4. (Quantum Mechanics)

A particle of mass m and charge q is confined to a circular ring of radius R lying in the x-y plane. There are also constant electric and magnetic fields: $\vec{E} = \mathcal{E}\hat{y}$, $\vec{B} = B\hat{z}$.

(a) Write a Schrödinger equation for the energy levels of this system.

(b) Compute the energy spectrum in the regime where $\mathcal{E}$ is negligible compared to B. Hint: this will be simpler with the right choice of gauge.

(c) By making a suitable approximation in the Hamiltonian, compute the energy spectrum in the regime where $\mathcal{E}$ is large and B is negligible. Hint: think about where the wavefunction is concentrated in this limit.

Solution: *Solution by Jonah Hyman (jthyman@g.ucla.edu)*

(a) The time-independent Schrödinger equation, which is the equation for the energy levels of this system, is

$$H\psi = E_n\psi \tag{114}$$

so this question boils down to finding the Hamiltonian H for this system.

To incorporate the constant electric field $\mathbf{E} = \mathcal{E}\hat{\mathbf{y}}$, we need to know its associated potential energy. The electric potential ϕ is defined by $\mathbf{E} = -\nabla\phi$, so we can write

$$\phi = -\mathcal{E}y = -\mathcal{E}R\sin\varphi \quad \text{where } \varphi \text{ is the angle around the ring} \tag{115}$$

The associated potential energy is just the electric potential multiplied by q:

$$V(\varphi) = -q\mathcal{E}R\sin\varphi \tag{116}$$

To incorporate the magnetic field, we'll need to add it to the Hamiltonian via the vector potential. What is the vector potential for the constant magnetic field $\mathbf{B} = B\hat{\mathbf{z}}$? There are many different answers due to different gauge choices. We should select the most rotationally symmetric gauge, since this problem takes place on a circular ring and we are looking forward to the hint in part (b). Using the fact that $\nabla \times \mathbf{A} = \mathbf{B}$, we can reverse-engineer an answer

$$\mathbf{A} = \frac{B}{2}(-y\hat{\mathbf{x}} + x\hat{\mathbf{y}}) \tag{117}$$

(This form of the vector potential for a constant magnetic field is the most useful for rotationally symmetric problems, and you should be familiar with it. You should also be familiar with the alternate vector potentials for a constant magnetic field $\mathbf{A} = -By\,\hat{\mathbf{x}}$ and $\mathbf{A} = Bx\,\hat{\mathbf{y}}$, which are useful for the derivation of "Landau levels" for a particle in a constant magnetic field.)

In this problem, the variable describing our location in space is φ, the angle around the ring. We can write (117) in a more convenient form by recalling the definition of the unit vectors in polar coordinates:

$$\hat{\mathbf{r}} = \cos\varphi\,\hat{\mathbf{x}} + \sin\varphi\,\hat{\mathbf{y}} = \frac{1}{r}(x\hat{\mathbf{x}} + y\hat{\mathbf{y}}) \tag{118}$$

$$\hat{\varphi} = -\sin\varphi\,\hat{\mathbf{x}} + \cos\varphi\,\hat{\mathbf{y}} = \frac{1}{r}(-y\hat{\mathbf{x}} + x\hat{\mathbf{y}}) \tag{119}$$

Since $r = R$ on the ring, what we have is

$$\mathbf{A} = \frac{BR}{2}\,(-\sin\varphi\,\hat{\mathbf{x}} + \cos\varphi\,\hat{\mathbf{y}}) = \frac{BR}{2}\hat{\varphi} \tag{120}$$

Next, we need to incorporate this vector potential into the Hamiltonian. (We'll work in SI units throughout.) Here is the prescription for doing so:

Incorporating a vector potential into the Lagrangian:
SI units: Add $+q\dot{\mathbf{r}} \cdot \mathbf{A}$.
Gaussian units: Add $+\frac{q\dot{\mathbf{r}}\cdot\mathbf{A}}{c}$.
Incorporating a vector potential into the Hamiltonian:
SI units: Replace $\mathbf{p}$ with $\mathbf{p} - q\mathbf{A}$.
Gaussian units: Replace $\mathbf{p}$ with $\mathbf{p} - \frac{q\mathbf{A}}{c}$.

