

3. Statistical Mechanics

Consider two non-interacting identical Fermions, in a system (closed with respect to the number of particles, but in contact with a heat bath) where they can access three different single-particle states, of energy 0, ϵ , 2ϵ .

(a) Calculate, at $T = 0$ and at finite T , the energy of the system and the average occupation numbers of the single-particle states.
(Hint: use the partition sum).

(b) Now consider that the system can exchange particles with a "reservoir" of the same particles characterized by a chemical potential $\mu = 2\epsilon + \alpha T$. What are now the average occupation numbers of the single-particle states of the original system, and the average number of particles in the original system, at $T = 0$?
(Hint: use occupation numbers).

(a) $Z = e^{-\beta(0\epsilon)} + e^{-\beta(\epsilon+2\epsilon)} + e^{-\beta(2\epsilon+\epsilon)}$ two non-interacting, identical Fermions
Particle state w/ E_i for ϵ_i
 n_1 n_2 n_3

Particle state	0	ϵ	2ϵ
	x	x	
	x	x	
	x	x	

- can't share same state

- Ignore spin

- can't distinguish particle occupation of a single state

$\langle E(0) \rangle = \epsilon @ T=0$ ← needs to be in G.S. as $T \rightarrow 0$

$$\langle E(T) \rangle = \sum_n p_n \epsilon_n = \sum_i \frac{\epsilon_i e^{-\beta \epsilon_i}}{Z} = \frac{\epsilon e^{-\beta \epsilon} + 2\epsilon e^{-\beta 2\epsilon} + 3\epsilon e^{-\beta 3\epsilon}}{Z}$$

$$\langle n_r \rangle = \sum_i p_i n_i = \frac{\sum_{n_r \in r} n_r e^{-\beta \epsilon_r}}{Z} \quad \epsilon_r = [n_1 \epsilon_1 + n_2 \epsilon_2 + n_3 \epsilon_3]_r \leftarrow \text{for state } r$$

$$\langle n_1(T) \rangle = \frac{e^{-\beta \epsilon} + e^{-\beta 2\epsilon}}{Z} \quad (n_1=1, n_2=1, n_3=0) \& (n_1=1, n_2=0, n_3=1)$$

$$\langle n_2(T) \rangle = \frac{e^{-\beta \epsilon} + e^{-\beta 3\epsilon}}{Z} \quad (n_1=1, n_2=1, n_3=0) \\ (n_1=0, n_2=1, n_3=1)$$

$$\langle n_3(T) \rangle = \frac{e^{-\beta 2\epsilon} + e^{-\beta 3\epsilon}}{Z} \quad (n_1=1, n_2=0, n_3=1) \\ (n_1=0, n_2=1, n_3=1)$$

Since we need to be in G.S. @ $T=0$, $\langle n_1(0) \rangle \approx e^{-\beta \epsilon}/Z \approx 1 \quad \langle n_2(0) \rangle \approx 0$
 $\langle n_3(0) \rangle \approx e^{-\beta 3\epsilon}/Z \approx 1 @ T=0$

Now, we consider exchange w/ Particle bath:

- $\mu = 2\epsilon + \alpha kT$ (they set $k=1$)

- $Z_{GC} = \sum_{N=0}^3 e^{\mu N} Z_N$

$$Z = e^{3\mu}$$

	n_1	n_2	n_3
	0	ϵ	2ϵ
"	x	x	
"	x	x	

3 possible (original sys)

$$\begin{aligned} \bullet Z_{GC} &= \sum_{n=0}^{\infty} e^{\beta E_n} t_n \\ &\quad \text{---} \quad \begin{array}{ccc} x & x & \\ x & x & \\ x & & \end{array} \left. \begin{array}{l} 3 \text{ possible (original sys)} \\ 1 \text{ possible} \\ 3 \text{ possible} \\ 1 \text{ possible} \end{array} \right\} \text{possible due to exchange w/ reservoir} \\ Z_{ac} &= 1 + z(1 + e^{-\beta E} + e^{-2\beta E}) \\ &\quad + z^2(e^{-3\beta E} + e^{-2\beta E} + e^{-\beta E}) \\ &\quad + z^3(e^{-3\beta E}) \\ &= 1 + e^\alpha(e^{3\beta E} + e^{\beta E} + 1) \\ &\quad + e^{2\alpha}(e^{3\beta E} + e^{2\beta E} + e^{\beta E}) \\ &\quad + e^{3\alpha}e^{\beta E} \end{aligned}$$

We may now calculate the $\langle n(\tau) \rangle$ occupation #s

$$\langle n_1(\tau) \rangle = \frac{z e^0 + z^2 (e^{-\beta E} + e^{-2\beta E}) + z^3 e^{-3\beta E}}{Z_{ac}} = \frac{e^\alpha e^{\beta E} + e^{2\alpha} (e^{3\beta E} + e^{2\beta E}) + e^{3\alpha} e^{\beta E}}{Z_{ac}}$$

$$\langle n_2(\tau) \rangle = \frac{z^2 (e^{-\beta E} + e^{-2\beta E}) + z^3 e^{-3\beta E}}{Z_{ac}} = \frac{e^\alpha e^{\beta E} + e^{2\alpha} (e^{3\beta E} + e^{2\beta E}) + e^{3\alpha} (e^{3\beta E})}{Z_{ac}}$$

$$\langle n_3(\tau) \rangle = \frac{z^3 (e^{-2\beta E} + e^{-3\beta E}) + z^2 e^{-\beta E} + z^3 e^{-3\beta E}}{Z_{ac}} = \frac{e^\alpha + e^{2\alpha} (e^{3\beta E} + e^{2\beta E}) + e^{3\alpha} e^{\beta E}}{Z_{ac}}$$

Now find in $\lim T \rightarrow 0$:

$$\begin{aligned} \lim T \rightarrow 0, \quad e^{\beta E} &\rightarrow \infty \quad \text{for any } E \\ t^{-\beta E} &\rightarrow 0 \end{aligned}$$

$$\text{s.t. } Z_{GC}(T \rightarrow 0) \sim (e^{2\alpha} + e^{3\alpha}) e^{3\beta E} \quad \text{for FD @ low-temp } E_F = \mu \sim 2E + O(T)$$

$$\langle n_1(T \rightarrow 0) \rangle = \langle n_2(T \rightarrow 0) \rangle \approx \frac{(e^{2\alpha} + e^{3\alpha}) e^{3\beta E}}{Z_{ac}} \approx 1$$

$$\langle n_3(T \rightarrow 0) \rangle \approx \frac{e^{3\alpha}}{e^{2\alpha} + e^{3\alpha}} = \frac{e^\alpha}{e^\alpha + 1}$$

$$\langle n_1 \rangle_{2E} \xrightarrow[\alpha \rightarrow \infty]{\beta \rightarrow 0} \begin{cases} 1 & \alpha \rightarrow 0 \\ 1 & \alpha \rightarrow \infty \end{cases}$$

$$\langle n_2 \rangle_{2E} \xrightarrow[\alpha \rightarrow \infty]{} 1$$

$$\langle n_3 \rangle_{2E} \xrightarrow[\alpha \rightarrow \infty]{} 1$$