Fall 2018

Physics Comprehensive Exam

September 17, 2018 (Part 1) 9:00am – 1:00pm

<u>Part 1: Quantum Mechanics and Classical Mechanics</u>

6 Total Problems/20 Points Each/Total 120 Points

- Closed book exam.
- Calculators not allowed.
- Begin your solution on the question page.
- Use paper provided for additional pages. **<u>Use one side only.</u>**
- Write your name on EACH of your response pages, including the question page.
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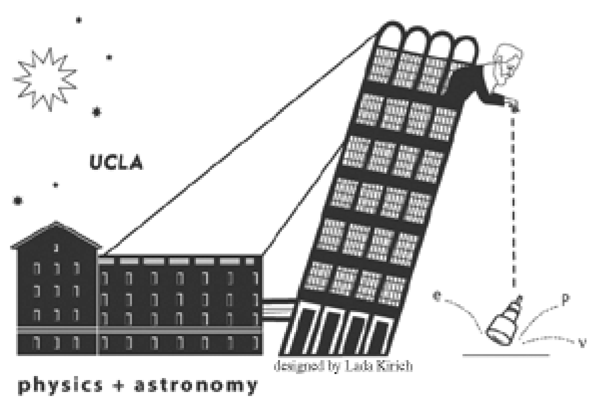


Name: ______________________________

1. Quantum Mechanics

A particle of charge q is subjected to a magnetic field $\mathbf{B} = B\hat{z}$.

(a) Consider the symmetric gauge for the vector potential

$$\mathbf{A} = \frac{B}{2}(-y\hat{x} + x\hat{y})$$

and show that it correctly gives the magnetic field. Write down the Hamiltonian in the symmetric gauge and define

$$Q = \frac{1}{qB}\left(cp_x + qyB/2\right), \; P = (p_y - qBx/2c)$$

Show that the commutator $[Q, P] = i\hbar$. (c is the velocity of light)

(b) Show that H in terms of P and Q becomes a one-dimensional harmonic oscillator problem, where $\omega = qB/mc$. Find the energy eigenvalues.

(c) Write down the harmonic oscillator annihilation operator a in terms of the complex coordinates $z = x + iy$ and $z^* = x - iy$ and show that the ground state wave function is given by $\psi_0(z, z^*) = u(z, z^*)\exp[-qBzz^*/4\hbar c]$ and u is an arbitrary analytic function $\frac{\partial}{\partial z^*}u(z, z^*) = 0$, for example, $u(z, z^*) = z^n$. (n is an arbitrary positive integer).

Hint: The Cauchy-Riemann conditions for the analyticity of a function f is $f = U(x, y) + V(x, y)$ is $\frac{\partial U}{\partial x} = \frac{\partial V}{\partial y}$ and $\frac{\partial V}{\partial x} = -\frac{\partial U}{\partial y}$.

Problem 1.

$$\underset{\sim}{B} = \underset{\sim}{\nabla} \times \underset{\sim}{A} = B\hat{z}$$

$$H = \frac{(p_x + qyB/2c)^2}{2m} + \frac{(p_y - qxB/2c)^2}{2m} + \frac{p_z^2}{2m}$$

$$\psi(x,y,z) = \psi(x,y)e^{ik_z z}$$

$$H(x,y)\psi(x,y) = \left(E - \frac{\hbar^2 k_z^2}{2m}\right)\psi(x,y)$$

verify that

$$[Q, P] = i\hbar$$

The Hamiltonian can now be written as

$$H = \frac{P^2}{2m} + \frac{1}{2}m\omega^2 Q^2$$

$$E = (n, k_z) = \left(n + \frac{1}{2}\right)\hbar\omega + \frac{\hbar^2 k_z^2}{2m}, \quad \omega = \frac{qB}{mc}$$

$$a = \left(\frac{c}{2\hbar qB}\right)\left\{\frac{\hbar}{i}\frac{\partial}{\partial x} + i\frac{\hbar}{i}\frac{\partial}{\partial y} + \frac{qB}{2c}(y - ix)\right]$$

$$z = x + iy, \quad z^* = x - iy$$

$$a = \left(\frac{2\hbar c}{qB}\right)^{1/2} i\left\{\frac{\partial}{\partial z^*} + \frac{qB}{4\hbar c}z\right\}$$

$$\Rightarrow \quad a|0\rangle = 0 \quad \text{gives}$$

$$\left(\frac{\partial}{\partial z^*} + \frac{qB}{4\hbar c}z\right)\psi_0(z, z^*) = 0$$

If we use $\psi_0(z,z^*) = u(z,z^*)\, e^{-\frac{qB}{4\hbar c} z z^*}$
then,

$$\frac{\partial}{\partial z^*} \cancel{z} u(z,z^*) = 0,$$

an analytic function of z; ~~as~~ an example

$$u = z^n$$

and satisfies the Cauchy-Riemann condition.

Name: ______________________________

2. Quantum Mechanics

A spin-1/2 particle precesses in a magnetic field $B_0\hat{z}$ at the frequency $\omega_0 = \gamma B_0$. Now we turn on a small transverse radiofrequency field given by

$$\mathbf{B} = B_1 \cos(\omega t)\hat{x} - B_1 \sin(\omega t)\hat{y}.$$

so that the total field is $\mathbf{B} = B_1 \cos(\omega t)\hat{x} - B_1 \sin(\omega t)\hat{y} + B_0\hat{z}$

(a) Construct the 2×2 Hamiltonian matrix for this system.

(b) Let $\chi(t) = \binom{a(t)}{b(t)}$ be the two component spinor at time t. Show that

$$\frac{da}{dt} = \frac{i}{2}\left(\Omega e^{i\omega t} b + \omega_0 a\right), \quad \frac{db}{dt} = \frac{i}{2}\left(\Omega e^{-i\omega t} a - \omega_0 b\right),$$

where $\Omega = \gamma B_1$.

