Comprehensive exam, Fall 2009

1. *Quantum Mechanics*

A particle of mass m in three dimensions is subject to the following potential

$$V = -\frac{A}{r} + Br^2 \ , \qquad r^2 = x^2 + y^2 + z^2$$

We can form the following dimensionless combination: $\lambda = \frac{m^3 A^4}{\hbar^6 B}$.

a) Compute the ground state energy for $\lambda \ll 1$, including the first subleading correction for small λ.

b) Repeat for $\lambda \gg 1$.

2. *Quantum Mechanics*

A particle of mass m and charge q sits in a three-dimensional harmonic oscillator potential, $V = \frac{1}{2}k(x^2 + y^2 + z^2)$. At $t = -\infty$ the particle is in the ground state. It is then perturbed by a spatially uniform time dependent electric field, $\vec{E}(t) = E_0 e^{-(t/\tau)^2}\hat{n}$, where A and τ are constants, and $\hat{n}$ is a unit vector. Calculate to lowest order in perturbation theory the probability that the oscillator is in an excited state at $t = +\infty$.

3. *Quantum Mechanics*

Consider a one-dimensional simple harmonic oscillator (SHO) governed by a Hamiltonian $\hat{H}_0$ and known eigenvalues ϵ_n and eigen-kets $|n\rangle$.

$$\begin{aligned}
\hat{H}_0 &= \frac{\hat{p}^2}{2m} + \frac{m\omega_0^2}{2}\hat{x}^2 \\
\hat{H}_0|n\rangle &= \epsilon_n|n\rangle \\
\epsilon_n &= \hbar\omega_0(n+1/2)
\end{aligned}$$

We add a small perturbation, $\hat{H}_0 \to \hat{H} = \hat{H}_0 + \hat{V}$ with $\hat{V} = \lambda\hat{x}^2$ and $\lambda > 0$.

a) Compute the matrix elements $\langle n|\hat{V}|n'\rangle$.

b) Compute the energy shifts in first and second order perturbation energy.

c) The SHO with the perturbation $\hat{V}$, $\hat{H}$, is again a simple harmonic oscillator. Compute the exact eigenvalues of $\hat{H}$ and compare with your result of b).

4. *Quantum Mechanics*

Consider a system of two distinguishable spin 1/2 particles, whose spins are labelled $\mathbf{S}_1$ and $\mathbf{S}_2$, and whose dynamics is governed by the Hamiltonian

$$H = J S_1^z S_2^z + B(S_1^x + S_2^x)$$

where J and B are real couplings. Calculate the spectrum exactly for all values of B and J.

5. *Quantum Mechanics*

Confinement of a heavy quark anti-quark pair may be described by an attractive potential proportional to their inter-distance. We study a one-dimensional model for this effect, whose Hamiltonian in the center of mass of the two quarks is given by,

$$H = \frac{p^2}{2\mu} + \frac{a^3}{2\mu}|x|$$

Here, μ is the reduced mass of the system, and $a > 0$ is a constant.

a) By using the variational method and a one parameter family of trial wave functions,

$$\psi(x;\lambda) = \begin{cases} (\lambda - |x|) & |x| < \lambda \\ 0 & |x| > \lambda \end{cases}$$

for $\lambda > 0$, estimate the ground state energy.

b) How can the variational method be used to estimate the energy of the first excited state as well ? Present an appropriate one-parameter family of trial wave functions.

c) Estimate the energy of the first excited state using (b).

6. *Statistical Mechanics*

A wire of length ℓ and mass per unit length μ is fixed at both ends and tightened to tension τ. What is the rms fluctuation, assuming classical statistics, of the midpoint of the wire when it is in equilibrium at temperature T?

