10. (Electromagnetism)

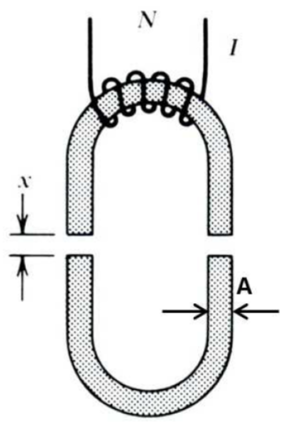
Consider an electromagnet with an iron core. Each segment of iron has length L, constant cross-sectional area A and permeability $\mu \gg \mu_0$ where μ_0 is the permeability of free space. The two halves of the magnet are separated by a small distance $x \ll L$. The magnet is powered by a coil of N turns carrying a constant current I.

(a) Determine the magnetic fields in the iron when $x = 0$ (the gap is closed).
Hint: since $\mu \gg \mu_0$, the magnetic field lines follow the shape of the iron (no magnetic flux leakage), and you can assume that the magnetic field strength is constant inside the iron core.

(b) Determine the fields H and B in the gap when x is non-zero, but very small, so that you can still assume that the magnetic field vanishes outside of the iron core and the small gap region.

(c) Determine the total magnetic field energy as a function of $x \ll L$.

(d) Calculate the force (magnitude and direction) between the two halves for vanishing small gap x.



Solution: *Solution by Jonah Hyman (jthyman@g.ucla.edu)*

This problem is a magnetostatics problem: You are given a steady current, and you need to solve for the magnetic field. Since we are given practically no information about the steady current or the geometry of the setup, the Biot-Savart law, finding the vector potential, and scalar potential theory won't work. That leaves Ampere's law as the only viable strategy for the problem.

In order to use Ampere's law, the setup must have quite a lot of symmetry. With this unusual geometry, you might think there isn't enough. However, the problem tells us that there is enough symmetry, under the guise of providing hints and assumptions:

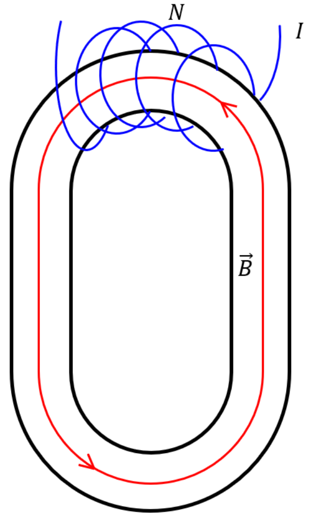
- In part (a), we are told that "the magnetic field strength is constant inside the iron core." Even though we are not explicitly told to use this assumption in part (b), we should do so because the problem is unsolvable otherwise.
- For part (b), we are told that the gap is of very small width $x \ll L$. Therefore, we can approximate the magnetic field strength in the gap by a constant value.
- In both parts (a) and (b), we are told that "the magnetic field lines follow the shape of the iron (no magnetic flux leakage)" and "the magnetic field vanishes outside of the iron core and the small gap region." This means that the magnetic field line at the edge of the electromagnet must be parallel to the edge (otherwise magnetic flux would "leak" out of the electromagnet). Since magnetic field lines never cross, all the magnetic field lines must therefore be parallel to the edge of the electromagnet. Similar logic applies even when the small gap is present in part (b).

As usual, we choose an Amperian loop that follows a magnetic field line of constant magnetic field strength. Here, that means a loop that parallels the edge of the electromagnet. We won't worry too much about the orientation of the Amperian loop, since we are only interested in the magnitude of the magnetic fields (and the problem doesn't tell us about the direction of the external current anyway).