(c) Now simplify the equations by the substitutions $a(t) = A(t)e^{i\omega_0 t/2}$ and $b(t) = B(t)e^{-i\omega_0 t/2}$ to find the equations for $A(t)$ and $B(t)$. Solve these equations at the resonance by setting $\omega = \omega_0$. Decouple them by taking another derivative. Apply the the initial condition $a(0) = 1$ and $b(0) = 0$ and sketch the probability of a transition to spin down, as a function of time. Can arbitrarily small B_1 flip the spin at resonance? Explain your answer.

Hint: The spin-1/2 matrices are:

$$S_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, S_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, S_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Problem 2

$$H = -\frac{\hbar\gamma}{2}\left[\sigma_x B_x + \sigma_y B_y + \sigma_z B_z\right]$$

$$= -\frac{\hbar}{2}\begin{bmatrix} \omega_0 & \Omega e^{i\omega t} \\ \Omega e^{-i\omega t} & \omega_0 \end{bmatrix}, \quad \Omega = \gamma B_1, \quad \omega_0 = \gamma B_0$$

$$i\hbar \frac{d\chi}{dt} = H\chi(t)$$

$$\frac{da}{dt} = \frac{i}{2}\left[\Omega e^{i\omega t} b + \omega_0 a\right]$$

$$\frac{db}{dt} = \frac{i}{2}\left[\Omega e^{-i\omega t} a - \omega_0 b\right]$$

Now, set $a(t) = A(t)e^{i\omega_0 t}$ and $b(t) = B(t)e^{-i\omega_0 t}$.

Simplify to get

$$\frac{dA(t)}{dt} = \frac{i\Omega}{2} e^{i(\omega-\omega_0)t} B(t), \quad \frac{dB(t)}{dt} = \frac{i\Omega}{2} e^{-i(\omega-\omega_0)t} A(t)$$

At resonance $\omega = \omega_0$. Now take one more derivative

$$\frac{d^2A}{dt^2} = \frac{i\Omega}{2}\frac{i\Omega}{2}A(t) = -\frac{\Omega^2}{4}A(t)$$

~~A(t)~~ Similarly $\frac{d^2B}{dt^2} = -\frac{\Omega^2}{4}B(t)$

$$A(t) = C\cos\left(\frac{\Omega t}{2}\right) + D\sin\left(\frac{\Omega t}{2}\right)$$

$$B(t) = \frac{2}{i\Omega}\frac{dA}{dt} = i\left[C\sin\frac{\Omega t}{2} - D\cos\frac{\Omega t}{2}\right]$$

$a(0) = A(0) = 1, \quad b(0) = B(0) = 0 \Rightarrow C = 1, \; D = 0$

$A(t) = \cos\frac{\Omega t}{2}, \quad B(t) = i \sin\frac{\Omega t}{2}$

$P_{\uparrow \to \downarrow}(t) = |b(t)|^2 = \sin^2\frac{\Omega t}{2}$

$P_{\uparrow}(t) = |a(t)|^2 = \cos^2\frac{\Omega t}{2}$

Simple to sketch it. Indeed, arbitrary small B_1 can flip the spin at resonance.

[There are many other ways of solving this problem]

3. Quantum Mechanics

a) Consider the lattice translation operator

$$T(a) = \exp(-iaP/\hbar)$$

Where a is a constant and P is the momentum operator Show

that
$$T^{\dagger}(a)\ X\ T(a) = x + a$$

b) Show that $T(a)$ is unitary and show that T(a) has eigenvalues of of the form $e^{i\phi}$ where ϕ is real (You can assume that P is hermitian).

c) Consider the family of Hamiltonians which are periodic under shifts by a:

$$H = \frac{P^2}{2m} + \sum_{n=-\infty}^{\infty} V(x - na)$$

Here you can assume that $V(x)$ goes exponentially fast to zero as $|x| \to \infty$ (This assumption makes the sum over n convergent). You can also assume that $V(x)$ can be expanded in a power series.

By examining the lattice translational symmetry of H or otherwise, prove that the Hamiltonian H has eigenstates $\mid E, k\rangle$ where k is a real parameter, which satisfy

$$H \mid E, k\rangle \quad = \quad E \mid E, k\rangle \tag{0.1}$$

and are such that the wave functions in position space defined by the following combinations

$$u_k(x) = e^{-ikx}\langle x \mid E, k\rangle$$

are periodic functions with period a, i.e.

$$u_k(x + a) = u_k(x)$$

What is the significance of the parameter k?
This is Bloch's theorem for periodic potentials (i.e. an energy eigenstate can be written as a Bloch wave times a periodic function).

4. Quantum Mechanics

Consider a two-level system with unperturbed energy levels such that

$$\begin{aligned} \mathbf{H_0} \left|1\right\rangle &= \epsilon \left|1\right\rangle \\ \mathbf{H_0} \left|2\right\rangle &= -\epsilon \left|2\right\rangle . \end{aligned}$$

Add a perturbation with off-diagonal elements only

$$\mathbf{V} = \begin{pmatrix} \langle 1|V|1\rangle & \langle 1|V|2\rangle \\ \langle 2|V|1\rangle & \langle 2|V|2\rangle \end{pmatrix} = \begin{pmatrix} 0 & V \\ V^* & 0 \end{pmatrix} .$$

a) What are the exact eigenvalues of the total Hamiltonian, $\mathbf{H} = \mathbf{H_0} + \mathbf{V}$?

b) Assuming the perturbation is small, i.e. $V \ll \epsilon$, expand the exact energies to 2nd order in $\frac{V}{\epsilon}$.

c) Show that this agrees with the results of 2nd order non-degenerate perturbation theory.

d) If $\epsilon \to 0$ the levels are degenerate. How do the exact energy eigenvalues depend on V in this case?

e) Show that for $\epsilon \neq 0, V \gg \epsilon$ the energy eigenvalues are linear in V.

f) In atoms and molecules, the field-free energy eigenstates are also eigenstates of parity. Show that the perturbation from applying an electric field $\mathcal{E}$ in the dipole approximation (i.e. $\mathbf{V} = -\mathbf{d} \cdot \mathcal{E} = -e\mathbf{r} \cdot \mathcal{E}$) has only off-diagonal elements.

g) In atoms, opposite parity states have energy separations much bigger than the Stark energy shifts due to an electric field that can be applied in the laboratory. How does the Stark shift depend on $\mathcal{E}$ in this case?

h) In chemistry you are typically taught that molecules have dipole moments and thus energy shifts linear in an electric field (i.e. $\Delta E = -\mathbf{d} \cdot \mathcal{E}$). Given that truly degenerate energy levels rarely (if ever) exist, how do you reconcile this?