The following series may be helpful:

$$\sum_{m=0}^{\infty} \frac{1}{(2m+1)^2} = \frac{\pi^2}{8}\ , \qquad \sum_{m=0}^{\infty} \frac{1}{(2m+1)^4} = \frac{\pi^4}{96}\ , \qquad \sum_{m=1}^{\infty} \frac{1}{m^2} = \frac{\pi^2}{6}\ , \qquad \sum_{m=1}^{\infty} \frac{1}{m^4} = \frac{\pi^4}{90}$$

7. *Statistical Mechanics*

Show that for independent particles the fluctuations in occupation number n_k of the single-particle state k is given by:

$$\langle (\Delta n_k)^2 \rangle = T \frac{\partial \overline{n}_k}{\partial \mu}$$

for Boltzmann, Fermi, and Bose statistics. Here $\langle (\Delta n_k)^2 \rangle = \langle (n_k - \overline{n}_k)^2 \rangle$, $\overline{n}_k = \langle n_k \rangle$, and μ is the chemical potential.

8. *Statistical Mechanics*

We are interested in the magnetic susceptibility, χ, of a system composed of N identical molecules. The ground state of the molecule is diamagnetic, i.e. the two electrons occupying the highest molecular orbital are in a spin singlet state. The first excited state is a triplet state with excitation energy of Δ. All other excited states and the interaction between the molecules can be neglected.

a) Calculate the temperature dependence of the susceptibility. Discuss the low and high temperature limits.

b) What difference would you expect if the first excited state puts one of the electrons onto the next orbital whose energy is higher (while allowing the two electrons to be in either the singlet or triplet configuration)?

c) Can these two scenarios be distinguished on the basis of a measured $\chi(T)$ curve? If yes, how would you analyze the data?

9. *Statistical Mechanics*

A rubber band can be modeled by a one-dimensional chain of linked segments, as shown in the figure. The segments cannot come apart. Any given element has two states: the "long" state, contributing a distance a to the total length of the band, and a "short" one contributing a distance b. The chain is in contact with a thermal reservoir of temperature T. The only thermal motion of the elements is flipping between the long and short states. Let us denote the number of segments in the long and short states by N_l and N_s respectively. The total number of segments, $N = N_l + N_s$ is constant.

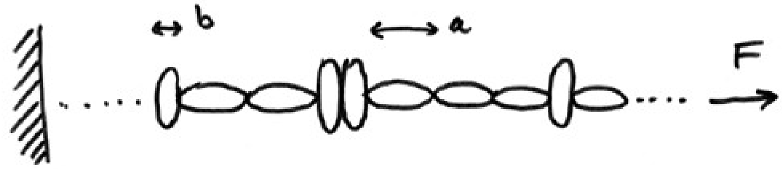

Figure 1:

a) What is the total length of the chain, if there is no energy difference between the long and short states?

b) One end of the chain is fixed, and the other end is pulled by an external force F. What is the length L of the chain in the limit of $T = 0$? What is the length if the temperature is very high? What is the sign of the thermal expansion coefficient? (Justify your answers with short statements; no calculation is necessary.)

c) We characterize the state of the system by a single parameter, $x = (N_l - N_s)/N$. For a given value of x, determine the entropy of the chain.

d) Determine the length of the chain at arbitrary temperature. Discuss the low and high temperature limits.

e) In what temperature range is Hookes's law ($F = \kappa \Delta L/L$, where ΔL is the change of length due to the application of force) valid? What is the spring constant κ?

10. *Electromagnetism*

A particle of mass m_1 and charge q_1 approaches with velocity v and impact parameter r an initially stationary particle of mass m_2 and charge q_2. Assume $v \ll c$ and $m_1 v^2$, $m_2 v^2 \gg q_1 q_2 / r$. Calculate the total energy radiated away by the particles in the dipole approximation.