The safest method for writing the Hamiltonian correctly is to start with the Lagrangian, but it is more efficient to start with the Hamiltonian. We will show both methods:

Lagrangian method:
The Lagrangian for a particle in an electric and magnetic field is

$$L = \underbrace{\frac{1}{2}m\dot{\mathbf{r}}^2}_{\text{Kinetic term}} - \underbrace{V(\varphi)}_{\text{Scalar potential}} + \underbrace{q\mathbf{A}\cdot\dot{\mathbf{r}}}_{\text{Vector potential}} \tag{121}$$

In this case, since the particle is confined to a ring of radius R, we have

$$\mathbf{r} = R\left(\cos\varphi\,\hat{\mathbf{x}} + \sin\varphi\,\hat{\mathbf{y}}\right) = R\,\hat{\mathbf{r}} \tag{122}$$

$$\dot{\mathbf{r}} = R\dot{\varphi}\left(-\sin\varphi\,\hat{\mathbf{x}} + \cos\varphi\,\hat{\mathbf{y}}\right) = R\dot{\varphi}\,\hat{\varphi} \tag{123}$$

This gives us

$$\dot{\mathbf{r}}^2 = R^2\dot{\varphi}^2\left(\sin^2\varphi + \cos^2\varphi\right) = R^2\dot{\varphi}^2 \tag{124}$$

Using (120), we also have

$$\mathbf{A}\cdot\dot{\mathbf{r}} = \frac{1}{2}BR^2\dot{\varphi}\left(\sin^2\varphi + \cos^2\varphi\right) = \frac{1}{2}BR^2\dot{\varphi} \tag{125}$$

Putting everything together (using (116 for the scalar potential), we get the final form of the Lagrangian:

$$L(\varphi,\dot{\varphi}) = \frac{1}{2}mR^2\dot{\varphi}^2 + q\mathcal{E}R\sin\varphi + \frac{1}{2}qBR^2\dot{\varphi} \tag{126}$$

To get the Hamiltonian, start by finding the momentum canonically conjugate to φ:

$$p_\varphi \equiv= \frac{\partial L}{\partial\dot{\varphi}} = mR^2\dot{\varphi} + \frac{1}{2}qBR^2 \tag{127}$$

Then apply the Legendre transform and eliminate $\dot{\varphi}$ to get the Hamiltonian

$$\begin{aligned}
H(p_\varphi,\varphi) &\equiv p_\varphi\dot{\varphi} - L \\
&= \left(mR^2\dot{\varphi} + \frac{1}{2}qBR^2\right)\dot{\varphi} - \left(\frac{1}{2}mR^2\dot{\varphi}^2 + q\mathcal{E}R\sin\varphi + \frac{1}{2}qBR^2\dot{\varphi}\right) \\
&= \frac{1}{2}mR^2\dot{\varphi}^2 - q\mathcal{E}R\sin\varphi \\
&= \frac{1}{2}mR^2\left(\frac{1}{mR^2}\left(p_\varphi - \frac{1}{2}qBR^2\right)\right)^2 - q\mathcal{E}R\sin\varphi \\
&= \frac{1}{2mR^2}\left(p_\varphi - \frac{1}{2}qBR^2\right)^2 - q\mathcal{E}R\sin\varphi
\end{aligned} \tag{128}$$

To canonically quantize the Hamiltonian, note that if we have a coordinate φ and its conjugate momentum p_φ, the operator p_φ is defined by

$$p_\varphi = \frac{\hbar}{i}\frac{\partial}{\partial\varphi} \tag{129}$$

Therefore, the quantum Hamiltonian is

$$H = \frac{1}{2mR^2}\left(\frac{\hbar}{i}\frac{\partial}{\partial\varphi} - \frac{1}{2}qBR^2\right)^2 - q\mathcal{E}R\sin\varphi \tag{130}$$

Hamiltonian method:
The Hamiltonian for a particle in an electric and magnetic field is

$$H = \underbrace{\frac{1}{2m}\left(\mathbf{p} - q\mathbf{A}\right)^2}_{\text{Kinetic term + vector potential}} + \underbrace{V(\varphi)}_{\text{Scalar potential}} \tag{131}$$

In this case, the scalar and vector potentials are given by (116) and (120), and the momentum $\mathbf{p}$ is given by

$$\mathbf{p} = \frac{L_z}{R}\hat{\varphi} \tag{132}$$

where L_z is the magnitude of the angular momentum about the $\hat{\mathbf{z}}$-axis. Plugging all this in, we get

$$\begin{aligned} H &= \frac{1}{2m}\left(\frac{L_z}{R} - \frac{1}{2}qBR\right)^2 - qER\sin\varphi \\ &= \frac{1}{2mR^2}\left(L_z - \frac{1}{2}qBR^2\right)^2 - q\mathcal{E}R\sin\varphi \end{aligned} \tag{133}$$

