Fall 2019
Physics Comprehensive Exam

September 16, 2019 (Part 1) 9:00 – 1:00pm

<u>Part 1: Quantum Mechanics and Classical Mechanics</u>

6 Total Problems/20 Points Each/Total 120 Points

- Closed book exam.
- Calculators not allowed.
- Begin your solution on the question page.
- Use paper provided for additional pages. **<u>Use one side only.</u>**
- Write your name on EACH of your response pages, including the question page.
- Return the question page as the first page of your answers.
- When submitting, please clip all pages together in question # order.
- If a part of any question seems ambiguous to you, state clearly your interpretations and answer the question accordingly.

Question 1: Quantum Mechanics

A particle of mass m in two dimensions is confined by an isotropic harmonic oscillator potential of frequency ω, while subject to a weak and anisotropic perturbation of strength $\alpha \ll 1$. The total Hamiltonian describing the motion of this particle is

$$H = H_0 + V = \frac{p_x^2}{2m} + \frac{p_y^2}{2m} + \frac{1}{2}m\omega^2(x^2 + y^2) + \alpha\, m\omega^2\, xy \tag{1}$$

a. (2 pts.) What are the energies and degeneracies of the three lowest-lying unperturbed states?

b. (5 pts.) Use perturbation theory to correct the energies to first order in α.

c. (5 pts.) Find the exact spectrum of H.

d. (4 pts.) Check that the perturbative results in part b. are recovered.

e. (4 pts.) Assume that 2 identical electrons are subject to the same anisotropic Hamiltonian (1). Write down the explicit wave-functions and degeneracies of the 2 lowest energy states.

Solution

a. The unperturbed states of H_0 are two dimensional harmonic oscillator states $|n_x n_y>$ with energies

$$E_{n_x n_y} = \hbar\omega(n_x + n_y + 1) \tag{2}$$

The three lowest energy states and their degeneracy d are

$$\begin{aligned} &|00> && \hbar\omega && d = 1 && (3)\\ &|01> && 2\hbar\omega && d = 2 && (4)\\ &|10> && 2\hbar\omega && d = 2 && (5) \end{aligned}$$

b. The state $|00>$ is non-degenerate. The energy shift is given by non-degenerate first order perturbation theory:

$$\Delta E_{00} =< 00|V|00 >= \alpha m\omega^2 < 0|x|0 >< 0|y|0 >= 0 \tag{6}$$

The states $|10>$ and $|01>$ are doubly degenerate with $d = 2$. Their energy shifts follow from degenerate perturbation theory. The interaction V in the degenerate subspace is

$$V_{2\times 2} = \begin{pmatrix} < 01|V|01 > & < 01|V|10 > \\ < 10|V|01 > & < 10|V|10 > \end{pmatrix} = \frac{1}{2}\alpha\hbar\omega \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{7}$$

Thus, the $2\hbar\omega$ states are split

$$\begin{aligned} &2\hbar\omega + \frac{1}{2}\alpha\hbar\omega && \frac{1}{\sqrt{2}}(|10> + |01>) && (8)\\ &2\hbar\omega - \frac{1}{2}\alpha\hbar\omega && \frac{1}{\sqrt{2}}(|10> - |01>) && (9)\\ &&&&& (10) \end{aligned}$$

c. By changing to the canonical variables: $X = (x + y)/\sqrt{2}$ and $Y = (x - y)/\sqrt{2}$ we can rewrite H

$$H = \frac{P_X^2}{2m} + \frac{P_Y^2}{2m} + \frac{1}{2}m\omega^2\left((1+\alpha)X^2 + (1-\alpha)Y^2\right) \tag{11}$$

and the exact spectrum is

$$E_{(n_X, n_Y)} = \hbar\omega\sqrt{1+\alpha}(n_X + \frac{1}{2}) + \hbar\omega\sqrt{1-\alpha}(n_Y + \frac{1}{2}) \tag{12}$$

d. The exact states in Taylor expansion are

$$E_{00} = \frac{1}{2}\hbar\omega\left(\sqrt{1+\alpha}+\sqrt{1-\alpha}\right) \approx \hbar\omega \tag{13}$$

$$E_{10} \approx \hbar\omega(1+\frac{\alpha}{2})(1+\frac{1}{2}) + \hbar\omega(1-\frac{\alpha}{2})(0+\frac{1}{2}) = 2\hbar\omega + \frac{1}{2}\alpha\hbar\omega \tag{14}$$

$$E_{01} \approx \hbar\omega(1+\frac{\alpha}{2})(0+\frac{1}{2}) + \hbar\omega(1-\frac{\alpha}{2})(1+\frac{1}{2}) = 2\hbar\omega - \frac{1}{2}\alpha\hbar\omega \tag{15}$$

in agreement with perturbation theory.

