Fall 2009 Problem 11

Linearly polarized plane wave with frequency ω polarized in $\hat{x}$ travels in $+\hat{z}$ in a conductor with ε, μ, and real conductivity σ, $\sigma \gg \varepsilon\omega$.

a) Find the instantaneous and time averaged power loss per unit volume due to resistive heating for any z.

$$\frac{d^4 W}{dt\,dV} = \vec{J}\cdot\vec{E} \qquad \text{homogeneous, isotropic, linear medium: } \vec{J}_f = \sigma\vec{E}$$

$$\nabla\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}, \quad \nabla\times\vec{B} = \mu\vec{J}_f + \mu\varepsilon\frac{\partial\vec{E}}{\partial t} = \mu\sigma\vec{E} + \mu\varepsilon\frac{\partial\vec{E}}{\partial t}$$

$$\nabla\times\nabla\times\vec{E} = -\frac{\partial}{\partial t}\left(\mu\sigma\vec{E} + \mu\varepsilon\frac{\partial\vec{E}}{\partial t}\right) \rightarrow \text{Fourier w/ } \vec{k} = k\hat{z}$$

$$k^2 = i\omega\mu\sigma - \omega^2\mu\varepsilon \rightarrow k = \left(\mu\omega(i\sigma - \omega\varepsilon)\right)^{1/2}, \quad \sigma \gg \varepsilon\omega$$

$$k \approx \sqrt{\mu\omega\sigma}\,\sqrt{i}, \quad \sqrt{i} = \frac{1+i}{\sqrt{2}} \longrightarrow \vec{J}\cdot\vec{E} = \sigma\vec{E}\cdot\vec{E} = \sigma E_0^2 e^{-2\kappa z} e^{2i(\kappa z-\omega t)}$$

$$= (1+i)\kappa$$

$$\frac{d^4 W}{dt\,dV} = \vec{J}\cdot\vec{E} = \sigma E_0^2 e^{-2\kappa z} e^{i(\kappa z-\omega t)}$$

$$k^2 \approx i\mu\omega\sigma \rightarrow \frac{k^2c^2}{\omega^2} \approx i\frac{\mu\sigma}{\omega}\frac{1}{\mu_0\varepsilon_0}$$

$$n^2 = \frac{\mu\varepsilon}{\mu_0\varepsilon_0} = \frac{c^2k^2}{\omega^2} \rightarrow \varepsilon = i\frac{\sigma}{\omega}$$

$$\left\langle\frac{d^4 W}{dt\,dV}\right\rangle = \frac{1}{2}\sigma E_0^2 e^{-2\kappa}$$

b) Find the total power loss per unit area $\left\langle\frac{d^3 W}{dt\,dx\,dy}\right\rangle$ between $z=0$ and ∞

$$\left\langle\frac{d^3 W}{dt\,dx\,dy}\right\rangle = \int_0^\infty dz\left\langle\frac{d^4 W}{dt\,dV}\right\rangle = \int_0^\infty dz\,\sigma E_0^2 e^{-2\kappa z} = -\frac{\sigma E_0^2}{2\kappa}\left(e^{-2\kappa z}\Big|_0^\infty\right) = -\frac{\sigma E_0^2}{2\kappa}$$

c) Find the time averaged Poynting vector at any z

$$\vec{B} = \frac{1}{c}\hat{k}\times\vec{E} = \hat{y}\frac{E_0}{c}e^{-\kappa z}e^{i(\kappa z-\omega t)} = \hat{y}\frac{\omega E_0}{k}e^{-\kappa z}e^{i(\kappa z-\omega t)}$$

$$\langle\vec{S}\rangle = \frac{1}{2\mu}\mathrm{Re}\left(\vec{E}\times\vec{B}^*\right) = \frac{1}{2\mu}\mathrm{Re}\left(\frac{1}{c}E_0^2 e^{-2\kappa z}\hat{z}\right) = \frac{1}{2}\mathrm{Re}\left(\sqrt{\frac{\varepsilon}{\mu}}E_0^2 e^{-2\kappa z}\hat{z}\right)$$

$$\sqrt{\frac{\varepsilon}{\mu}} = \sqrt{\frac{i\sigma}{\mu\omega}} = (1+i)\sqrt{\frac{\sigma}{2\mu\omega}} \longrightarrow \langle\vec{S}\rangle = \frac{1}{2}\sqrt{\frac{\sigma}{2\mu\omega}}E_0^2 e^{-2\kappa z}\hat{z}$$

d) Compare the value of your result for b) with the magnitude of your result in c) evaluated at $z=0$. Is the answer reasonable?

Poynting's theorem: $\frac{\partial}{\partial t}(u_{mech} + u_{em}) + \nabla \cdot \vec{S} + \vec{J} \cdot \vec{E} = 0$

$$\nabla \cdot \langle \vec{S} \rangle = \frac{\partial}{\partial z}\left(\frac{1}{2} E_0^2 \sqrt{\frac{\sigma}{2\mu\omega}}\, e^{-2\kappa z}\right) = -\kappa \sqrt{\frac{\sigma}{2\mu\omega}}\, E_0^2 e^{-2\kappa z} = -\sqrt{\frac{\sigma\mu\omega}{2}}\sqrt{\frac{\sigma}{2\mu\omega}}\, E_0^2 e^{-2\kappa z}$$

$$= -\frac{\sigma}{2} E_0^2 e^{-2\kappa z}$$ ← this is the same as part (a)!

↳ Power lost from $z=0$ to $z=\infty$ $= -\frac{\sigma E_0^2}{2\kappa}$ from (b)

at $z=0$ $\quad \langle \vec{S} \rangle = \frac{1}{2} E_0^2 \sqrt{\frac{\sigma}{2\mu\omega}} = \frac{\kappa}{\kappa}\sqrt{\frac{\sigma}{2\mu\omega}}\,\frac{E_0^2}{2} = \frac{\sigma}{4\kappa} E_0^2$

This is $-\frac{1}{2}$ the power lost in the half space from (b), because $\langle \vec{S} \rangle$ is the power per unit area passing all space.