Solution

a) $H = \begin{pmatrix} \epsilon & V \\ V^* & -\epsilon \end{pmatrix}$ Solving the characteristic equation

$$(\epsilon-\lambda)(-\epsilon-\lambda) - |V|^2 = 0$$

$$\lambda^2 - \epsilon^2 - |V|^2 = 0$$

$$\lambda = \pm\sqrt{\epsilon^2 + |V|^2}$$

b) $\lambda = \pm\epsilon\sqrt{1+\frac{|V|^2}{\epsilon^2}} \simeq \pm\epsilon\left(1+\frac{|V|^2}{2\epsilon^2}\right)$

$$= \pm\left(\epsilon + \frac{|V|^2}{2\epsilon}\right)$$

c) The 1st order corrections are zero since

$$\langle 1|V|1\rangle = \langle 2|V|2\rangle = 0$$

The 2nd order corrections are given by

$$\Delta E_i = \sum_j \frac{\langle i|V|j\rangle\langle j|V|i\rangle}{E_i - E_j}.$$

$$\Rightarrow \Delta E_{1,2} = \pm\frac{|V|^2}{2\epsilon}.$$

d) $H = \begin{pmatrix} 0 & V \\ V^* & 0 \end{pmatrix}$ for $\epsilon \to 0$.

$$\Rightarrow \lambda = \pm|V|$$

The energies are linear in the magnitude of the perturbation

e) $\lambda = \pm\sqrt{\epsilon^2+|V|^2} = \pm|V|\sqrt{1+\frac{\epsilon^2}{|V|^2}}$

$$\overset{|V|\gg|\epsilon|}{\simeq} \pm\left(|V| + \frac{\epsilon^2}{2|V|}\right)$$

f) The operator $\vec{r}$ is odd under parity. Diagonal matrix elements $\langle i|\vec{r}|i\rangle$ are zero since the initial and final states have the same parity. More formally

$$\langle i|\vec{r}|i\rangle = \langle i|\pi^{\dagger}\pi\,\vec{r}\,\pi^{\dagger}\pi|i\rangle$$

$$= \left(\langle i|\pm\right)\left(-\vec{r}\right)\left(\pm|i\rangle\right) = -\langle i|\vec{r}|i\rangle$$

$$\Rightarrow \langle i|\vec{r}|i\rangle = 0$$

where π is the parity operator.

g) This implies $\sqrt{|V|^2+\epsilon^2} - \epsilon \ll \epsilon$

$$\Rightarrow |V|^2 \ll \epsilon^2$$

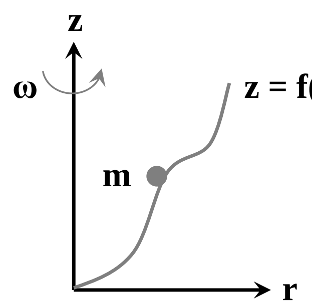
Therefore $\Delta E \simeq \dfrac{|V|^2}{2\epsilon} = \dfrac{e^2\mathcal{E}^2|\langle 1|r|2\rangle|^2}{2\epsilon}$.

There is a quadratic Stark shift.

h) If the Stark shift is larger than the energy separation the shift is linear. Therefore molecules must have closely spaced levels of opposite parity.

5. Classical Mechanics

Consider a point mass m sliding on a wire defined by the function $z = f(r)$, where $r = \sqrt{x^2 + y^2}$. The wire has a fixed shape and is rotating about the z-axis with an angular velocity ω. Consider the gravitational acceleration g (acting towards $-\hat{z}$) and ignore any friction.



(a) Write down the Lagrangian $L(r, \dot{r}, t)$ for the mass.

(b) Using (a), find the equation of motion for $r(t)$. Then let r_0 be the (constant) radius of a fixed circular orbit. Derive the expression of a requirement in terms of $f(r)$ at $r = r_0$. (Hint: show that $f'(r_0)/r_0$ equals to some constant.)

(c) Consider a small change in the circular orbit $r(t) = r_0 + \epsilon(t)$. What is the condition on $f(r)$ in order to have a stable circular orbit at $r = r_0$?

(d) From the Lagrangian, find the Hamiltonian $H(r, p, t)$, where p is the canonical momentum. Is the Hamiltonian conserved?

(a)

Without the wire constraint, the Lagrangian is:

$$L = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}m\dot{z}^2 + \frac{1}{2}m\omega^2 r^2 - mgz.$$

With the wire constraint, it becomes:

$$L = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mf'(r)^2\dot{r}^2 + \frac{1}{2}m\omega^2 r^2 - mgf(r).$$

(b)

Using the Lagrange's equation $\frac{d}{dt}\frac{\partial L}{\partial \dot{r}} - \frac{\partial L}{\partial r} = 0$, we find the equation of motion:

$$\ddot{r}[1 + f'(r)^2] + \dot{r}^2 f'(r)f''(r) - \omega^2 r + gf'(r) = 0.$$

On a fixed orbit $r = r_0$, $\dot{r} = \ddot{r} = 0$. Together with the equation of motion, it thus requires

$$\frac{f'(r_0)}{r_0} = \frac{\omega^2}{g}.$$

(c)

Expanding in terms of $r = r_0 + \epsilon$ to the leading order in ϵ, the equation of motion becomes:

$$\ddot{\epsilon}[1 + f'(r_0)^2] + [gf''(r_0) - \omega^2]\epsilon = 0.$$

In order to have a stable orbit, the effective spring constant that is proportional to $\left[gf''(r_0) - \omega^2\right] / \left[1 + f'(r_0)^2\right]$ must be positive, therefore

$$f''(r_0) > \omega^2/g.$$

(d)

Using the canonical momentum $p = \frac{\partial L}{\partial \dot{r}} = m\dot{r}[1 + f'(r)^2]$, the Hamiltonian can be obtained:

$$H = p\dot{r} - L = \frac{p^2}{2m[1 + f'(r)^2]} - \frac{1}{2}m\omega^2 r^2 + mgf(r),$$

which is time-independent (with fixed ω and $f(r)$) and thus conserved.