11. *Electromagnetism*

A linearly polarized plane wave of frequency ω with electric field parallel to the x-axis travels in the positive z direction in a conductor with ε, μ, and real conductivity σ, such that $\sigma \gg \varepsilon\omega$. *Hint: The notation* $\kappa = \sqrt{\frac{\omega\mu\sigma}{2}}$ *is convenient in this problem.*

a) Find the instantaneous and time averaged power loss per unit volume, $\frac{d^4W}{dtdV}$, due to resistive heating for any z.

b) Find the total power loss per unit area, $\left\langle \frac{d^3W}{dtdxdy} \right\rangle$, between $z = 0$ and $z = \infty$.

c) Find the time averaged Poynting vector at any z.

d) Compare the value of your result for b) with the magnitude of your result in c) evaluated at $z = 0$. Is the answer reasonable? Explain

12. *Electromagnetism*

A conducting sphere of radius R is placed in the field of a point charge q at a distance $a > R$ from the center of the sphere. Find the potential φ and the induced surface charge σ on the sphere, if

a) The potential of the sphere is V (assuming that potential $\varphi\,(r \to \infty) = 0$).

b) The total charge of the sphere is Q.

c) Finally, represent the potential φ as a superposition of the contribution of several point charges.

13. *Electromagnetism*

A non-relativistic particle of charge q, mass m, and energy E is scattered on a conducting sphere of radius R placed at the origin. The sphere doesn't move and is grounded. Find the cross-section for the particle to fall onto the sphere (neglect radiation losses by the particle). *Hint: The notation* $\alpha = \left(\frac{1}{4\pi\epsilon_0}\frac{q^2}{R}\right)\frac{1}{E}$ *is convenient for this problem.*

14. *Electromagnetism*

Consider a gas-filled, rectangular waveguide of transverse dimensions $a = 10$ cm, $b = 6$ cm. The relative dielectric permittivity of the gas is $\epsilon/\epsilon_0 = 1.0005$, and one may neglect magnetic effects.

a) A wave is launched in the lowest cutoff frequency ω_c TE_{10}-mode, having angular frequency $\omega = 1.25\omega_c$. What is ω? Find the wavelength in the guide.

b) At a certain point, the gas ionizes, producing a plasma having electron density n_0. Assuming that the permittivity is described only by the plasma response, what is the minimum value of n_0 needed to cut-off propagation?

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$\frac{e^2}{4\pi\varepsilon_0 a} = \frac{\alpha^2 mc^2}{[illegible]}$

$a_0 = \frac{\hbar}{m_e e^2}$?

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$a =$

$\alpha = \frac{e^2}{4\pi\varepsilon_0 \hbar c}$ (e^2)

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1. *Quantum Mechanics*

$\alpha^2 m_e c^2$

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We can form the following dimensionless combination: $\lambda = \frac{m^3 A^4}{\hbar^6 B}$.

a) Compute the ground state energy for $\lambda \ll 1$, including the first subleading correction for small λ.

b) Repeat for $\lambda \gg 1$.

$(e^2) =$

$\frac{e^2}{4\pi\varepsilon_0 a} = \frac{e^4 m}{16\pi^2\varepsilon_0^2 \hbar^2}$

$4\pi\varepsilon_0$

$A^4 = \lambda \cdot \frac{\hbar^4 B}{m^3}$ $\times \frac{2B}{\pi^2\hbar^2}$

$a = \frac{(4\pi\varepsilon_0)\hbar^2}{me^2}$

$\frac{\sqrt{\frac{2Bm^2}{\pi\hbar^3}}}{[illegible]\sqrt{\frac{2B}{[illegible]}}}$

$= \frac{2\lambda \hbar^4 B^2}{\pi^2 m^2}$

$\frac{\frac{B\hbar^6}{m^3 A^4}}{\frac{A^2 [illegible]}{[illegible]}}$

$\sqrt{}$

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a) $\lambda \ll 1$

$$\hat{H}_0 = \frac{p^2}{2m} + Br^2 \quad , \quad \hat{H}' = -\frac{A}{r}$$

let $B = \frac{m\omega^2}{2} \Rightarrow \omega \equiv \sqrt{\frac{2B}{m}}$

$$\Rightarrow \hat{H}_0 = \frac{P_x^2}{2m} + \frac{P_y^2}{2m} + \frac{P_z^2}{2m} + \frac{m\omega^2}{2}(x^2+y^2+z^2)$$

$$E^0_{n_x n_y n_z} = \left(n_x + n_y + n_z + \frac{3}{2}\right)\hbar\omega$$

$$E^0_G = \frac{3}{2}\hbar\omega$$

$$\psi^0_G(\vec{r}) = N e^{-\frac{m\omega r^2}{2\hbar}}$$, N is a normalization factor