e. Let $\varphi_{1,2}(x)$ be the wave function associated with the deformed Hamiltonian (1) with the 2 lowest eigenvalues $\hbar\omega$ and $2\hbar\omega - \alpha\hbar\omega/2$ respectively. For 2 identical electrons the two lowest energy states are

$$d = 1 \qquad E = 2\hbar\omega : \qquad \varphi_1(1)\varphi_1(2)\frac{1}{\sqrt{2}}(|\uparrow\downarrow> - |\downarrow\uparrow>) \tag{16}$$

$$\begin{aligned} & d = 4 \qquad E = 3\hbar\omega - \frac{1}{2}\alpha\hbar\omega : \\ & \frac{1}{\sqrt{2}}(\varphi_1(1)\varphi_2(2) + \varphi_2(1)\varphi_1(2))\frac{1}{\sqrt{2}}(|\uparrow\downarrow> - |\downarrow\uparrow>) \\ & \frac{1}{\sqrt{2}}(\varphi_1(1)\varphi_2(2) - \varphi_2(1)\varphi_1(2))\left(|\uparrow\uparrow>, \frac{1}{\sqrt{2}}(|\uparrow\downarrow> + |\downarrow\uparrow>), |\downarrow\downarrow>\right) \end{aligned} \tag{17}$$

Question 2: Quantum Mechanics

An electron of charge e and mass m_e is subject to a uniform magnetic field $B_0\hat{z}$ and has its spin along the positive z-axis. At $t = 0$ an additional time-dependent magnetic field is switched on in the transverse plane with

$$B_{\perp}(\cos(\omega t)\hat{x} + \sin(\omega t)\hat{y} \tag{1}$$

a) Write down the Schrödinger equation for this time-dependent problem and solve it.

b) What is the probability in time to find the electron with its spin along the negative z-axis, and for what frequency is the spin flip maximum?

(Assume zero orbital momentum)

Solution

a. The Schrödinger equation in the spin-$\frac{1}{2}$ basis is

$$i\hbar\frac{\partial|\Psi(t)\rangle}{\partial t} = -\frac{e\hbar}{2m_e}\begin{pmatrix} B_0 & B_\perp e^{-i\omega t} \\ B_\perp e^{i\omega t} & -B_0 \end{pmatrix}|\Psi(t)\rangle \tag{3}$$

with the initial condition $|\Psi(0)\rangle = |\uparrow\rangle$. The solution is

$$|\Psi(t)\rangle = \begin{pmatrix} e^{-i\omega t/2}(\cos(\gamma t/2) + i((\omega - 2\omega_0)/\gamma)\sin(\gamma t/2) \\ -2ie^{i\omega t/2}(\omega_\perp/\gamma)\sin(\gamma t/2) \end{pmatrix} \tag{4}$$

with $\omega_0 = |e|B_0/2m_e$ and $\omega_\perp = |e|B_\perp/2m_e$ and

$$\gamma = \left((\omega - 2\omega_0)^2 + 4\omega_\perp^2\right)^{\frac{1}{2}} \tag{5}$$

b. The spin flip probability in time is

$$P_\downarrow(t) = \frac{4\omega_\perp^2}{(\omega - 2\omega_0)^2 + 4\omega_\perp^2}\sin^2\left(\frac{\gamma t}{2}\right) \tag{6}$$

The spin flip is resonant or maximum for $\omega = 2\omega_0$.

Question 3: Quantum Mechanics

A particle in an infinite cubic well.

(a) Find the exact energies and wave functions of the ground and the first excited states and specify the degeneracies for the infinite cubic potential.

$$V(x,y,z) = \begin{cases} 0 & 0 < x < L,\, 0 < y < L,\, 0 < z < L \\ \infty & \text{otherwise} \end{cases}$$

Now add the perturbation to the infinite cubic well:

$$H_p = V_0 L^3 \delta(x - L/4)\delta(y - 3L/4)\delta(z - L/4)$$

(b) Using the first order perturbation theory, calculate the energy of the groundstate.

(c) Using first order degenerate perturbation theory, calculate the energy of the first excited state.

1. (a) Find the exact energies and wave functions of the ground ~~st~~ and first excited states and specify their degeneracies for the infinite cubic potential well

$$V(x,y,z) = \begin{cases} 0 & \text{if } 0<x<L,\ 0<y<L,\ 0<z<L \\ \infty & \text{otherwise} \end{cases}$$

Now add the perturbation to the infinite cubic well:

$$H_p = V_0 L^3 \delta\left(x-\frac{L}{4}\right)\delta\left(y-\frac{3L}{4}\right)\delta\left(z-\frac{L}{4}\right)$$

(b) Using first order perturbation theory, calculate the energy of the ground state.

(c) using first order degenerate perturbation theory, calculate the energy of the first excited state

(a) $E^{exact}_{111} = \frac{3\pi^2\hbar^2}{2mL^2}$, $\phi_{111}(x,y,z) = \sqrt{\frac{8}{L^3}}\sin\left(\frac{\pi x}{L}\right)$

$\sin\left(\frac{\pi y}{L}\right)\sin\left(\frac{\pi z}{L}\right)$, etc.

(b) $E^{(1)}_1 = V_0$

first order correction to the ground state energy.

Note that the exact first excited state is three fold degenarate : E_112= E_121=E_211 in an obvious notation: phi_(n_x, n_y, n_z)(x,y,z).

(c) To calculate the energy of the first excited state form the matrix

$$\begin{pmatrix} V_{11} & V_{12} & V_{13} \\ V_{21} & V_{22} & V_{23} \\ V_{31} & V_{32} & V_{33} \end{pmatrix}$$

We get

$$V = 2V_0 \begin{pmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix}$$

$$\begin{vmatrix} 1-\lambda & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{vmatrix} = \lambda^2(\lambda-3)$$

$$E_1^{(1)} = E_2^{(1)} = 0 \quad , \quad E_3^{(1)} = 6V_0$$

$$E_1 = E_2 = \frac{3\pi^2\hbar^2}{mL^2} \quad , \quad E_3 = \frac{3\pi^2\hbar^2}{mL^2} + 6V_0$$

The degeneracy is partially lifted in first order.