6. Classical Mechanics

(This is a classical mechanics problem where we do not consider radiation effects).

Consider a particle of charge q and mass m in a uniform magnetic field $\mathbf{B} = B\,\hat{\mathbf{z}}$.

a) Write the non-relativistic Hamiltonian for the particle and find and integrate the equations of motion.

b) Write the relativistic energy – momentum relation for the particle. Guess or anyway write down the relativistic Hamiltonian for the particle in the uniform magnetic field, and find the equations of motion.

Comp. exam 2018 solutions

a) starting from $H = \frac{p^2}{2m}$,

the Hamiltonian is obtained with the substitution
$p \to \vec{P} - \frac{q}{c}\vec{A}$ where $\vec{P}$ is the generalized momentum.
We choose the gauge $\vec{A} = (-By, 0, 0)$

then $H(\vec{P}, \vec{x}) = \frac{1}{2m}\left[\left(P_x + \frac{qB}{c}y\right)^2 + P_y^2 + P_z^2\right\}$

Using $\dot{P}_i = -\frac{\partial H}{\partial x_i}$, $\dot{x}_i = \frac{\partial H}{\partial P_i}$ we find:

$\dot{P}_z = 0$, $\dot{P}_x = 0$, $\dot{P}_y = -\frac{1}{m}\left(P_x + \frac{qB}{c}y\right)\frac{qB}{c}$

$\dot{z} = \frac{1}{m}P_z = \text{const.}$, $\dot{x} = \frac{1}{m}\left(P_x + \frac{qB}{c}y\right)$, $\dot{y} = \frac{1}{m}P_y$

so $\dot{P}_y = -\frac{qB}{c}\dot{x}$, and $\left.\begin{array}{l} m\ddot{x} = \frac{qB}{c}\dot{y} \\ m\ddot{y} = -\frac{qB}{c}\dot{x} \end{array}\right\}$

which is the familiar $m\ddot{\vec{x}} = q\frac{\dot{\vec{x}}}{c} \times \vec{B}$ etc.

b) Since the modulus square of the energy-momentum 4 vector $(E/c, \vec{p})$ is a Lorentz invariant, we find: $\left(\frac{E}{c}\right)^2 - |\vec{p}|^2 = m^2c^2$ (the RHS in the rest frame) or $E = c\sqrt{p^2+m^2c^2}$, therefore:

$H = c\sqrt{(\vec{p}-\frac{q}{c}\vec{A})^2+m^2c^2}$ in general, and in

our case: $H = c\left[\left(p_x+\frac{qB}{c}y\right)^2+p_y^2+p_z^2+m^2c^2\right]^{1/2}$

The eqs. of motion are now:

$\dot{p}_z = 0$, $\dot{p}_x = 0$, $\dot{p}_y = -c[\ ]^{-1/2}\left(p_x+\frac{qB}{c}y\right)\frac{qB}{c}$

$\dot{z} = c[\ ]^{-1/2}p_z$, $\dot{y} = c[\ ]^{-1/2}p_y$, $\dot{x} = c[\ ]^{-1/2}\left(p_x+\frac{qB}{c}y\right)$

So again $\dot{p}_y = -\frac{qB}{c}\dot{x}$. Since we know that

$|\vec{v}|^2 = $ const., let's calculate $\dot{x}^2+\dot{y}^2+\dot{z}^2$

$$= \frac{c^2}{[\ ]}\left\{\left(p_x+\frac{qB}{c}y\right)^2+p_y^2+p_z^2\right\} = c^2\,\frac{[\ ]-m^2c^2}{[\ ]}$$

$$\Rightarrow \frac{v^2}{c^2}-1 = -\frac{m^2c^2}{[\ ]} \quad \text{or} \quad [\ ] = \frac{m^2c^2}{1-\frac{v^2}{c^2}} = (\gamma mc)^2$$

so we find
$$\left\{\begin{aligned}\dot{x} &= \frac{1}{\gamma m}\left(p_x+\frac{qB}{c}y\right)\\ \dot{y} &= \frac{1}{\gamma m}p_y\end{aligned}\right. \Rightarrow \left\{\begin{aligned}\gamma m\ddot{x} &= \frac{qB}{c}\dot{y}\\ \gamma m\ddot{y} &= -\frac{qB}{c}\dot{x}\end{aligned}\right\}$$

$$\gamma = \frac{1}{\sqrt{1-v^2/c^2}}$$

Fall 2018
Physics Comprehensive Exam

September 18, 2018 (Part 2) 9:00am – 1:00pm

Part 2: Electromagnetism and Statistical Mechanics
6 Total Problems/20 Points Each/Total 120 Points

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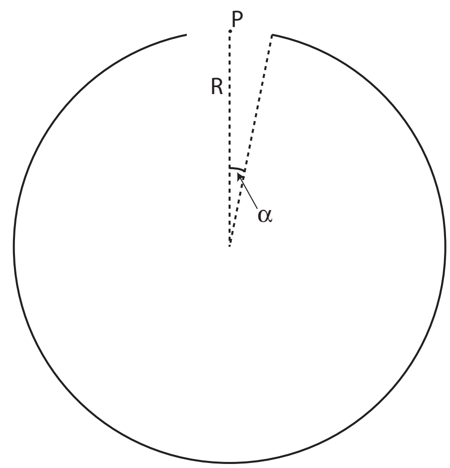
Name: ________________________________

