the total radiated power is the same is equivalent to finding out when the total cross-section is the same. The cross-section we calculated in part (a) is

$$\sigma_a = \frac{4}{9}a^6k^4\chi^2\left(\frac{8\pi}{3}\right) \tag{286}$$

In part (b), we didn't calculate the cross-section, but as an intermediate step, we did calculate the differential cross-section for general $\theta$ and $\varphi$. (Note that our final answer for part (b) is only valid for the $xz$-plane, which is not enough to calculate the cross-section.) What we got was

$$\frac{d\sigma_b}{d\Omega} = \frac{2}{9}a^6k^4\chi^2\left[1+\cos\left(kb\left(1-\cos\theta\right)\right)\right]\left(\sin^2\theta\cos^2\varphi + \cos^2\theta\right) \tag{287}$$

To get the total cross-section for part (b), we take the integral over the solid angle:

$$\begin{aligned}
\sigma_b &= \int \frac{d\sigma_b}{d\Omega} d\Omega \\
&= \frac{2}{9} a^6 k^4 \chi^2 \int_{-1}^{1} d(\cos\theta) \int_0^{2\pi} d\varphi \left[1 + \cos\left(kb\left(1 - \cos\theta\right)\right)\right] \left(\sin^2\theta \cos^2\varphi + \cos^2\theta\right) \\
&= \frac{2}{9} a^6 k^4 \chi^2 \left[ \int_{-1}^{1} d(\cos\theta) \int_0^{2\pi} d\varphi \left(\sin^2\theta \cos^2\varphi + \cos^2\theta\right) \right. \\
&\qquad \left. + \int_{-1}^{1} d(\cos\theta) \int_0^{2\pi} d\varphi \cos\left(kb\left(1 - \cos\theta\right)\right) \left(\sin^2\theta \cos^2\varphi + \cos^2\theta\right) \right]
\end{aligned}$$

We already took the first integral in brackets in part (a); it is equal to $8\pi/3$. Bearing in mind that $\int_0^{2\pi} d\varphi \cos^2\varphi = \int_0^{2\pi} d\varphi \, \frac{1}{2}(1 + \cos\varphi) = \pi$ and taking the second integral over $\varphi$, we can write this as

$$\begin{aligned}
\sigma_b &= \frac{2}{9} a^6 k^4 \chi^2 \left[ \frac{8\pi}{3} + \int_{-1}^{1} d(\cos\theta) \cos\left(kb\left(1 - \cos\theta\right)\right) \left(\pi \sin^2\theta + 2\pi \cos^2\theta\right) \right] \\
&= \frac{2}{9} a^6 k^4 \chi^2 \left[ \frac{8\pi}{3} + \pi \int_{-1}^{1} du \cos\left(kb\left(1 - u\right)\right) \left(1 + u^2\right) \right] \quad \text{for } u \equiv \cos\theta \qquad (288)
\end{aligned}$$

In the last step, we used the Pythagorean identity $\sin^2\theta = 1 - \cos^2\theta$. This integral can be partially evaluated using a sum-to-product identity:

$$\begin{aligned}
&\int_{-1}^{1} du \cos\left(kb\left(1 - u\right)\right) \left(1 + u^2\right) \\
&\qquad = \int_{-1}^{1} du \left(\cos(kb)\cos(kbu) + \sin(kb)\sin(kbu)\right) \left(1 + u^2\right) \\
&\qquad = \cos(kb) \int_{-1}^{1} du \cos(kbu)(1 + u^2) \quad \text{canceling out the integral of an odd function} \\
&\qquad = \cos(kb) \left[ \int_{-1}^{1} du \cos(kbu) + \int_{-1}^{1} du \, u^2 \cos(kbu) \right] \\
&\qquad = \cos(kb) \left[ \frac{2}{kb} \sin(kb) + \int_{-1}^{1} du \, u^2 \cos(kbu) \right]
\end{aligned}$$