Question 4: Quantum Mechanics

Find the differential and total cross sections of slow particles (small velocity) from a spherical delta potential $V(r) = V_0\delta(r - a)$. You may use partial wave analysis.

2. Find the differential and total cross sections from a spherical delta potential $V(r) = V_0 \delta(r-a)$. You may use partial wave analysis.

When the incident particles have small velocities, only the s-waves, $l=0$ contribute. The differential and total cross sections are given for $l=0$ by

$$\frac{d\sigma}{d\Omega} = |f_0|^2 = \frac{\sin^2\delta_0}{k^2}, \quad \sigma = \frac{4\pi}{k^2}\sin^2\delta_0$$

Find the phase shift δ_0.

$$-\frac{\hbar^2}{2m}\frac{d^2u(r)}{dr^2} + \left[V_0\delta(r-a) + \frac{l(l+1)}{2mr^2}\right]u(r) = Eu(r)$$

where $u(r) = rR(r)$. For s-state

$$\frac{d^2u(r)}{dr^2} = -k^2u(r), \quad k^2 = \frac{2mE}{\hbar^2}$$

acceptable solutions are

$$u(r) = \begin{cases} u_1(r) = A\sin kr, & 0<r<a \\ u_2(r) = B\sin(kr+\delta_0), & r>a \end{cases}$$

continuity gives

$$B\sin(ka+\delta_0) = A\sin(ka)$$

On the other hand,

$$\left.\frac{du_2(r)}{dr}\right|_{r=a} - \frac{du_1(r)}{dr} - \frac{2mV_0}{\hbar^2}u_2(a) = 0$$

Therefore

$$k\cot(ka+\delta_0) - \frac{2mV_0}{\hbar^2} = k\cot(ka)$$

$$\tan(ka+\delta_0) = \frac{1}{\tan ka + \frac{2mV_0}{k\hbar^2}}$$

If the incident particles have small velocities $ka \ll 1$ and $\tan ka \simeq ka$, $\tan(ka+\delta_0) \approx \tan\delta_0$

$$\tan\delta_0 \simeq \frac{ka}{1+2mV_0a/\hbar^2}, \quad \sin^2\delta_0 \simeq \frac{k^2a^2}{k^2a^2+(1+2mV_0/\hbar^2)^2}$$

$$\frac{d\sigma}{d\Omega} \simeq \frac{a^2}{k^2a^2+(1+2mV_0/\hbar^2)^2}$$

$$\sigma_0 \simeq \frac{4\pi a^2}{k^2a^2+(1+2mV_0/\hbar^2)^2}$$

Question 5: Classical Mechanics

A non-relativistic particle with mass m and electric charge e moves in a two-dimensional plane (with Cartesian coordinates x, y), under the influence of a constant uniform magnetic field pointing in the z-direction, $\mathbf{B} = (0, 0, B)$, and an *inverted* harmonic oscillator potential,

$$V(x, y) = -\frac{1}{2}m\omega^2(x^2 + y^2)$$

a) Write down the Lagrangian for this system in xy-variables.
b) Derive and solve the corresponding Euler-Lagrange equations.
c) Discuss the stability of motion near the point $(x, y) = (0, 0)$ as a function of m, e, ω, B.

a) The Lagrangian is given by the coupling of a particle of mass m and electric charge e to a constant magnetic field B perpendicular to the xy-plane,

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + e\mathbf{A} \cdot \dot{\mathbf{x}} - V(x, y)$$

We choose the gauge $A_x = -\frac{1}{2}By$ and $A_y = +\frac{1}{2}Bx$, so that the Lagrangian is given by,

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}eB(x\dot{y} - y\dot{x}) + \frac{1}{2}m\omega^2(x^2 + y^2)$$

b) The Euler-Lagrange equations are given by,

$$\begin{aligned} m\ddot{x} - eB\dot{y} - m\omega^2 x &= 0 \\ m\ddot{y} + eB\dot{x} - m\omega^2 y &= 0 \end{aligned}$$

c) This system of equations is linear with constant coefficients, so the solutions are,

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} e^{\lambda t} \qquad\qquad \begin{pmatrix} m(\lambda^2 - \omega^2) & -eB\lambda \\ +eB\lambda & m(\lambda^2 - \omega^2) \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = 0$$

where x_0 and y_0 are independent of t. The four eigenvalues λ solve the quartic equation,

$$\left(\left(\lambda - \frac{ieB}{2m} \right)^2 + \frac{e^2B^2}{4m^2} - \omega^2 \right) \left(\left(\lambda + \frac{ieB}{2m} \right)^2 + \frac{e^2B^2}{4m^2} - \omega^2 \right) = 0$$

When $\omega^2 < e^2B^2/4m^2$, the four eigenvalues are purely imaginary, all solutions are oscillatory and thus "stable". When $\omega^2 > e^2B^2/4m^2$ there are two a run-away mode which drives the particle indefinitely off the potential hill, as would be the case for zero magnetic field, and the system is "unstable". A sufficiently strong magnetic field has a stabilizing effect.