1. Electromagnetism

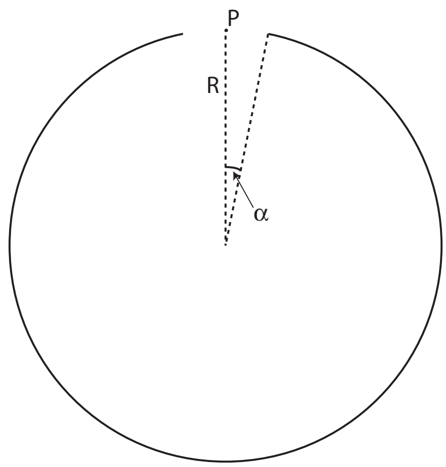
A spherical shell of radius R is uniformly charged so that the charge per unit area on the surface is σ. You take a sword and chop off the very top of the sphere, so that there is a hole at the apex with polar opening angle α, as shown below.



(a) In the limit that the angle α is small (so that the diameter of the opening is much smaller than the radius of the sphere), calculate the electric field at the center of the sphere (magnitude and direction)?

(b) Assuming the same limit, calculate the electric field at point P in the diagram (in the opening, at the location where the apex of the sphere used to be before I sliced off the top)?

[2.] A spherical shell of radius R is uniformly charged so that the charge per unit area on the surface is σ. You take a sword and chop off the very top of the sphere, so that there is a hole at the apex with polar opening angle α, as shown below.



(a) If the angle α is small (so that the diameter of the opening is much smaller than the radius of the sphere), what is the electric field at the center of the sphere (magnitude and direction)?

SOLUTION: We solve this by superposition. We superimpose the electric field generated by two different charge distributions: (1) E_1 due to a complete spherical shell with uniform charge density σ and (2) E_2 due to a uniformly negatively charged spherical cap (charge density σ) with radius of curvature R and with extent in the polar angle α. Adding these two together gives me the charge distribution shown in the figure.

The first of these two objects produces no electric field at the center of the sphere (in fact, it produces no electric field for any $r < R$. So we just need to deal with the negative spherical cap.

The problem statement lets us consider the leading order contribution from the cap (we are told that the cap is very small compared to R). The leading term in the electric field from the cap will be the monopole term; so we can treat the field like that from a point charge located at the apex of the sphere. The total charge of the cap is the surface area times $-\sigma$. The surface area of the cap is:

$$A = R^2 \int_0^{2\pi} d\phi \int_0^{\alpha} \sin\theta d\theta = \sigma 2\pi R^2(1 - \cos\alpha) \approx \pi\alpha^2 R^2$$

The field at the center of the sphere is then:

$$\mathbf{E} \approx \frac{\sigma A}{4\pi\epsilon_o R^2}\hat{z}$$

The field points up, toward the opening.

(b) What is the electric field at point P in the diagram (in the opening, at the location where the apex of the sphere used to be before I sliced off the top)?

SOLUTION: The field at point P is again the superposition of the fields from the two charge distributions. For whole spherical shell, the electric field just above the sphere at point P is:

$$\mathbf{E}_{1,\text{above}} = \hat{z}\frac{\sigma}{\epsilon_o}$$

You can get the above from knowing the field of the whole sphere is the same as if a point charge of total charge $\sigma 4\pi R^2$ was sitting at the origin (center of the sphere). You can also get it from the jump condition for E at the surface.

$$\hat{z} \cdot (\mathbf{E}_{1,\text{above}} - \mathbf{E}_{1,\text{below}}) = \frac{\sigma}{\epsilon_o}$$

And you use $E_{1,\text{below}} = 0$; the field below is made zero by contributions from elsewhere on the sphere.

The field at point P due to the cap can be obtained the same way. If the cap is small, so that we can treat it as a small disk/plane, then the field just above and just below the plane is:

$$E_{2,\text{above}} = -\frac{\sigma}{2\epsilon_o}\hat{z}$$

and

$$E_{2,\text{below}} = \frac{\sigma}{2\epsilon_o}\hat{z}$$

If the cap were large, this answer would not be correct – the cap would have too much curvature and the field would not be the same as an infinite plane (e.g. take the limit of the cap being the whole sphere, and you get zero electric field under the surface at point P).

We add these two together to find the same field above and below point P (we had better as there is no charge there to introduce a discontinuity!):

$$\mathbf{E}_P = \frac{\sigma}{2\epsilon_o}\hat{z}$$

2. Electromagnetism

A solenoid of radius R with n turns per unit of length carries a stationary current I. Two hollow cylinders of length l are fixed coaxially and freely rotating. One cylinder of radius a is inside the coil ($a< R$) and carries the uniformly distributed charge Q. The outer cylinder of radius b ($b>R$) carries the charge $-Q$. If the current is switched off the cylinders start to rotate.

a) Calculate the angular momentum of each cylinder.
b) Calculate the total angular momentum at the end and explain where it is coming from.

Electromagnetic field angular momentum

a) - Solenoid field $\vec{B} = \mu_0 n I \hat{z}$

When current is switched off, induction field arises

@ $r=a$ $\qquad 2\pi a E_\vartheta = -\mu_0 n \pi a^2 \frac{dI}{dt}$

$$E_\vartheta = -\frac{\mu_0 n a}{2}\frac{dI}{dt}$$

Inner cylinder will start to rotate clockwise

$$\vec{L}_{in} = \int \hat{r} \times \vec{F}\, dt = \int \hat{r} \times q\vec{E}\, dt = a q \frac{\mu_0 n a}{2} I \hat{z}$$

