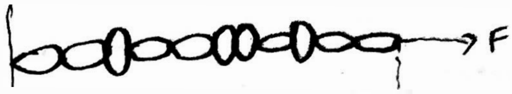
Lim 2012



Consider a one-dimensional chain consisting of $n \gg 1$ segments as illustrated in the figure. Let the length of each segment be a when the long dimension of the segment is parallel to the chain and zero when the segment is vertical (i.e., long dimension normal to the chain direction). Each segment has just two states, a horizontal orientation and a vertical orientation, and each of these states is not degenerate. The distance between the chain ends is nx.

(a) Find the entropy of the chain as a function of x.

(b) Obtain a relation between the temperature T of the chain and the tension F which is necessary to maintain the distance nx, assuming the joints turn freely.

(c) Under which conditions does your answer lead to Hook's law?

(*Princeton*)

QUESTION 4 [14 points]

(a) For any system of fermions at chemical potential μ and temperature T, show that the probability for finding an occupied state of energy $\varepsilon + \mu$ is the same as that for finding an unoccupied state of energy $\mu - \varepsilon$.

Consider now a system of non-interacting Dirac fermions of spin $1/2$ and mass m. One-particle states at momentum $\mathbf{k}$ come in pairs of positive and negative energy,

$$\varepsilon_\pm(\mathbf{k}) = \pm\sqrt{m^2c^4 + \mathbf{k}^2c^2}$$

for each value of the spin quantum number. At $T = 0$, all negative energy Dirac states are filled (the so-called Dirac-sea), and all positive energy states are empty, so that $\mu(T = 0) = 0$.

- **(b)** Using the result of (a) compute the chemical potential at arbitrary temperature T.
- **(c)** Compute (an integral representation for) the mean excitation energy $E(T) - E(0)$
- **(d)** Evaluate the integral in part (c) for $m = 0$ and evaluate the specific heat C_V;
- **(e)** Describe qualitatively the dependence on m in the specific heat at low temperature when $m \neq 0$.

Problem 4.2. Consider a heteronuclear diatomic molecule with moment of inertia I. In this problem, only the rotational motion of the molecule should be considered.

a) Using classical statistical mechanics, calculate the specific heat $C(T)$ of this system at temperature T.

b) In quantum mechanics, this system has energy levels

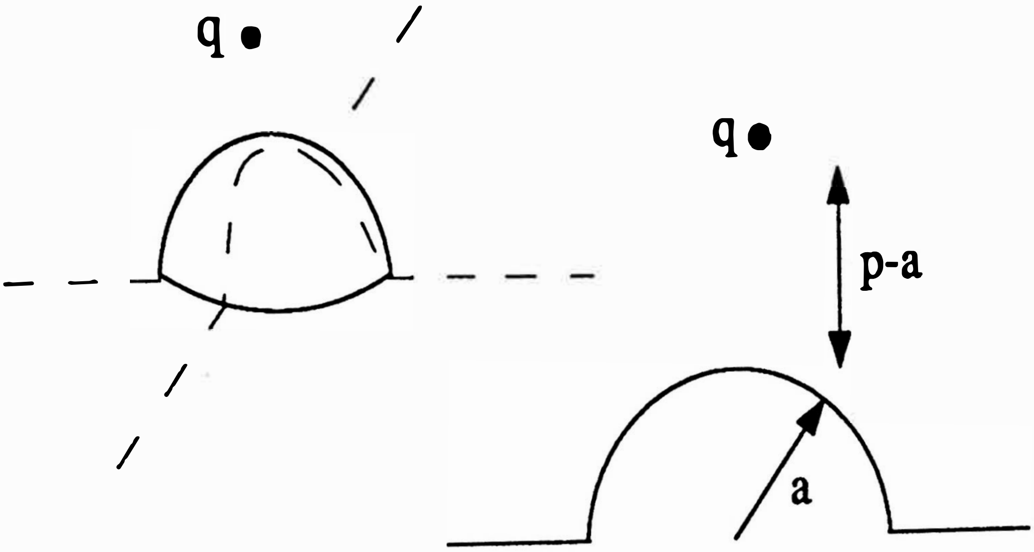
$$E_j = \frac{\hbar^2}{2I} j(j+1) \quad j = 0, 1, 2, \ldots \tag{4.1}$$

Each j level is $(2j + 1)$-fold degenerate. Using quantum statistical mechanics, find expressions for the partition function $\mathcal{Z}$ and the average energy $\langle E \rangle$ of this system, as a function of temperature. Do not attempt to evaluate these expressions.