Question 6: Classical Mechanics

Consider a particle of mass ‘m’ moving in a plane under the action of a central force with potential U(r).

Write down the generating function of a canonical transformation the description of the motion to a reference frame rotating counter clockwise at an angular frequency Ω . Write down the new Hamiltonian and the relationship between the new and old coordinates and momenta. Write down the equations of motion in the rotating frame.

Consider a particle of mass ‘m’ moving in a plane under the action of a central force with potential U(r).

Write down the generating function of a canonical transformation the description of the motion to a reference frame rotating counter clockwise at an angular frequency Ω . Write down the new Hamiltonian and the relationship between the new and old coordinates and momenta. Write down the equations of motion in the rotating frame.

$$H = \frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} + U(r)$$

In the rotating frame the relation between new and old coordinates is:
$R = r;\ \Theta = \theta - \Omega t$

Take for the form of the generating function $F(r, \theta; P_R, P_\Theta)$

$$\partial F / \partial P_\Theta = \Theta = \theta - \Omega t$$

$$F = P_\Theta(\theta - \Omega t) + rP_R$$

$$\partial F / \partial \theta = p_\theta = P_\Theta;\ \partial F / \partial r = p_r = P_R.$$

$$H' = H + \partial F / \partial t = H - \Omega P_\Theta$$

$$H = \frac{P_R^2}{2m} + \frac{P_\Theta^2}{2mR^2} + U(R) - \Omega P_\Theta$$

$$\dot{\Theta} = \frac{P_\Theta}{mR^2} - \Omega;\ \dot{P}_\Theta = 0$$

$$\dot{P}_R = -\frac{\partial U}{\partial R} + \frac{P_\Theta^2}{mR^3};\ \dot{R} = P_R / m$$

Fall 2019
Physics Comprehensive Exam

September 17, 2019 (Part 2) 9:00 – 1:00pm

Part 2: Electromagnetism and Statistical Mechanics

6 Total Problems/20 Points Each/Total 120 Points

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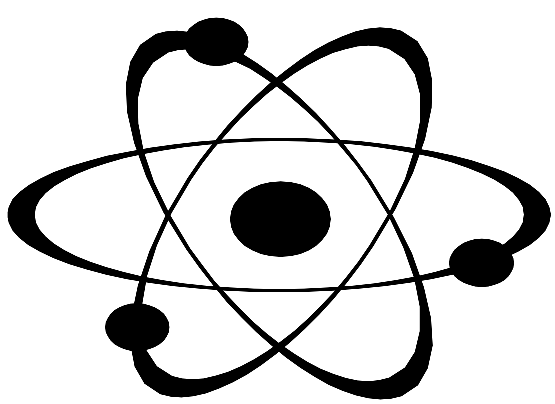
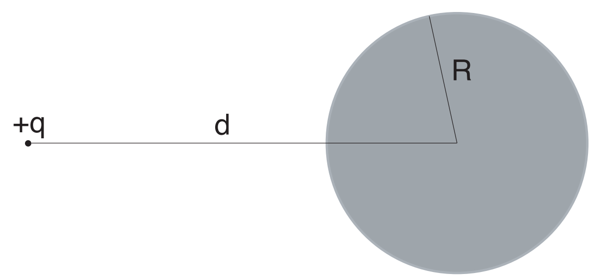


Question 7: Electromagnetism

A point charge q is located a distance d from the center of a conducting sphere of radius R. What must the total charge on the conducting sphere be for the force on the point charge to be zero?



Solution

Image charge solution. There are two images, one located a distance $d' = R^2/d$ from the center of the sphere with $q' = -qR/d$, and one at the center of the sphere (q'').

The force on the point charge is zero if:

$$\frac{qR/d}{(d - R^2/d)^2} = \frac{q''}{d^2}$$

Or

$$q'' = \frac{qRd^3}{(d^2 - R^2)^2}$$

The total charge on the sphere is then:

$$Q = q'' - \frac{qR}{d} = \frac{qR}{d}\left(\frac{d^4}{(d^2 - R^2)^2} - 1\right)$$

Question 8: Electromagnetism

An electron is bound to a spring with spring constant k, the electron is free to move in three dimensions.

(a) Calculate the scattering cross section for linearly-polarized EM waves of frequency ω incident on the electron.

(b) In what limit for the incident wave frequency should the total cross section equal the Thomson scattering cross section? Take this limit and confirm that it results in the Thomson scattering cross section.

(c) In what limit should the total cross section yield Rayleigh scattering? Take this limit and confirm that the cross section is consistent with Rayleigh scattering (what is the frequency dependence you expect?)