@ $r=b$ $\qquad 2\pi b E_\vartheta = -\mu_0 n \pi R^2 \frac{dI}{dt}$

$$\vec{L}_{OUT} = \int \hat{r} \times -q\vec{E}\, dt = -\frac{\cancel{b}\mu_0 n \cancel{\pi} R^2}{2\cancel{b}\cancel{\pi}} q I \hat{z}$$

$$\vec{L}_{tot} = -\frac{\mu_0 n (R^2 - a^2)}{2} q I \hat{z}$$

b) Field momentum

$$\vec{E} = \frac{\lambda}{2\pi r \varepsilon_0}\hat{r} \quad a<r<b$$

with $\int \lambda dz = q$

$$\vec{L} = \varepsilon_0 \int \vec{r} \times (\vec{E} \times \vec{B})\, dv$$

$$= \cancel{\varepsilon_0}\left(\int_a^R \cancel{r}\frac{q}{\cancel{2\pi r \varepsilon_0}} \mu_0 n I \cancel{2\pi r}\, dr\right) - \hat{z}$$

$$= \mu_0 n I q \frac{R^2 - a^2}{2}(-\hat{z})$$

3. Electromagnetism

Calculate the scattering cross section for unpolarized light of a small dielectric sphere (electric susceptibility χ) of radius a. ($a << \lambda$)

a) First calculate the induced dipole moment of a sphere in an external field.

b) Then use this induced dipole to calculate the radiated power and then the cross section

c) What changes for $a \sim \lambda$? (Do not calculate the cross section, just qualitatively explain the difference with respect to the main case of this problem and what approach you would follow to calculate this).

d) Can you estimate the cross section in the opposite limit $a >> \lambda$?

Scattering from small dielectric sphere

a) sphere is small compared to wavelength
$\vec{E}_{inc}$ can be assumed uniform

Then we have to solve

it can be done in many ways,
for example assuming

$$\phi_{out} = \frac{B_1 \cos\vartheta}{r} - E_0 r\cos\vartheta$$

$$\phi_{in} = A_1 r\cos\vartheta$$

and solving to get

$$\vec{E}_{in} = \frac{3}{3+\chi}\vec{E}_0 \Rightarrow \vec{P} = \frac{3\chi\varepsilon_0}{3+\chi}\vec{E}_0$$

induced dipole $\quad \vec{p} = \frac{4\pi}{3}a^3\vec{P} = \frac{4\pi}{3}a^3\frac{3\varepsilon_0\chi}{3+\chi}\vec{E}_0$

b) Start from power radiated from dipole

$$\frac{dP}{d\Omega} = \frac{\mu_0}{16\pi^2 c}\left|\hat{r}\times\ddot{\vec{p}}\right|^2 = \frac{\mu_0}{\cancel{16\pi^2} c}\,\omega^4\,\frac{\cancel{16\pi^2}}{\cancel{9}}\,a^6\,\cancel{9}\varepsilon_0^2\left(\frac{\chi}{3+\chi}\right)^2 E_0^2\left|\hat{r}\times\hat{e}_d\right|^2$$

$$\frac{d\sigma}{d\Omega} = \frac{\frac{dP}{d\Omega}}{\frac{1}{2}\varepsilon_0 c E_0^2} = \frac{\varepsilon_0\mu_0\,\omega^4 a^6}{c^2}\left(\frac{\chi}{3+\chi}\right)^2\left|\hat{k}\times\hat{e}_0\right|^2$$

for unpolarized light $\quad " \frac{1}{2}\left(1+\cos^2\vartheta\right)$

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c) for $a \sim \lambda$ we can't assume $\vec{E}$ as uniform in dielectric sphere.
we could try Born approximation assuming $\vec{p} = \varepsilon_0 \chi \vec{E}_{inc}$

d) for $a \gg \lambda$ we are in geometric optic limit

$\frac{d\sigma}{d\Omega}$ = sum of perfectly conducting sphere (backscatter)
+ perfectly absorbing (extinguish transmission)

$$= 2\sigma_{geo} = 2\pi a^2$$

Name: ________________________________

4. Electromagnetism

a) An infinitely long straight wire of negligible cross-sectional area with a uniform linear charge density q_0 is at rest in the inertial frame K'. The frame K' moves with a speed $v = \beta c$, where c is the speed of light, along the direction of the wire with respect to the laboratory frame, K. Write down the electric and magnetic fields in cylindrical coordinates in the rest frame of the wire. Using the Lorentz transformation properties of the fields, find the components of the electric and magnetic fields in the laboratory.

b) What are the charge and current densities associated with the wire in its rest frame? In the laboratory?

c) From the laboratory charge and current densities, calculate directly the electric and magnetic fields in the laboratory. Compare with the results of part a).

5. Statistical Mechanics

Consider a gas of classical and non-interacting atoms in thermal equilibrium at temperature T in a container of volume V and surface area A. Each atom in the 3D bulk has zero potential energy, but when absorbed on the surface, has a negative potential energy $-E_0$ and can be treated as a 2D ideal gas. Each atom has a mass m.

(a) Find the free energy F_B and chemical potential μ_B for the bulk gas with N_B atoms.

(b) Find the free energy F_S and chemical potential μ_S for the surface gas with N_S atoms.

(c) Compute the surface density $\sigma(\rho, T) = N_S/A$ in terms of the bulk density $\rho = N_B/V$ and T. What is the value of σ in the limit of $\hbar \to 0$?