Now, recall the fact that the quantum mechanical angular momentum operator L_z is given by

$$L_z = \frac{\hbar}{i}\frac{\partial}{\partial\varphi} \tag{134}$$

Therefore, the quantum Hamiltonian is

$$H = \frac{1}{2mR^2}\left(\frac{\hbar}{i}\frac{\partial}{\partial\varphi} - \frac{1}{2}qBR^2\right)^2 - q\mathcal{E}R\sin\varphi \tag{135}$$

Whichever method we pick, we can write down the time-independent Schrödinger equation $H\psi = E_n\psi$ using it. If E_n is the nth energy level and $\psi(\varphi)$ is the wavefunction, we have

$$\boxed{\left(\frac{1}{2mR^2}\left(\frac{\hbar}{i}\frac{\partial}{\partial\varphi} - \frac{1}{2}qBR^2\right)^2 - q\mathcal{E}R\sin\varphi\right)\psi(\varphi) = E_n\psi(\varphi)} \tag{136}$$

(b) If $\mathcal{E}$ is negligible compared to B, we can drop the scalar potential from the time-dependent Schrödinger equation:

$$\frac{1}{2mR^2}\left(\frac{\hbar}{i}\frac{\partial}{\partial\varphi} - \frac{1}{2}qBR^2\right)^2\psi(\varphi) = E_n\psi(\varphi) \tag{137}$$

The way to solve this differential equation is by already knowing the answer. For a particle confined to a ring, the wavefunction must be periodic ($\psi(\varphi) = \psi(\varphi + 2\pi)$), which means that it can be written as a sum of complex exponentials $e^{in\varphi}$, where n is an integer. To find the energy of such a state, plug it into the differential equation (137):

$$\begin{aligned} \frac{1}{2mR^2}\left(\frac{\hbar}{i}\frac{\partial}{\partial\varphi} - \frac{1}{2}qBR^2\right)^2 e^{in\varphi} &= \frac{1}{2mR^2}\left(\frac{\hbar}{i}\frac{\partial}{\partial\varphi} - \frac{1}{2}qBR^2\right)\left(\frac{\hbar}{i}\frac{\partial}{\partial\varphi} - \frac{1}{2}qBR^2\right)e^{in\varphi} \\ &= \frac{1}{2mR^2}\left(\frac{\hbar}{i}\frac{\partial}{\partial\varphi} - \frac{1}{2}qBR^2\right)\left(\hbar n - \frac{1}{2}qBR^2\right)e^{in\varphi} \\ &= \frac{1}{2mR^2}\left(\hbar n - \frac{1}{2}qBR^2\right)\left(\hbar n - \frac{1}{2}qBR^2\right)e^{in\varphi} \\ &= \frac{1}{2mR^2}\left(\hbar n - \frac{1}{2}qBR^2\right)^2 e^{in\varphi} \\ &= \frac{\hbar^2}{2mR^2}\left(n - \frac{qBR^2}{2\hbar}\right)^2 e^{in\varphi} \end{aligned}$$

so the energy spectrum is

$$\boxed{E_n = \frac{\hbar^2}{2mR^2}\left(n - \frac{qBR^2}{2\hbar}\right)^2 \quad \text{for } n \text{ an integer}} \tag{138}$$

Since n is any integer, we could also write this as

$$\boxed{E_n = \frac{\hbar^2}{2mR^2}\left(-n - \frac{qBR^2}{2\hbar}\right)^2 = \frac{\hbar^2}{2mR^2}\left(n + \frac{qBR^2}{2\hbar}\right)^2 \quad \text{for } n \text{ an integer}} \tag{139}$$

While the problem doesn't ask for this, it is interesting to note that if we change the magnetic flux through the ring, $\Phi_{\mathbf{B}} = \pi R^2 B$, by an integer multiple of the flux quantum $\Phi_0 \equiv \frac{2\pi\hbar}{q}$ (i.e., if we change B by an integer multiple of $\frac{2\hbar}{qR^2}$), then the energy spectrum is unchanged.

(c) If $\mathcal{E}$ is large and B is negligible, then we can drop the vector potential term from the quantum Hamiltonian:

$$\left(\frac{1}{2mR^2}\left(\frac{\hbar}{i}\frac{\partial}{\partial\varphi}\right)^2 - q\mathcal{E}R\sin\varphi\right)\psi(\varphi) = E_n\psi(\varphi) \tag{140}$$

Simplifying, we get

$$\left(-\frac{\hbar^2}{2mR^2}\frac{\partial^2}{\partial\varphi^2} - q\mathcal{E}R\sin\varphi\right)\psi(\varphi) = E_n\psi(\varphi) \tag{141}$$

Now consider the hint: think about where the wavefunction is concentrated in this limit. If the electric field $\mathcal{E}\hat{y}$ is of large magnitude, the particle is most likely to be found in the part of the ring closest to the point $R\hat{\mathbf{y}}$. In other words, since φ is measured from the $+\hat{x}$-axis, the wavefunction will be concentrated around $\varphi = \frac{\pi}{2}$.

