

3. Quantum Mechanics (Spring 2005)

A beam of particles scatters off an impenetrable sphere of radius a . That is, the potential is zero outside the sphere, and infinite inside. The wave function must therefore vanish at $r = a$.

(a) What is the S-wave ($l = 0$) phase shift as a function of the incident energy or momentum?

(b) What is the total cross section in the limit of zero incident kinetic energy?

$$V(r) = \begin{cases} \infty & r < a \\ 0 & r > a \end{cases} \quad \begin{aligned} &\bullet \text{sph symm; azimuthal symm (no } \varphi\text{-dep)} \\ &\bullet \text{ang mom cons} \\ &\bullet |\vec{k}_0|^2 \& |\vec{k}_{\text{sc}}| = |\vec{k}| \text{ cons of linear mom.} \end{aligned}$$

Review:

$$1. \psi(r, \theta) = \phi_{\text{inc}} + \phi_{\text{sc}} = e^{i\vec{k}_0 \cdot \vec{r}} + f(\theta) \left(\frac{e^{i\vec{k} \cdot \vec{r}}}{r} \right)$$

$$\bullet \phi_{\text{inc}} = e^{i\vec{k}_0 \cdot \vec{r}} = e^{i\vec{k} \cdot \vec{r} \cos \theta} = \sum_l A_l j_l(kr) P_l(\cos \theta), \quad A_l = (2l+1) i^l = (2l+1) e^{i(\pi/2)l}$$

$$\bullet \phi_{\text{sc}} = f(\theta) \frac{e^{i\vec{k} \cdot \vec{r}}}{r} = \left[\sum_l (2l+1) f_l(k) P_l(\cos \theta) \right] \frac{e^{i\vec{k} \cdot \vec{r}}}{r} \quad \text{n. we've expanded } f(\theta) \text{ in terms of } P_l(\cos \theta)$$

2. We will show that asymptotic limit $r \gg a$ (detector is "far") leads to desired result!

$$j_l(kr) \rightarrow \frac{\sin(kr - \pi/2)}{kr} = \frac{e^{i(kr - \pi/2)} - e^{-i(kr - \pi/2)}}{2ikr} \Rightarrow \text{apply to } \phi_{\text{inc}}$$

$$\begin{aligned} \psi(r, \theta) \xrightarrow{\text{large } r} & e^{i\vec{k} \cdot \vec{r} \cos \theta} + f(\theta) \frac{e^{i\vec{k} \cdot \vec{r}}}{r} = \sum_l (2l+1) P_l(\cos \theta) \left(\frac{1}{2ikr} \right) \left\{ \left(e^{i\vec{k} \cdot \vec{r}} - e^{-i(kr - \pi/2)} \right) + 2ikf_l(k) e^{i\vec{k} \cdot \vec{r}} \right\} \\ & = \sum_l (2l+1) P_l(\cos \theta) \left(\frac{1}{2ikr} \right) \left\{ (1 + 2ikf_l(k)) e^{i\vec{k} \cdot \vec{r}} - e^{-i(kr - \pi/2)} \right\} \end{aligned}$$

Notice only outgoing wave has scattering amp.

$$\text{let } 1 + 2ikf_l(k) = e^{2i\delta_l} \text{ s.t. } f_l(k) = \frac{e^{i\delta_l} \sin \delta_l}{2k}$$

$$3. \text{ But! } \psi(r, \theta) = \sum_l A_l R_{ll}(r) P_l(\cos \theta) \xrightarrow{\text{total wfn}} \psi^{m=0} \quad (\text{General sol'n of spherical TISE w/ } V(r))$$

where R_{ll} obeys Radial Schrö Egn: Let $u(r) = r R(r)$

$$-\frac{\hbar^2}{2m} \frac{d^2 u(r)}{dr^2} + \left[V(r) + \frac{l(l+1)\hbar^2}{2mr^2} \right] u(r) = E u(r)$$

$$\Rightarrow \text{sol'ns } R_{ll}(r) = A_{ll} j_l(kr) + B_{ll} n_l(kr) \quad \text{for } V(r)=0 \quad (r>a)$$

$$= C_l^{(1)} h_l^{(1)} + C_l^{(2)} h_l^{(2)} \quad (\text{Hankel fcns: } h_{ll} = j_{ll} \pm i n_{ll})$$

$$\rightarrow \psi(r, \theta) = \sum_l A_l \left\{ C_l^{(1)} h_l^{(1)} + C_l^{(2)} h_l^{(2)} \right\} P_l(\cos \theta) \quad \text{for } r>a$$

$$\begin{aligned} \text{using } n_l(kr) \xrightarrow{\text{large } r} & \frac{-\cos(kr - \pi/2)}{kr} = \frac{\left(e^{i(kr - \pi/2)} + e^{-i(kr - \pi/2)} \right)}{2kr} \\ h_l^{(2)} \xrightarrow{\text{large } r} & \pm \frac{e^{\pm i(kr - \pi/2)}}{ikr} \end{aligned}$$