(Hint: $N! \approx N^N/e^N$)

(a)

For the 3D bulk gas with N_B atoms, the energy is simply kinetic, i.e. $E = \frac{p_x^2+p_y^2+p_z^2}{2m}$. The canonical partition function is:

$$Z_B = \frac{1}{N_B!}\left[\int \frac{dp}{\hbar/L} e^{-\frac{p^2}{2mk_BT}}\right]^{3N_B} = \frac{1}{N_B!}\left[\frac{(2\pi m k_B T)^{3/2}}{\hbar^3/V}\right]^{N_B} = \frac{1}{N_B!}\left(\frac{V}{\lambda^3}\right)^{N_B},$$

where $\lambda = \hbar/\sqrt{2\pi m k_B T}$ is the thermal wave length. This leads to a bulk free energy (using $N! \approx N^N/e^N$):

$$F_B = -k_BT \ln Z_B \approx N_B k_B T \ln\left(\frac{N_B\lambda^3}{eV}\right),$$

and a bulk chemical potential:

$$\mu_B = \frac{\partial F_B}{\partial N_B} \approx k_BT \ln\left(\frac{N_B\lambda^3}{V}\right).$$

(b) We proceed similarly to calculate the same quantities for the 2D surface gas with an energy per atom $E = \frac{p_x^2+p_y^2}{2m} - E_0$. We have:

$$\begin{aligned}
Z_S &= \frac{1}{N_S!}\left[\int \frac{dp}{\hbar/L} e^{-\frac{p^2}{2mk_BT}}\right]^{2N_S} e^{\frac{N_S E_0}{k_BT}} = \frac{1}{N_S!}\left(\frac{A}{\lambda^2} e^{\frac{E_0}{k_BT}}\right)^{N_S}, \\
F_S &= -k_BT \ln Z_S \approx N_S k_B T \ln\left(\frac{N_S\lambda^2}{eA}\right) - N_S E_0, \\
\mu_S &= \frac{\partial F_S}{\partial N_S} \approx k_BT \ln\left(\frac{N_S\lambda^2}{A}\right) - E_0.
\end{aligned}$$

(c)

Equating the chemical potentials, we have:

$$\ln\left(\frac{N_B\lambda^3}{V}\right) = \ln\left(\frac{N_S\lambda^2}{A}\right) - \frac{E_0}{k_BT}.$$

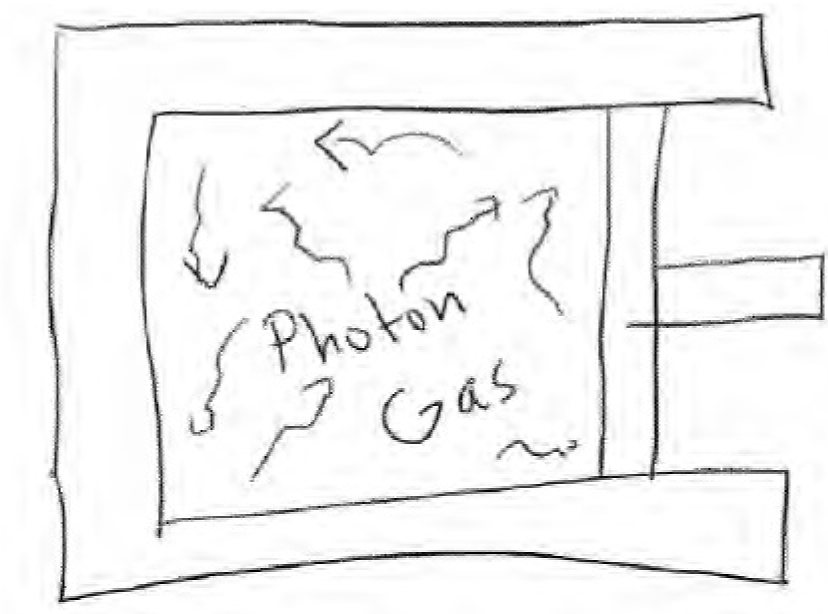
Rewriting with $N_S = \sigma A$ and $N_B = \rho V$ and rearranging terms, we arrive at

$$\sigma(\rho, T) = \frac{\hbar\rho}{\sqrt{2\pi m k_B T}} e^{\frac{E_0}{k_BT}}.$$

Thus, in the limit of $\hbar \to 0$, $\sigma \to 0$.

6. Statistical Mechanics

Carnot cycle with a photon gas.



a) Consider a photon gas in a piston of volume V with walls in equilibrium with a reservoir at temperature T. Show that the energy density, U/V, is proportional to T^4. Hint: Consider the energies of the photon modes of a box and their occupation probability. Also you may use

$$\int_0^\infty \frac{x^3}{e^x - 1}\, dx = \frac{\pi^4}{15}.$$

b) Show that the pressure is given as $P = \frac{1}{3} U/V$. Does the pressure depend on volume at constant T? Why or why not?

c) Show that under adiabatic expansion $VT^\gamma = const$. What is γ?

d) Prove whether or not an engine can be built, using the photon gas as a fluid, that realizes the maximum Carnot cycle efficiency of

$$\eta = \frac{\text{Useful work}}{\text{Heat flow into the system}} = 1 - \frac{T_c}{T_h}$$

where T_c and T_h are the temperature of the cold and hot reservoirs, respectively.

Carnot cycle w/ Photon gas

a) Show that $P = \frac{1}{3} U/V$ and comment on V dependence

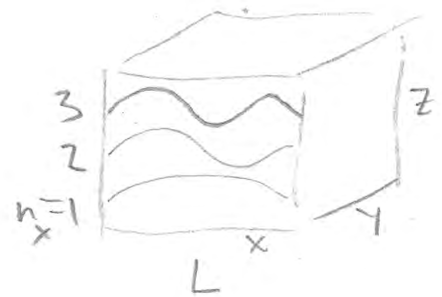
There are a lot of ways to do this.
The students should probably know

$\frac{U}{V} \propto T^4$ (maybe give this to them?)