For that reason, we'll define

$$\alpha \equiv \varphi - \frac{\pi}{2} \tag{142}$$

and expand the Hamiltonian about small α:

$$\begin{aligned}
H &= -\frac{\hbar^2}{2mR^2}\frac{\partial^2}{\partial\alpha^2} - q\mathcal{E}R\sin\left(\frac{\pi}{2}+\alpha\right) \\
&= -\frac{\hbar^2}{2mR^2}\frac{\partial^2}{\partial\alpha^2} - q\mathcal{E}R\left[\sin\left(\frac{\pi}{2}\right) + \cos\left(\frac{\pi}{2}\right)\alpha - \frac{1}{2}\sin\left(\frac{\pi}{2}\right)\alpha^2 + \mathcal{O}(\alpha^3)\right] \quad \text{Taylor expanding} \\
&= -\frac{\hbar^2}{2mR^2}\frac{\partial^2}{\partial\alpha^2} - q\mathcal{E}R\left[1 - \frac{1}{2}\alpha^2 + \mathcal{O}(\alpha^3)\right] \\
&\approx -\frac{\hbar^2}{2mR^2}\frac{\partial^2}{\partial\alpha^2} + \frac{1}{2}q\mathcal{E}R\alpha^2 - q\mathcal{E}R
\end{aligned} \tag{143}$$

The α^2 dependence of the second term should remind you of the harmonic oscillator. If we use the prescription for the canonical momentum operator in quantum mechanics

$$p_\alpha = \frac{\hbar}{i}\frac{\partial}{\partial\alpha} \tag{144}$$

we can write

$$H = \frac{p_\alpha^2}{2mR^2} + \frac{1}{2}q\mathcal{E}R\alpha^2 - q\mathcal{E}R \tag{145}$$

We want this to closely match the harmonic oscillator potential $\frac{p^2}{2m} + \frac{1}{2}m\omega^2x^2$, since we already know that its spectrum is $E_n = \hbar\omega\left(n + \frac{1}{2}\right)$. To do this, first define

$$\overline{m} \equiv mR^2 \tag{146}$$

This makes the first term in (145) look just like the kinetic term of the harmonic oscillator. We will also incorporate it into the second term in an attempt to make it look more like the potential term of the harmonic oscillator.

$$H = \frac{p_\alpha^2}{2\overline{m}} + \frac{1}{2}\overline{m}\frac{q\mathcal{E}}{mR}\alpha^2 - q\mathcal{E}R \tag{147}$$

As a final step, define

$$\overline{\omega} \equiv \sqrt{\frac{q\mathcal{E}}{mR}} \tag{148}$$

Then, (145) exactly matches the Hamiltonian for a harmonic oscillator of mass $\overline{m}$ and natural frequency $\overline{\omega}$, with a constant energy shift of $-q\mathcal{E}R$:

$$H = \frac{p_\alpha^2}{2\overline{m}} + \frac{1}{2}\overline{m}\,\overline{\omega}^2 \frac{q\mathcal{E}}{mR}\alpha^2 - q\mathcal{E}R \tag{149}$$

Using the formula for the energy eigenstates of the harmonic oscillator, and subtracting the constant energy shift of $-q\mathcal{E}R$, we get the energy spectrum in this approximation:

$$E_n = \hbar\overline{\omega}\left(n + \frac{1}{2}\right) - q\mathcal{E}R \quad \text{for } n = 0, 1, 2, \ldots \tag{150}$$

Substituting our expression for $\overline{\omega}$, we get our final answer for the energy spectrum:

$$\boxed{E_n = \hbar\sqrt{\frac{q\mathcal{E}}{mR}}\left(n + \frac{1}{2}\right) - q\mathcal{E}R \quad \text{for } n = 0, 1, 2, \ldots} \tag{151}$$

Part (c) of this problem is really the "quantum pendulum" in disguise: R represents the length of the pendulum, and the constant electric field acts like a constant gravitational field. As you might expect, in the small-angle approximation (when the field is large), the quantum pendulum has a spectrum similar to that of the quantum harmonic oscillator.