$$1 = N^2 \int_0^\infty e^{-\frac{m\omega r^2}{\hbar}} \cdot r^2 dr \cdot 4\pi \quad , \quad u \equiv \sqrt{\frac{m\omega}{\hbar}}\, r$$

$$du = \sqrt{\frac{m\omega}{\hbar}}\, dr$$

$$= N^2 4\pi \left(\frac{\hbar}{m\omega}\right)^{3/2} \cdot \int_0^\infty u^2 e^{-u^2} du$$

$$= N^2 \frac{4\pi}{2} \left(\frac{\hbar}{m\omega}\right)^{3/2} \frac{\partial}{\partial\alpha}\left[-\int_{-\infty}^\infty e^{-\alpha u^2} du\right]_{\alpha=1}$$

$$= -N^2\, 2\pi \left(\frac{\hbar}{m\omega}\right)^{3/2} \cdot \frac{\partial}{\partial\alpha}\sqrt{\frac{\pi}{\alpha}}\Big|_{\alpha=1}$$

$$= +N^2 \frac{\cancel{2}\pi}{\cancel{2}} \cdot \sqrt{\pi} \times \left(\frac{\hbar}{m\omega}\right)^{3/2} = N^2 \times \left(\frac{\pi\hbar}{m\omega}\right)^{3/2}$$

$$\Rightarrow N = \left(\frac{\pi\hbar}{m\omega}\right)^{-\frac{3}{4}} = \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{3}{4}}$$

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$$E_G^{(1)} = \langle \psi_G^0 | \hat{H}' | \psi_G^0 \rangle = -N^2 A \int_0^\infty \frac{1}{r} e^{-\frac{m\omega r^2}{\hbar}} \cdot r^2 dr \cdot 4\pi$$

$$= -4\pi A N^2 \int_0^\infty e^{-u^2} \cdot u\, du \cdot \frac{\hbar}{m\omega} \qquad , \text{let } x = u^2 \quad dx = 2u\,du$$

$$= -\frac{2\pi\hbar}{m\omega} A N^2 \int_0^\infty e^{-x} dx$$

$$\therefore E_G^{(1)} = -\frac{2\pi\hbar}{m\omega} A \cdot \left(\frac{m\omega}{\pi\hbar}\right)^{3/2}$$

$$= \frac{-2\pi\hbar A}{(\pi\hbar)^{3/2}} \cdot \sqrt{m\omega}$$

$$= \frac{-2A}{\sqrt{\pi\hbar}} \cdot \sqrt{m \cdot \sqrt{\frac{2B}{m}}}$$

$$= \frac{-2A}{\sqrt{\pi\hbar}} \cdot \sqrt[4]{2Bm}$$

$$\therefore E_G \simeq \frac{3}{2}\hbar \cdot \sqrt{\frac{2B}{m}} - 2A \sqrt[4]{\frac{2Bm}{\pi^2\hbar^2}} \qquad A^4 = \frac{\lambda \hbar^6 B}{m^3}$$

$$= \frac{3}{2}\hbar\sqrt{\frac{2B}{m}} - 2\left(\frac{2BA^4 m}{\pi^2\hbar^2}\right)^{\frac{1}{4}}$$

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$$= \frac{3}{2}\hbar\sqrt{\frac{2B}{m}} - 2\left[\frac{2\lambda\hbar^4 B^2}{\pi^2 m^2}\right]^{1/4}$$

$$= \frac{3}{2}\hbar\sqrt{\frac{2B}{m}} - 2\left(\frac{\hbar^2 B}{\pi m}\right)^{1/2} \cdot (2\lambda)^{1/4}$$

$$= \hbar\sqrt{\frac{B}{m}}\left[\frac{3}{\sqrt{2}} - \frac{2}{\sqrt{\pi}}(2\lambda)^{1/4}\right]$$