Since we are dealing with magnetic materials, we should use the version of Ampere's law relating the $\mathbf{H}$ field to the *free* current $\mathbf{J}_f$:

$$\nabla \times \mathbf{H} = \mathbf{J}_f \Longleftrightarrow \int_{\text{loop}} \mathbf{H} \cdot d\boldsymbol{\ell} = I_{f,\text{encl}} \quad \text{where } \mathbf{B} = \mu \mathbf{H} \tag{270}$$

(a) Here is the setup for this part of the problem:



Each half of the electromagnet is of length L, so the circulation of $\mathbf{H}$ about this loop is

$$\int_{\text{loop}} \mathbf{H} \cdot d\boldsymbol{\ell} = H(2L) \tag{271}$$

The free current enclosed by the loop is provided by the coil. Each turn of wire has current I, and there are N turns, so

$$I_{f,\text{encl}} = NI \tag{272}$$

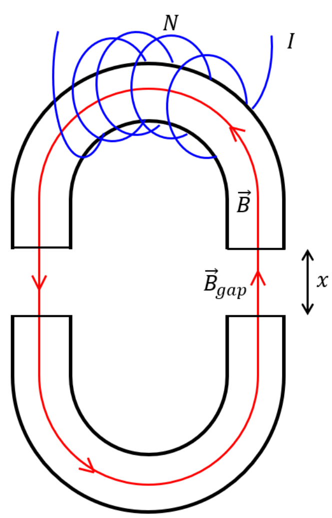
Setting (271) and (272) equal to one another by the integral version of Ampere's law, we can solve for H inside the electromagnet:

$$H(2L) = NI$$
$$H = \frac{NI}{2L} \tag{273}$$

The magnetic field inside the electromagnetic B is defined by $B = \mu H$, so we have our answer:

$$\boxed{H = \frac{NI}{2L} \quad \text{and} \quad B = \frac{\mu NI}{2L}} \tag{274}$$

(b) Here is the setup for this part of the problem:



Note that for this part of the problem, we expect both B and H to be different in the electromagnet and in the gap. We will use B and H to describe the magnetic field in the electromagnet, and we will use B_{gap} and H_{gap} to describe the magnetic field in the gap.

Each half of the electromagnet is of length $2L$, and the Amperian loop crosses the width-x gap twice, so the circulation of $\mathbf{H}$ about this loop is

$$\int_{\text{loop}} \mathbf{H} \cdot d\boldsymbol{\ell} = H(2L) + H_{\text{gap}}(2x) \tag{275}$$

The free current enclosed by the loop is provided by the coil. Each turn of wire has current I, and there are N turns, so

$$I_{f,\text{encl}} = NI \tag{276}$$

Setting (275) and (276) equal to one another by the integral version of Ampere's law, we can get an expression relating H to H_{gap}:

$$H(2L) + H_{\text{gap}}(2x) = NI \tag{277}$$

We now need another condition that relates H to H_{gap}. This is the boundary condition for the perpendicular component of the magnetic field at an interface:

$$B_{1,\perp} = B_{2,\perp} \quad \text{at the interface between media 1 and 2} \tag{278}$$

(This relation comes from the no-magnetic-monopoles rule $\nabla \cdot \mathbf{B} = 0$, as applied to a small "Guassian pillbox" at the interface.) Since $\mathbf{B} = \mu\mathbf{H}$, this becomes a condition on the perpendicular component of $\mathbf{H}$ at the interface:

$$\mu_1 H_{1,\perp} = \mu_2 H_{2,\perp} \quad \text{at the interface between media 1 and 2} \tag{279}$$

Here, the gap has magnetic permeability μ_0 and the electromagnet has magnetic permeability μ. The magnetic field lines are parallel to the edge of the electromagnet, so they are perpendicular to the interface between the electromagnet and the gap. (279) tells us that

$$\mu H = \mu_0 H_{\text{gap}} \tag{280}$$

To solve for H_{gap}, combine (277) with (280):

$$\begin{aligned}
H(2L) + H_{\text{gap}}(2x) &= NI \quad \text{by (277)} \\
\left(\frac{\mu_0}{\mu} H_{\text{gap}}\right)(2L) + H_{\text{gap}}(2x) &= NI \quad \text{using (280)} \\
H_{\text{gap}}\left[\frac{\mu_0}{\mu}(2L) + (2x)\right] &= NI \\
H_{\text{gap}} &= \frac{NI}{\frac{\mu_0}{\mu}(2L) + (2x)}
\end{aligned}$$

