6. (Classical Mechanics)

Consider a particle of mass m moving in a plane under the action of a central force with potential $U(r)$.

Write down the generating function of a canonical transformation of the the description of the motion from the lab frame to a reference frame rotating counterclockwise at an angular frequency Ω. Write down the new Hamiltonian and the relationship between the new and old coordinates and momenta. Write down the equations of motion in the rotating frame.

Solution: *Solution by Jonah Hyman (jthyman@g.ucla.edu)*

Because we are dealing with a central potential and a rotating reference frame, it makes the most sense to work in polar coordinates in the plane of motion. Let (r, θ) be the coordinates of the particle in the lab frame, and let (R, Θ) be the coordinates of the particle in the rotating frame. Since the rotating frame is moving counterclockwise at angular frequency Ω with respect to the lab frame, we have

$$R = r \quad \text{and} \quad \Theta = \theta - \Omega t \tag{9}$$

Note that since the rotating frame moves counterclockwise with respect to the lab frame, a particle at rest in the lab frame will move clockwise in the rotating frame.

Now, we need to select which generating function we want to use. Our options are $F_1(q, Q, t)$, $F_2(q, P, t)$, $F_3(p, Q, t)$, and $F_4(p, P, t)$ (note that each contains one parameter from the old coordinate system and one from the new coordinate system).

Normally, you should select the generating function for which it is easiest to isolate the variables that do not appear in the generating function. In this case, though, it is easy to isolate all the coordinates (r, R, θ, Θ). For convention's sake, we will therefore use $F_2(q, P, t)$ (this is also the generating function for which it is easiest to avoid sign errors). F_2 defines the old momenta, new coordinates, and new Hamiltonian as follows:

$$p_i = \frac{\partial F_2}{\partial q_i} \tag{10}$$

$$Q_i = \frac{\partial F_2}{\partial P_i} \tag{11}$$

$$H' = H + \frac{\partial F_2}{\partial t} \tag{12}$$

We know very little about the momenta at this point, so our starting point is equation (11):

$$r = R = \frac{\partial F_2}{\partial P_R} \Longrightarrow F_2(r, \theta, P_R, P_\Theta, t) = rP_R + f_1(r, \theta, P_\Theta, t) \tag{13}$$

$$\theta - \Omega t = \Theta = \frac{\partial F_2}{\partial P_\Theta} \Longrightarrow F_2(r, \theta, P_R, P_\Theta, t) = (\theta - \Omega t)P_\Theta + f_2(r, \theta, P_R, t) \tag{14}$$

where f_1 and f_2 are unknown functions not set by the differential equations. Combining these constraints, we get that

$$F_2(r, \theta, P_R, P_\Theta, t) = rP_R + (\theta - \Omega t)P_\Theta + f(r, \theta, t) \tag{15}$$

As long as a transformation comes from a generating function, it is a canonical transformation. Thus, we can select any arbitrary function $f(r, \theta, t)$, and we will still have a canonical transformation that obeys (9). For simplicity, let's select $f(r, \theta, t) = 0$, which gives us

$$\boxed{F_2(r, \theta, P_R, P_\Theta, t) = rP_R + (\theta - \Omega t)P_\Theta} \tag{16}$$

To find the new Hamiltonian, we start by deriving the old Hamiltonian from the old Lagrangian:

$$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - U(r) \tag{17}$$

The canonical momenta are given by

$$p_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r} \quad \text{and} \quad p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta} \tag{18}$$

The relationships between new and old coordinates are given by equations (10) and (11):

$$p_r = \frac{\partial F_2}{\partial r} = P_R \qquad R = \frac{\partial F_2}{\partial P_R} = r \tag{19}$$

$$p_\theta = \frac{\partial F_2}{\partial r} = P_\Theta \qquad \Theta = \frac{\partial F_2}{\partial P_\Theta} = \theta - \Omega t \tag{20}$$

The Hamiltonian is defined by

$$\begin{aligned} H &= p_r \dot{r} + p_\theta \dot{\theta} - L \\ &= m\dot{r}^2 + mr^2\dot{\theta} - \left(\frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - U(r)\right) \\ &= \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta} + U(r) \\ H &= \frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} + U(r) \end{aligned} \tag{21}$$

We may now use equation (12) to calculate the new Hamiltonian:

$$\begin{aligned} H' &= H + \frac{\partial F_2}{\partial t} \\ &= \frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} + U(r) - \Omega P_\Theta \end{aligned}$$

Rewriting in terms of the new coordinates, we get that

$$\boxed{H' = \frac{P_R^2}{2m} + \frac{P_\Theta^2}{2mR^2} - \Omega P_\Theta + U(R)} \tag{22}$$

We can now find the Hamilton's equations of motion in the rotating frame using the formulas

$$\dot{Q}_i = \frac{\partial H}{\partial P_i} \quad \text{and} \quad \dot{P}_i = -\frac{\partial H}{\partial Q_i} \tag{23}$$

This yields the following equations of motion:

$$\dot{R} = \frac{P_R}{m} \qquad \dot{P}_R = \frac{P_\Theta^2}{mR^3} - \frac{dU}{dR} \tag{24}$$

$$\dot{\Theta} = \frac{P_\Theta}{mR^2} - \Omega \qquad \dot{P}_\Theta = 0 \tag{25}$$