It can also be derived with one tricky integral

<u>1st method deriving U/V for BBR</u>

Assume a box with modes



$$n_x \frac{\lambda_x}{2} = L \Rightarrow k_x = \frac{2\pi}{\lambda_x} = \frac{\pi n_x}{L}$$

$$E_{\vec{n}} = |\vec{p}|c = \hbar |\vec{k}| c$$

$$= \frac{hc|\vec{n}|}{2L} \qquad |\vec{n}| = \sqrt{n_x^2 + n_y^2 + n_z^2}$$

Total energy

$$U = \underset{\text{polariz.}}{2} \sum_{n_x, n_y, n_z} E_{\vec{n}} \underset{\text{Planck dist}}{\rho(\vec{n})} = \frac{hcn}{L}\left(\frac{1}{e^{\beta E_{\vec{n}}} - 1}\right)$$

convert to integrals ($n_x, n_y, n_z > 0$)

$$U = \int_0^\infty n^2 dn \int_0^{\pi/2} \sin\theta \, d\theta \int_0^{\pi/2} d\theta \, \frac{hcn}{L}\left(\frac{1}{e^{\beta E_n} - 1}\right)$$

$$= \frac{\pi hc}{2L} \int_0^\infty dn \, \frac{n^3}{e^{\beta E_n} - 1} \qquad \text{use } dE = \frac{hc}{2L} dn$$

$$U = L^3 \int_0^\infty \frac{8\pi \, E^3/(hc)^3}{e^{\beta E} - 1} dE$$

Let $x = \beta E$ $\left(\beta = \frac{1}{k_B T}\right)$

$$\Rightarrow \frac{U}{V} = \frac{8\pi (k_B T)^4}{(hc)^3} \underbrace{\int_0^\infty \frac{x^3}{e^x - 1}\, dx}_{\pi^4/15 \text{ give?}}$$

$$\Rightarrow \frac{U}{V} = \frac{8\pi^5 (k_B T)^4}{15 (hc)^3} \Rightarrow \frac{U}{V} \propto T^4$$

From here use thermo relations $\left(\text{start here if } \frac{U}{V} \propto T^4 \text{ given}\right)$

$$dU = -P\,dV + T\,dS$$

$$\Rightarrow P = -\frac{\partial U}{\partial V}\bigg|_S$$

What's S? For constant V $dU = T\,dS$

Using $U = bVT^4$

$dU = 4bVT^3\,dT$ for const. V

$$\Rightarrow S = \int \frac{dU|_V}{T} = \int_0^T \frac{4bVT^3}{T}\,dt$$

$$= \frac{4}{3} bVT^3 \Rightarrow \frac{S}{V} \propto T^3$$

Now express U in terms of V and S

$$U \propto VT^4 \propto V\left(\frac{S}{V}\right)^{\frac{4}{3}} \propto S^{4/3} V^{-1/3}$$

$$\boxed{P = -\frac{\partial U}{\partial V}\bigg|_S = \frac{U}{3V}}$$

2nd method w/o deriving U/V

For any gas the force on a wall is

$$F = \underbrace{\underset{\text{density}}{n}\ \underset{\text{towards wall}}{v_x}\ \underset{\text{Area}}{A}}_{\text{hits per unit time}} \times \underbrace{2p_x}_{\text{change in momentum}}$$

$$\Rightarrow P = \frac{F}{A} = 2\,n\,v_x\,p_x$$

Average over momentum distribution (isotropic)

$$PV = \underset{\substack{\text{only}\\ \text{half towards}\\ \text{wall}}}{\tfrac{1}{2}}\,2\,nV\,\frac{\langle v\cdot p\rangle}{3}$$

For photons $v = c$ $p = E/c$

$$\Rightarrow PV = N\frac{\langle E\rangle}{3} = \frac{U}{3}$$

$$\boxed{\Rightarrow \frac{1}{3}\frac{U}{V} = P}$$

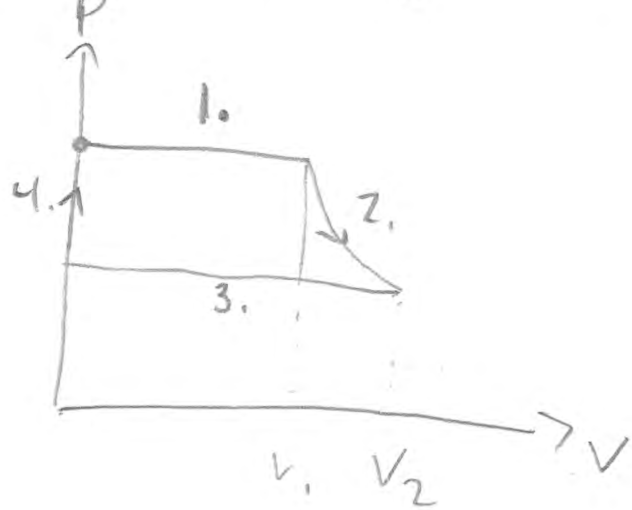
Since U/V is constant at const. T, P is as well

As volume changes, pressure doesn't. This is because the walls create photons. N is proportional to V unlike in an ideal gas.

b) Carnot cycle

1. Isothermal expansion at T_h
2. Adiabatic expansion
3. Isothermal compression at T_c
4. Adiabatic expansion.

Photon gas can go to $V=0$ so cycle is



$$PV = \frac{U}{3} \Rightarrow dU = 3P\,dV + 3\,dP\,V$$

For adiabatic $dS=0 \Rightarrow dU = -P\,dV$

combined $\quad 4P\,dV = -3V\,dP$

$$4\frac{dV}{V} + \frac{dP}{P} = 0$$

Integrate

$$\Rightarrow \frac{4}{3}\ln V + \ln P = \ln \text{const.}$$

$$\Rightarrow PV^{4/3} = \text{const.} \quad (\text{adiabatic})$$

Use 1st $\quad \Delta U = Q - W$

① $Q_h = \Delta U - W_1 = bV_1T_h^4 + \int_0^{V_1} P\,dV$

$$= bV_1T_4^4 + \frac{1}{3}b\int_0^{V_1} \frac{bVT_h^4}{V}\,dV = \frac{4}{3}b\,V_1T_h^4$$

Similarly

(3) $Q_c = -\frac{4}{3} b V_2 T_c^4$

(2) + (4) have $Q=0$

$$\Rightarrow \eta = \frac{Q_h - |Q_c|}{Q_h} = 1 - \frac{|Q_c|}{Q_h}$$

$$\eta = 1 - \frac{V_1 T_h^4}{V_2 T_c^4}$$

but in (2) we have $V_1 T_h^3 = V_2 T_c^3$

since $VT^3 = \text{const}$

$$\Rightarrow \boxed{\eta = 1 - \frac{T_h}{T_c}}$$

which is the same regardless of the gas.