Since $B_{\text{gap}} = \mu_0 H_{\text{gap}}$,

$$\boxed{H_{\text{gap}} = \frac{\mu NI}{2\left(\mu_0 L + \mu x\right)} \quad \text{and} \quad B_{\text{gap}} = \frac{\mu_0 \mu NI}{2\left(\mu_0 L + \mu x\right)}} \tag{281}$$

Note that when $x = 0$, this reduces to the part (a) answer (274). Technically speaking, these expressions cannot be further simplified: We have both $\mu \gg \mu_0$ and $L \gg x$, but that does not imply anything about how $\mu_0 L$ compares to μx.

However, if you know something about iron-core electromagnets, you might be able to determine that $\mu x \gg \mu_0 L$. For iron, $\mu/\mu_0 \approx 5000$. Reasonable values for L/x are around $10 - 100$ (e.g. $L = 10$ cm and $x = 1$ mm), so we have $\mu/\mu_0 \gg L/x$, or $\mu x \gg \mu_0 L$. Using this approximation, we can get

$$\boxed{H_{\text{gap}} = \frac{NI}{2x} \quad \text{and} \quad B_{\text{gap}} = \frac{\mu_0 NI}{2x} \quad \text{if } \mu x \gg \mu_0 L} \tag{282}$$

(c) We will start by using the exact answer from part (b) (281). By (278), the magnetic field strength inside the electromagnet is the same as the magnetic field strength in the gap:

$$B = \frac{\mu_0 \mu NI}{2\left(\mu_0 L + \mu x\right)} \quad \text{in the electromagnet} \tag{283}$$

The magnetic field energy density in a linear isotropic homogeneous magnetic material is given by the formula

$$u_{\text{magnetic}} = \frac{B^2}{2\mu} \tag{284}$$

Note the use of μ rather than μ_0 to account for the presence of the magnetic material. Therefore, since the magnetic field strength has the same value B (given by (283) in the entire electromagnet and gap, we have

$$u_{\text{magnetic}} = \begin{cases} \frac{B^2}{2\mu} & \text{in the electromagnet} \\ \frac{B^2}{2\mu_0} & \text{in the gap} \end{cases} \tag{285}$$

Since the magnetic energy density is uniform in the electromagnet, we can just multiply the magnetic energy density in the electromagnet by the volume of the electromagnet, which is the cross-sectional area times the total length for both halves: $V_{\text{electromagnetic}} = 2LA$. This gets us the magnetic energy stored in the electromagnet

$$U_{\text{magnetic, electromagnet}} = \frac{B^2}{2\mu}(2LA) \tag{286}$$

Similarly, we can multiply the magnetic energy density in the gap by the volume of the gap $V_{\text{gap}} = 2Ax$ to get the magnetic energy stored in the gap

$$U_{\text{magnetic, gap}} = \frac{B^2}{2\mu_0}(2xA) \tag{287}$$

Adding together the two contributions to the total mangnetic energy, we get

$$\begin{aligned} U_{\text{magnetic}} &= U_{\text{magnetic, electromagnet}} + U_{\text{magnetic, gap}} \\ &= \frac{B^2}{2\mu}(2LA) + \frac{B^2}{2\mu_0}(2xA) \\ &= B^2 A\left(\frac{L}{\mu} + \frac{x}{\mu_0}\right) \\ &= \left(\frac{\mu_0 \mu N I}{2(\mu_0 L + \mu x)}\right)^2 A\left(\frac{L}{\mu} + \frac{x}{\mu_0}\right) \\ &= \frac{\mu_0^2 \mu^2 (NI)^2 A}{4(\mu_0 L + \mu x)^2}\left(\frac{L}{\mu} + \frac{x}{\mu_0}\right) \\ &= \frac{\mu_0 \mu (NI)^2 A}{4(\mu_0 L + \mu x)^2}(\mu_0 L + \mu x) \end{aligned}$$

$$\boxed{U_{\text{magnetic}} = \frac{\mu_0 \mu (NI)^2 A}{4(\mu_0 L + \mu x)}} \tag{288}$$