We can take the remaining integral using repeated integration by parts:

$$\begin{aligned}
\int_{-1}^{1} du \, u^2 \cos(kbu) &= \int_{-1}^{1} du \left[ \left( \frac{1}{kb} u^2 \sin(kbu) \right)' - \frac{2}{kb} u \sin(kbu) \right] \\
&= \left[ \frac{1}{kb} u^2 \sin(kbu) \right]_{-1}^{1} - \frac{2}{kb} \int_{-1}^{1} du \, u \sin(kbu) \\
&= \frac{2}{kb} \sin(kb) - \frac{2}{kb} \int_{-1}^{1} du \left[ \left( -\frac{1}{kb} u \cos(kbu) \right)' + \frac{1}{kb} \cos(kbu) \right] \\
&= \frac{2}{kb} \sin(kb) + \frac{2}{(kb)^2} \left[u \cos(kbu)\right]_{-1}^{1} - \frac{2}{(kb)^2} \int_{-1}^{1} du \, \cos(kbu) \\
&= \frac{2}{kb} \sin(kb) + \frac{4}{(kb)^2} \cos(kb) - \frac{2}{(kb)^3} \left[\sin(kbu)\right]_{-1}^{1} \\
&= \frac{2}{kb} \sin(kb) + \frac{4}{(kb)^2} \cos(kb) - \frac{4}{(kb)^3} \sin(kb)
\end{aligned}$$

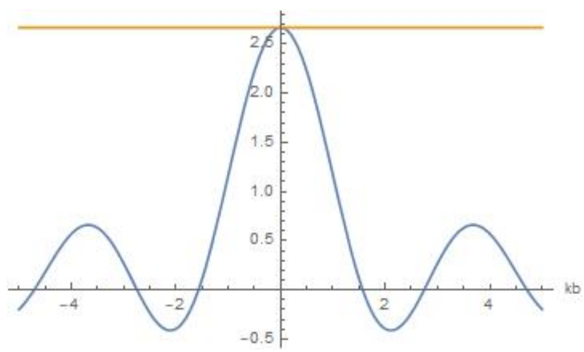
Plugging this in, we get

$$\begin{aligned}
&\int_{-1}^{1} du \cos\left(kb\left(1-u\right)\right)\left(1+u^2\right) \\
&\qquad = \cos(kb)\left[\frac{2}{kb}\sin(kb) + \left(\frac{2}{kb}\sin(kb) + \frac{4}{(kb)^2}\cos(kb) - \frac{4}{(kb)^3}\sin(kb)\right)\right] \\
&\qquad = \cos(kb)\left[\frac{4}{kb}\sin(kb) + \frac{4}{(kb)^2}\cos(kb) - \frac{4}{(kb)^3}\sin(kb)\right] \\
&\qquad = \frac{2}{kb}\sin(2kb) + \frac{4}{(kb)^2}\cos^2(kb) - \frac{2}{(kb)^3}\sin(2kb) \quad \text{using the identity } \sin(2x) = 2\sin x\cos x
\end{aligned}$$

Setting the cross-sections from part (a) and part (b) equal to one another, we get

$$\begin{aligned}
\sigma_a &= \sigma_b \\
\frac{4}{9}a^6k^4\chi^2\left(\frac{8\pi}{3}\right) &= \frac{2}{9}a^6k^4\chi^2\left[\frac{8\pi}{3} + \pi\int_{-1}^{1} du\cos\left(kb\left(1-u\right)\right)\left(1+u^2\right)\right] \\
\frac{16\pi}{3} &= \frac{8\pi}{3} + \pi\int_{-1}^{1} du\cos\left(kb\left(1-u\right)\right)\left(1+u^2\right) \\
\frac{8}{3} &= \int_{-1}^{1} du\cos\left(kb\left(1-u\right)\right)\left(1+u^2\right) \\
\frac{8}{3} &= \frac{2}{kb}\sin(2kb) + \frac{4}{(kb)^2}\cos^2(kb) - \frac{2}{(kb)^3}\sin(2kb) \qquad (289)
\end{aligned}$$

It turns out that the function on the right-hand side of this equation is maximized at $kb = 0$ only, where the right-hand side of the equation is equal to exactly $8/3$ (see graph below).




Therefore, the total radiated power from part (a) and (b) are the same only when $\boxed{kb = 0}$.

It isn't mentioned in the problem, but it is nice to observe that if we look for solutions where the part (b) power radiated is *half* the part (a) power radiated, we get

$$\begin{aligned}
\sigma_a &= 2\sigma_b \\
0 &= \int_{-1}^{1} du\cos\left(kb\left(1-u\right)\right)\left(1+u^2\right) \\
0 &= \frac{2}{kb}\sin(2kb) + \frac{4}{(kb)^2}\cos^2(kb) - \frac{2}{(kb)^3}\sin(2kb)
\end{aligned}$$

The values $kb = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots$ satisfy this equation, so $kb = \frac{n\pi}{2}$ for $n$ odd are values of $kb$ where the part (b) power radiated is half the part (a) power radiated.