(a) SOLUTION: Without loss of generality, I can consider a plane wave with polarization in the $\hat{x}$ direction. The equation of motion for the electron in the plane wave is then:

$$m\dot{v}_x = -kx - eE_o \exp(-i\omega t)$$

We'll assume that we have time harmonic plane waves and that the response of the electron in the long time limit is at the same frequency of the wave (strictly speaking we'd need some damping here to kill off the initial transient at the resonant frequency, but we can consider this as the limit of very small damping coefficient). Then the solution for the position of the electron as a function of time is:

$$x = \frac{-eE_o \exp(-i\omega t)}{m(\omega_o^2 - \omega^2)}$$

This gives us the oscillating dipole moment of the electron, which we can use to compute the time-averaged power radiated by the electron:

$$\langle P_{rad} \rangle = \left\langle \frac{\mu_o}{6\pi c} |\ddot{\mathbf{p}}|^2 \right\rangle = \frac{\mu_o e^4 E_o^2 \omega^4}{6\pi m^2 c \left(\omega_o^2 - \omega^2\right)^2} \left\langle \cos^2(\omega t) \right\rangle = \frac{\mu_o e^4 E_o^2 \omega^4}{12\pi m^2 c \left(\omega_o^2 - \omega^2\right)^2}$$

The total scattering cross section is defined as:

$$\sigma = \frac{\langle P_{rad} \rangle}{\langle S_{\text{inc}} \rangle}$$

Where $\langle S_{inc} \rangle$ is the time averaged Poynting flux of the incident wave:

$$\langle S_{inc} \rangle = \frac{1}{2} c \epsilon_o E_o^2$$

So the cross section is then:

$$\sigma = \frac{\mu_o e^4 \omega^4}{6\pi \epsilon_o m^2 c^2} \frac{1}{\left(\omega_o^2 - \omega^2\right)^2} = \frac{e^4 \omega^4}{6\pi \epsilon_o^2 m^2 c^4} \frac{1}{\left(\omega_o^2 - \omega^2\right)^2}$$

We can simplify this by using the classical electron radius:

$$r_e = \frac{e^2}{4\pi \epsilon_o m c^2}$$

Then the cross section becomes:

$$\sigma = \frac{8\pi r_e^2}{3} \frac{\omega^4}{\left(\omega_o^2 - \omega^2\right)^2}$$

(b) SOLUTION: The correct limit is $\omega \gg \omega_o$, where ω_o is the resonant frequency of the electron-spring system. In this limit, the electron behaves like a free particle. Taking this limit in our expression above yields:

$$\langle\sigma\rangle_{\text{unpol}} \xrightarrow[\omega\gg\omega_o]{} \frac{8\pi}{3} r_e^2$$

Which is the Thomson scattering cross-section.

(c) SOLUTION: The correct limit is $\omega \ll \omega_o$, where ω_o is the resonant frequency of the electron-spring system. In this limit, you find an ω^4 dependence in the scattering cross-section (leading to the blue sky explanation):

$$\langle\sigma\rangle_{\text{unpol}} \xrightarrow[\omega\ll\omega_o]{} \frac{8\pi}{3} r_e^2 \frac{\omega^4}{\omega_o^4}$$

Question 9: Electromagnetism

Consider a dielectric slab waveguide, i.e. a dielectric volume of index of refraction n_1 delimited by two planes $-a \leq x \leq a$ and infinitely wide in the other two directions surrounded by dielectric of index of refraction n_2 ($\mu_1 = \mu_2 = \mu_0$). Study the propagation of transverse electric (TE) waves in the z-direction in this system (i.e, assume $\boldsymbol{E}(x, z, t) = E(x)e^{i(hz-\omega t)}\boldsymbol{y}$).

a) Write down the wave equation for *E(x)* in each region of the slab.
b) Look for even solutions (i.e. invariant for x → -x) with fields decaying outside the guide. Apply the boundary conditions at the interfaces to obtain expressions for electric and magnetic fields in the guide.
c) Calculate the cut-off frequencies in this guide (i.e. the frequencies for which the wave is no longer guided by the dielectric slab). What is the lowest frequency that can propagate in this guide?

Start from Maxwell wave equation in homogeneous media

$$\nabla^2 \vec{E} - \varepsilon\mu\omega^2 \vec{E} = 0$$

$$\nabla^2 \vec{E} - n^2 \frac{\omega^2}{c^2} \vec{E} = 0$$

Substitute to get

$$\left(-\frac{d^2}{dx^2} + \left(h^2 - \frac{n^2\omega^2}{c^2}\right)\right) E(x) = 0$$

Solutions with even parity propagating in slab

$$E(x) = E_1 \cos \gamma_1 x$$

$$\gamma_1^2 = \frac{n_1^2\omega^2}{c^2} - h^2 \qquad -a<x<a$$

$$= E_2 e^{-\gamma_2 |x|} \qquad |x|>a$$

$$\gamma_2^2 = h^2 - \frac{n_2^2\omega^2}{c^2}$$

$$E_{t,1} = E_{t,2}$$

$$E_1 \cos \gamma_1 a = E_2 e^{-\gamma_2 a}$$

Magnetic field $\quad \vec{H} = \frac{1}{i\omega\mu_0} \vec{\nabla} \times \vec{E}$

$$H_z = \frac{1}{i\omega\mu_0} \frac{\partial}{\partial x} E(x) e^{i hz - \omega t}$$

$$= \frac{-1}{i\omega\mu_0} \gamma_1 E_1 \sin \gamma_1 a \qquad -a< x<a$$

$$= \frac{-i}{i\omega\mu_0} \gamma_2 E_2 e^{-\gamma_2 x} \qquad |x|>a$$

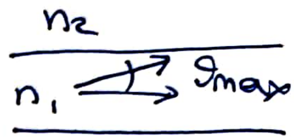
$$\tan \gamma_1 a = \frac{\gamma_2}{\gamma_1}$$