If we additionally make the approximation $\mu x \gg \mu_0 L$, discussed at the end of part (b), this answer simplifies to

$$\boxed{U_{\text{magnetic}} = \frac{\mu_0 (NI)^2 A}{4x} \quad \text{if } \mu x \gg \mu_0 L} \tag{289}$$

(d) Given the flow of this question, it is reasonable to expect that we should attempt to derive the force between the two halves using the stored magnetic energy as a function of x, which we found in part (c). This is indeed what we need to do, using the "method of virtual displacement." Before using this method, though, you must be warned about a common way to make sign errors:

Method of virtual displacement:
Consider an electromagnetic setup that depends on a one-dimensional displacement x. If $U(x)$ is the electromagnetic energy stored in the setup, then the (signed) force associated with this potential energy is

$$F = -\frac{dU}{dx} \quad \textbf{if the setup is } \textbf{\textit{not}} \textbf{ connected to a battery} \tag{290}$$

This is the formula most familiar from elementary mechanics. (As an example, for a spring with potential energy $U = \frac{1}{2}kx^2$, the corresponding signed force is $F_{\text{spr}} = -\frac{dU}{dx} = -kx$.) It reflects the fact that if a force does positive work on a system, the system moves from higher to lower potential energy, so the change in potential energy is negative.
However, if the setup *is* connected to a battery, the battery can do work on the system. It turns out that the work that the battery does on the system is exactly enough to make the net force on the system switch signs from (290):

$$F = +\frac{dU}{dx} \quad \textbf{if the setup } \textbf{\textit{is}} \textbf{ connected to a battery} \tag{291}$$

(This heuristic is helpful in practice for avoiding sign errors, but it is somewhat inaccurate. If the parameter x never changes, then the system never "knows" whether there is a battery or not. So the force on the system as a function of x can't directly depend on the presence or absence of a battery. A more correct statement is that if you take the derivative of U while holding constant variables that would only be constant if a battery were connected, then you need to use (291). Connecting a battery holds the voltage and current constant, so if you intend to hold those variables constant, then use (291).)

In this problem, we are assuming the current in the wire I is constant, so we must be assuming a battery is connected to the system. (If the system wasn't connected to a battery, changing the width of the gap x would change the circulation around the Amperian loop (275) identified in part (b), so by Ampere's law, the enclosed current NI would also have to change.) Therefore, we must use (291) (with the somewhat unusual plus sign) to get the signed force

$$\begin{aligned} F &= +\frac{dU_{\text{magnetic}}}{dx} \\ &= \frac{d}{dx}\left(\frac{\mu_0\mu\,(NI)^2\,A}{4(\mu_0 L + \mu x)}\right) \quad \text{by (288)} \\ &= -\frac{\mu_0(\mu NI)^2 A}{4\,(\mu_0 L + \mu x)^2} \end{aligned}$$

The minus sign means that the force is in the direction of *decreasing* x, meaning that the force is attractive: It tends to close the gap. We therefore have

$$\boxed{F = -\frac{\mu_0(\mu NI)^2 A}{4\,(\mu_0 L + \mu x)^2} \quad \text{(direction: attractive force between the halves)}} \tag{292}$$

If we additionally make the approximation $\mu x \gg \mu_0 L$, discussed at the end of part (b), this answer simplifies to

$$\boxed{F = -\frac{\mu_0(NI)^2 A}{4x^2} \quad \text{(direction: attractive force between the halves)} \qquad \text{if } \mu x \gg \mu_0 L} \tag{293}$$