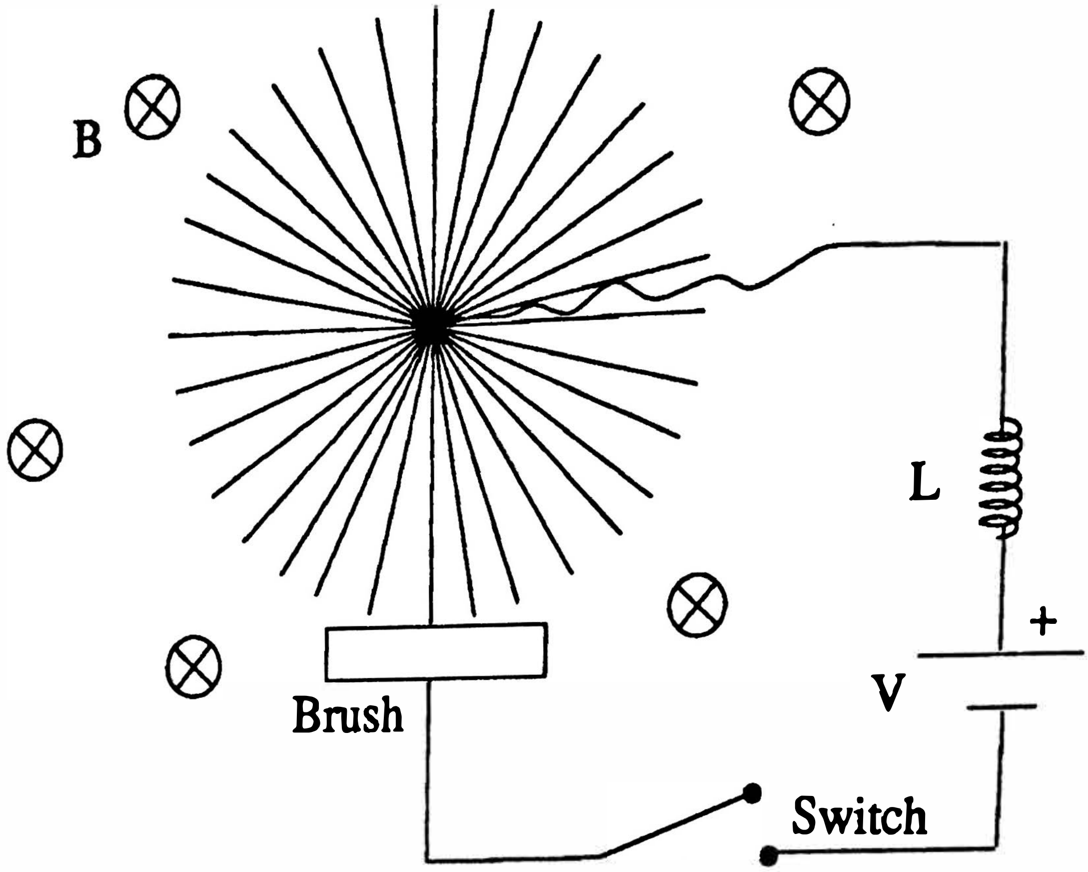
c) By simplifying your expressions in (b), derive an expression for the specific heat $C(T)$ that is valid at very low temperatures. In what range of temperatures is your expression valid?

d) By simplifying your answer to (b), derive a high-temperature approximation to the specific heat $C(T)$. What is the range of validity of your approximation?

Problem 2.1. A conductor at potential $V = 0$ has the shape of an infinite plane except for a hemispherical bulge of radius a (Figure 2.1). A charge q is placed above the center of the bulge, a distance p from the plane (or $p - a$ from the top of the bulge). What is the force on the charge?



Problem 2.5. As shown in Figure 2.3, a wheel consisting of a large number of thin conducting spokes is free to rotate about an axle. A brush always makes electrical contact with one spoke at a time at the



bottom of the wheel. A battery with voltage V feeds current through an inductor, into the axle, through a spoke, to the brush. A permanent magnet provides a uniform magnetic field **B** into the plane of the paper. At time $t = 0$ the switch is closed, allowing current to flow. The radius and moment of inertia of the wheel are R and J respectively. The total inductance of the current path is L, and the wheel is initially at rest. Neglecting friction and resistivity, calculate the battery current and the angular velocity of the wheel as functions of time.

[13.] An electron is bound to a spring with spring constant k.

- **(a)** Calculate the scattering cross-section for unpolarized EM waves incident on the elec-tron.
- **(b)** In what limit should this cross-section equal the Thomson scattering cross section? Take this limit and confirm that it results in the Thomson scattering cross section.
- **(c)** In what limit should this cross-section yield Rayleigh scattering? Take this limit and confirm that the cross section is consistent with Rayleigh scattering (what is the fre-quency dependence you expect?)

14. Electromagnetism

Consider a dielectric medium of infinite extent in all directions. The medium has a tensor dielectric (at zero frequency) given by

$$\overleftrightarrow{\varepsilon}=\begin{pmatrix} \varepsilon_{xx} & 0 & 0 \\ 0 & \varepsilon_{yy} & 0 \\ 0 & 0 & \varepsilon_{zz} \end{pmatrix}$$

with $\varepsilon_{xx} = \varepsilon_{yy} \equiv \varepsilon_{\perp} \neq \varepsilon_{zz}$, and where (x, y, z) refer to Cartesian coordinates. A point charge of charge q is placed at the origin of the coordinate system.

(a) Find the magnitude of the electric field at an arbitrary point (x, y, z), i.e., $|\vec{E}|$.

(b) Deduce the polarization charge density ρ_p induced on the dielectric at an arbitrary point (x, y, z).

(c) Find the total electrical energy density u_E at (x, y, z).