$$\Rightarrow \gamma_1 E_1 \sin \gamma_1 a = \gamma_2 E_2 e^{-\gamma_2 a}$$

$$H_{t,1} = H_{t,2}$$

Cutoff $\Rightarrow \gamma_2 = 0$ Non confined modes

$$\Rightarrow \gamma_1 a = m\pi$$



$$\gamma_1^2 + \gamma_2^2 = (n_1^2 - n_2^2)\frac{\omega^2}{c^2}$$

$$\omega_{cutoff}^2 = \frac{\gamma_1^2 c^2}{n_1^2 - n_2^2} = \frac{m^2\pi^2 c^2}{a^2(n_1^2 - n_2^2)}$$

at cutoff $\quad k_z^2 = h^2 = \dfrac{n_2^2\omega^2}{c^2}$

$$k_x^2 = \gamma_1^2 = (n_1^2 - n_2^2)\frac{\omega^2}{c^2}$$

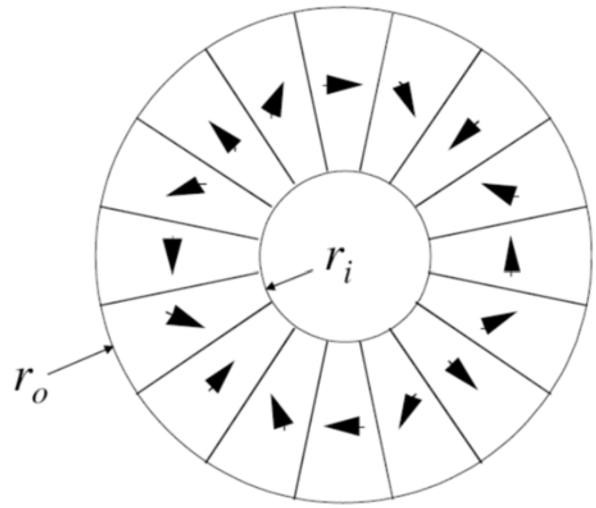
$$k^2 \sin^2\vartheta = (n_1^2 - n_2^2)\frac{\omega^2}{c^2}$$

$$\Rightarrow \sin^2\vartheta = \left(1 - \frac{n_2^2}{n_1^2}\right) \quad \Rightarrow \cos\vartheta_{max} = \frac{n_2}{n_1}$$

Total internal reflection condition

Question 10: Electromagnetism

A *Halbach quadrupole* is made by assembling segmented permanent magnet pieces where the magnetization vector is rotated through 6π as one travels around the azimuth, as shown below.



In order to calculate the magnetic field of this magnet, approximate the magnetization vector as a continuous function of φ as follows

$$\boldsymbol{M} = M_0(-\boldsymbol{\rho}\sin(2\varphi) + \boldsymbol{\varphi}\cos(2\varphi))$$

for $r_i < \rho < r_o$ and zero elsewhere. Assume the magnet to be infinitely long in the z direction.

a) Calculate the magnetization currents.
b) Calculate the magnetic field in the vicinity of the axis (i.e. for $\rho < r_i$)
c) Calculate the magnetic field outside the quadrupole (i.e. for $\rho > r_o$)

Hint: In order to solve this problem calculate first the magnetic field due to an azimuthal current sheet $\boldsymbol{K} = K_0 \sin 2\varphi \ \boldsymbol{\varphi}$ located at $\rho = a$ and recall the general solution of the Laplace equation for the magnetic scalar potential in 2D polar coordinates.

$$\vec{M} = M_0\left(-\hat{\rho}\sin 2\varphi + \hat{\varphi}\cos 2\varphi\right)$$

Magnetization currents

$$\vec{J}_M = \vec{\nabla}\times\vec{M} = \frac{M_0}{\rho}\left(\cos 2\varphi + 2\cos 2\varphi\right)\hat{z} = \frac{3M_0\cos 2\varphi}{\rho}\hat{z}$$

$$\text{from } \left(\vec{\nabla}\times\vec{M}\right)_z = \frac{1}{\rho}\left[\frac{\partial}{\partial\rho}\rho A_\varphi - \frac{\partial A_\rho}{\partial\varphi}\right]$$

$$\vec{K} = \begin{matrix} +M_0\cos 2\varphi\,\hat{z} & \rho = r_i \\ -M_0\cos 2\varphi\,\hat{z} & \rho = r_0 \end{matrix}$$

Now use hint:

assume $\vec{K} = K_0\cos 2\varphi \quad \rho = a$

Use multipole expansion for scalar magnetic potential

$$\psi_m = \begin{cases} A_2 \frac{\rho^2}{a^2}\sin 2\varphi & \text{inside } \rho < a \\ B_2 \frac{a^2}{\rho^2}\sin 2\varphi & \text{outside } \rho > a \end{cases}$$

and use boundary $B_{in,n} = B_{out,n}$

$$\left(\vec{B}_{out} - \vec{B}_{in}\right)\times\hat{n} = \mu_0 \vec{K}$$

$$\vec{B}_{in} = \hat{\rho}\frac{2A_2\rho}{a^2}\sin 2\varphi + \hat{\varphi}\frac{2A_2}{\rho}\frac{\rho^2}{a^2}\cos 2\varphi$$