Comparing large r , total wfn $\psi(r, \theta)$ from (2)

$$\begin{aligned} \psi(r, \theta) \xrightarrow{\text{large } r} & \sum_l (2l+1) P_l(\cos \theta) \left(\frac{1}{2ikr} \right) \left\{ e^{2i\delta_l} e^{i\vec{k} \cdot \vec{r}} - e^{-i(kr - \pi/2)} \right\} \\ & \quad \left(2l+1 \right) \left(\frac{1}{2ikr} \right) \left\{ e^{i\delta_l} e^{2i\delta_l} e^{i(kr - \pi/2)} - e^{-i(kr - \pi/2)} e^{i\delta_l} \right\} \\ & = \sum_l A_l \left\{ \frac{e^{2i\delta_l}}{2} \left(\frac{e^{i(kr - \pi/2)}}{ikr} + \frac{1}{2} \frac{-e^{-i(kr - \pi/2)}}{ikr} \right) \right\} \\ & \quad \left\{ C_l^{(1)} = \frac{e^{2i\delta_l}}{2} \quad \text{and} \quad C_l^{(2)} = \frac{1}{2} \right\} \end{aligned}$$

$$R_{\text{tot}}(r) = \left(\frac{e^{2i\delta_2}}{2}\right) h_2^{(1)} + \left(\frac{1}{2}\right) h_2^{(2)}$$

$$\text{for } r > a$$

$$= \left(\frac{1}{2}\right) \left[j_2(\lambda r) \{ e^{2i\delta_2} + 1 \} + i n_2(\lambda r) \{ e^{2i\delta_2} - 1 \} \right]$$

$$R_{\text{tot}} = e^{2i\delta_2} \left[\cos \delta_2 j_2(\lambda r) - \sin \delta_2 n_2(\lambda r) \right] \quad * \text{ useful}$$

(a) Method 1: the wavefn must vanish @ $r=a$

$$\Psi(a, \theta) = 0 \quad \text{s.t. } R_{\text{tot}}(r) \Big|_{r=a} = 0$$

$$\tan \delta_2 = \frac{j_2(\lambda a)}{n_2(\lambda a)}$$

For S-wave:

$$\tan \delta_0 = \frac{j_0(\lambda a)}{n_0(\lambda a)} = \frac{\sin(\lambda a)/\lambda a}{-\cos(\lambda a)/\lambda a} \Rightarrow \delta_0 = -\lambda a = -\sqrt{2mE} a / \hbar$$

Method 2: They only ask for s-states, problem simplifies

Starting from Review 2. (copied here) *if it's @ detector*

$$\Psi(r, \theta) \xrightarrow{\text{large } r} e^{i\lambda r \cos \theta} + f(\theta) \frac{e^{i\lambda r}}{r} = \sum (2l+1) P_l(\cos \theta) \left(\frac{1}{2i\lambda r}\right) \left\{ e^{2i\delta_l} e^{i\lambda r} - e^{-i\lambda r - i\delta_l} \right\}$$

$$\text{let } l \rightarrow 0 \text{ s.t. } \Psi_{l=0}(r, \theta) \xrightarrow{\text{large } r} \frac{1}{2i\lambda r} \left\{ e^{2i\delta_0} e^{i\lambda r} - e^{-i\lambda r} \right\}$$

$$= \frac{e^{2i\delta_0}}{2i\lambda r} \left\{ e^{2i\delta_0} e^{i\lambda r} - e^{-2i\delta_0} e^{-i\lambda r} \right\} = \frac{e^{2i\delta_0}}{\lambda r} \sin(\lambda r + \delta_0)$$

since $A_0 = 1$ and $P_0(\cos \theta) = 1$,

$$R_{\text{tot}} \rightarrow \begin{cases} C \sin(\lambda r + \delta_0) / r & \text{for } r > a \\ 0 & \text{for } r < a \end{cases}$$

for $r \neq a$ and s states: $\frac{d^2 U(r)}{dr^2} = -k^2 U(r)$ w/ solns that vanish @ $r=0$ and $r \rightarrow \infty$

$$U(r) = \begin{cases} U_1(r) = 0 & r < a \\ U_2(r) = C \sin(\lambda r + \delta_0) & r > a \end{cases}$$

by continuity @ $r=a$, $\sin(\lambda a + \delta_0) = 0$

$$e^{i\lambda a} e^{i\delta_0} - e^{-i\lambda a} e^{-i\delta_0} = 0$$

$$e^{2i\delta_0} e^{i\lambda a} = e^{-i\lambda a}$$

$$\delta_0 = -\lambda a$$

(b) Total Cross-section for zero incident KE

$$\frac{d\sigma}{d\Omega} = |f(\theta, \phi)|^2$$

$$f(\theta) = \frac{1}{2} \sum (2l+1) e^{2i\delta_l} \sin \delta_l P_l(\cos \theta) \quad (\text{from Review 2})$$

$$f_{l=0}(\theta) = \frac{1}{2} e^{2i\delta_0} \sin \delta_0$$

$$\sigma_0 = \int \frac{d\sigma}{d\Omega} d\Omega = \frac{4\pi}{\lambda^2} \sin^2(\lambda a)$$

$$\lim_{E \rightarrow 0} \sigma_0 = \frac{4\pi}{\lambda^2} (\lambda a)^2 = 4\pi a^2$$