$$\vec{B}_{out} = \hat{\rho}\frac{-2B_2 a^2}{\rho^3}\sin 2\varphi + \hat{\varphi}\frac{2B_2 a^2}{\rho^3}\cos 2\varphi$$

$$B_{in,\hat{\rho}} = B_{out,\hat{\rho}} \Rightarrow A_2 = -B_2$$

$$\left(B_{\varphi,out} - B_{\varphi,in}\right) = \mu_0 K \Rightarrow -2A_2 - 2A_2 = \mu_0 K a$$

$$A_2 = -\frac{\mu_0 K a}{4}$$

Finally, integrate over all shells to find out total magnetic field

$$\vec{B}_{in} = \frac{-\mu_0 K}{2a} \rho \left(\sin 2\varphi \, \hat{\rho} + \cos 2\varphi \, \hat{\varphi} \right)$$

$$\vec{B}_{in,\,total} = \int_{r_i}^{r_0} da \left(-\frac{\mu_0}{2} \frac{3M_0}{a^2} \right) - \frac{\mu_0 M_0}{2r_i} + \frac{\mu_0 M_0}{2r_0}$$

$$\frac{3}{2}\mu_0 M_0 \left(\frac{1}{r_0} - \frac{1}{r_i} \right) - \frac{\mu_0 M_0}{2r_i} + \frac{\mu_0 M_0}{2r_0} = 2\mu_0 M_0 \left(\frac{1}{r_i} - \frac{1}{r_0} \right)$$

$$\vec{B}_{out} = \frac{-\mu_0 K a^3}{2} \left(\frac{\hat{\rho} \sin 2\varphi}{\rho^3} + \frac{\hat{\varphi} \cos 2\varphi}{\rho^3} \right)$$

$$\vec{B}_{out,\,total} = \int_{r_i}^{r_0} \frac{3M_0}{\cancel{a}} \cdot \frac{-\mu_0 a^{\cancel{3}}}{2} da - \frac{\mu_0 M_0}{2} \left(r_i^3 - r_0^3 \right)$$

$$= \frac{-\mu_0 M_0}{2} \left(r_0^3 - r_i^3 \right) - \frac{\mu_0 M_0}{2} \left(r_i^3 - r_0^3 \right) = 0$$

Question 11: Statistical Mechanics

Consider an ideal gas of N_{tot} particles in a box of volume V_{tot}, in the classical regime. We focus on a fixed sub-volume, perhaps in a corner of the box, of volume $V << V_{tot}$. On average, it contains $\langle N \rangle = N_{tot}V/V_{tot}$ particles.

Derive an expression for the probability that the sub-volume contains exactly N particles, in terms of the parameters of the problem. Also, find an expression for the relative rms fluctuations of N, i.e. for the quantity $\langle (N - \langle N \rangle)^2 \rangle^{1/2} / \langle N \rangle$.

1) In the thermodynamic limit (large N_{tot}), with respect to the volume V, the rest of the box constitutes a "particle reservoire". For the classical ideal gas, the partition sum in the grand canonical ensemble is =

$$Q = \sum_{N=0}^{\infty} \sum_{E} e^{-(E-\mu N)/T} \quad ; \quad E(N) = \sum_{i=1}^{N} \frac{|\vec{p}_i|^2}{2m}$$

$$= \sum_{N=0}^{\infty} e^{\mu N/T} \underbrace{\sum_{\substack{\text{states with} \\ N \text{ particles}}} e^{-E(N)/T}}_{= Z_N}$$

canonical part. sum for the ideal gas

$$Z_N = \frac{1}{N!} Z_1^N \, , \quad Z_1 = \frac{V}{\lambda^3}$$

λ thermal wave length

$$\Rightarrow \quad Q = \sum_{N=0}^{\infty} \frac{1}{N!} \left(e^{\mu/T} \frac{V}{\lambda^3} \right)^N = \exp\left\{ e^{\mu/T} \frac{V}{\lambda^3} \right\}$$

From these expressions you see that:

$$P(N) = \frac{1}{N!} \left(e^{\mu/T} \frac{V}{\lambda^3} \right)^N (1/Q)$$

prob. of having N particles

$$\langle N \rangle = T \frac{\partial}{\partial \mu} \ln Q = e^{\mu/T} \frac{V}{\lambda^3}$$

$$\Rightarrow \quad Q = e^{\langle N \rangle}$$

$$\Rightarrow \quad \boxed{P(N) = \frac{1}{N!} e^{-\langle N \rangle} \langle N \rangle^N}$$

a Poisson distrib.

For us, $\boxed{\langle N \rangle = N_{tot} \frac{V}{V_{tot}}}$.

Further, $\langle N^2 \rangle = \frac{T^2}{Q} \frac{\partial^2 Q}{\partial \mu^2}$

$$\frac{\partial Q}{\partial \mu} = \frac{1}{T} Q \frac{V}{\lambda^3} e^{\mu/T} , \quad \frac{\partial^2 Q}{\partial \mu^2} = \frac{1}{T} \frac{V}{\lambda^3} e^{\mu/T} \frac{1}{T} \frac{V}{\lambda^3} e^{\mu/T}$$

$$+ \frac{1}{T^2} Q \frac{V}{\lambda^3} e^{\mu/T} \Rightarrow \frac{T^2}{Q} \frac{\partial^2 Q}{\partial \mu^2} = \left(\frac{V}{\lambda^3} e^{\mu/T} \right)^2 + \frac{V}{\lambda^3} e^{\mu/T}$$

$$\Rightarrow \langle N^2 \rangle = \langle N \rangle^2 + \langle N \rangle$$

So $\langle (N - \langle N \rangle)^2 \rangle = \langle N^2 \rangle - \langle N \rangle^2 = \langle N \rangle$

and $\frac{\langle (N - \langle N \rangle)^2 \rangle^{1/2}}{\langle N \rangle} = \frac{1}{\sqrt{\langle N \rangle}}$.

(relative width of the Poisson distribution)

Question 12: Statistical Mechanics

Consider a system of electrons whose density of states per unit volume (for one electron) is $D(\varepsilon)$ as a function of energy ε. We shall assume $D(\varepsilon) = 0$ for all $\varepsilon < 0$. The system is in equilibrium at arbitrary temperature T and chemical potential μ.
(a) Derive a formula for the specific heat at constant volume C_V in terms of the Fermi-Dirac occupation number and the density of states $D(\varepsilon)$.
(b) For arbitrary $D(\varepsilon)$ obtain C_V to leading non-zero order for strong degeneracy.
[Hint: Your result will be a function of the temperature, the volume, and the density of states at the Fermi energy.]
(c) Compute $D(\varepsilon)$ for a system of free non-relativistic electrons.
(d) Obtain C_V for a system of free non-relativistic electrons at low temperature.
The following integral may be useful,

$$\int_{-\infty}^{\infty} dx\, \frac{x^2}{(e^x + e^{-x})^2} = \frac{\pi^2}{24}$$

(a) In terms of the one-electron density of states per unit volume $D(\varepsilon)$ and the Fermi-Dirac occupation number the total number of electrons N and the internal energy E are,

$$N = V\int_0^\infty d\varepsilon \, \frac{D(\varepsilon)}{e^{\beta(\varepsilon-\mu)}+1} \qquad\qquad \beta = \frac{1}{kT}$$

$$E = V\int_0^\infty d\varepsilon \, \frac{\varepsilon\, D(\varepsilon)}{e^{\beta(\varepsilon-\mu)}+1}$$

Since the function $D(\varepsilon)$ is independent of temperature (but may involve V which is held constant in computing C_V), the specific heat is obtained by differentiating the Fermi-Dirac occupation number and we find,

$$C_V = \left.\frac{\partial E}{\partial T}\right|_{V,\mu} = \frac{V}{kT^2}\int_0^\infty d\varepsilon \, \frac{\varepsilon(\varepsilon-\mu)D(\varepsilon)}{\left(e^{\beta(\varepsilon-\mu)/2}+e^{-\beta(\varepsilon-\mu)/2}\right)^2}$$

(b) Strong degeneracy corresponds to low temperatures. The denominator is then responsible for concentrating the support of the integral over ε near μ, so we may extend the integration region to $-\infty$ and, to leading order, evaluate $D(\varepsilon)$ at $\varepsilon = \mu$. The parity of the remaining integral allows us to replace the factor ε in the numerator by $\varepsilon - \mu$, so that we end up with the following expression,

$$C_V = \frac{V D(\mu)}{kT^2}\int_{-\infty}^\infty d\varepsilon \, \frac{(\varepsilon-\mu)^2}{\left(e^{\beta(\varepsilon-\mu)/2}+e^{-\beta(\varepsilon-\mu)/2}\right)^2}$$

Changing variables from ε to x with $\varepsilon = \mu + 2kTx$ gives,

$$C_V = 8k^2 T V D(\mu)\int_{-\infty}^\infty dx \, \frac{x^2}{(e^x + e^{-x})^2}$$

Using the value of the integral stated in the problem, and the fact that for small temperatures we have $\mu = \mu_0 + \mathcal{O}(T^2)$, with μ_0 defined by,

$$N = V\int_0^{\mu_0} d\varepsilon\, D(\varepsilon)$$

we approximate this result by setting $D(\mu) = D(\mu_0)$, so that the final result is given by,

$$C_V = \frac{1}{3}\pi^2 k^2 T V D(\mu_0)$$

(c) The density of states for a non-relativistic free particle is given by

$$g\int \frac{d^3p\, d^3q}{(2\pi\hbar)^3} f(\varepsilon = p^2/2m) = \frac{2\pi g V(2m)^{3/2}}{(2\pi\hbar)^3}\int_0^\infty d\varepsilon\,\sqrt{\varepsilon} f(\varepsilon)$$

where $g = 2$ for the electron. Thus, we conclude that

$$D(\varepsilon) = \frac{2\pi g (2m)^{3/2}}{(2\pi\hbar)^3} \sqrt{\varepsilon}$$

d) For low T, formula (0.1) gives μ_0 as a function of the number density,

$$\frac{N}{V} = \frac{4\pi g}{3} \left(\frac{2m\mu_0}{(2\pi\hbar)^2} \right)^{3/2}$$

so that the specific heat takes the form,

$$\frac{C_V}{Nk} = \frac{mkT}{4\hbar^2} \left(\frac{4\pi g V}{3N} \right)^{2/3}$$

and is linear in T as expected.