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b) $\lambda \gg 1$

$$\hat{H}_0 = \frac{p^2}{2m} - \frac{A}{r} \quad , \quad \hat{H}' = Br^2$$

$\Rightarrow$ Coulomb potential $\oplus$ Br^2 perturbation

$$\therefore E_n^0 = -\frac{\alpha^2 mc^2}{2n^2}, \quad \text{with} \quad \alpha = \frac{A}{\hbar c}, \quad n = 1, 2, 3, \ldots$$

$$= -\frac{A^2 m \cancel{c^2}}{2\hbar^2 \cancel{c^2} n^2} = \boxed{\frac{-A^2 m}{2\hbar^2 n^2}}$$ ($\alpha = \frac{e^2}{\hbar c}$, originally, cgs unit $4\pi\varepsilon_0 \to 1$)

$$\psi_1^0(\vec{r}) = \frac{1}{\sqrt{\pi a^3}} e^{-\frac{r}{a}} \quad \text{with } a = \frac{\hbar^2}{mA} \quad \left(a_0 = \frac{\hbar^2}{me^2} \text{ originally} \right)$$

$$E_n^1 = \langle \psi_1^0 | \hat{H}' | \psi_1^0 \rangle$$

$$= \frac{\cancel{4} B}{\pi a^3} \iint_{\Omega} \int_0^\infty e^{-\frac{2r}{a}} \cdot r^2 \cdot r^2 dr\, \cancel{d\Omega} \qquad u \equiv \frac{2r}{a}, \quad du = \frac{2}{a} dr$$

$$= \frac{4B}{a^3} \left(\frac{a}{2}\right)^5 \underbrace{\int_0^\infty u^4 e^{-u} du}_{4!}$$

$$= \frac{Ba^2}{\cancel{8}} \times \cancel{4} \times 3 \times \cancel{2}$$

$$= 3Ba^2 = \frac{3B\hbar^4}{m^2 A^2}$$

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$$\therefore E_n^{(0)} + E_n^{(1)} = \frac{-A^2 m}{2\hbar^2} + \frac{3B\hbar^4}{m^2 A^2} = -\frac{A^2 m}{\hbar^2}\left[\frac{1}{2} - \frac{3B\hbar^6}{m^3 A^4}\right] = -\frac{mA^2}{\hbar^2}\left(\frac{1}{2} - \frac{3}{\lambda}\right)$$

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uniform field

$\vec{E}(t) = E_0 e^{-(\frac{t}{\tau})^2} \hat{n}$ $\Rightarrow$ let $\hat{n} = \hat{z}$

$V(t) = -q E_0 z e^{-(\frac{t}{\tau})^2}$

$\hat{H}_0 = \frac{P_x^2 + P_y^2 + P_z^2}{2m} + \frac{1}{2} k (x^2 + y^2 + z^2)$

let $k = m\omega^2 \Rightarrow \omega \equiv \sqrt{\frac{k}{m}}$

$E^0_{n_x n_y n_z} = (n_x + n_y + n_z + \frac{3}{2}) \hbar\omega$

$\therefore E_i^0 = \frac{3}{2}\hbar\omega$. $|\psi_i\rangle_{t=-\infty} = |0\rangle$

$C_{n_x n_y n_z}(t) \simeq \delta_{n_x 0}\delta_{n_y 0}\delta_{n_z 0} - \frac{i}{\hbar} \int_{-\infty}^{t} \langle n_x n_y n_z | V(t) | i \rangle e^{i\omega_{ni} t'} dt'$, where $\omega_{ni} \equiv \frac{E_{n_x n_y n_z} - \frac{3}{2}\hbar\omega}{\hbar} = (n_x + n_y + n_z)\omega$

nonvanishing $\langle n_x n_y n_z | V(t) | i \rangle = -qE_0 \langle n_x\ n_y\ n_z | z | 0,0,0 \rangle e^{-\frac{t^2}{\tau^2}}$

$= -qE_0 e^{-\frac{t^2}{\tau^2}} \delta_{n_x 0} \delta_{n_y 0} \langle n_z | z | 0 \rangle$

$z = \sqrt{\frac{\hbar}{2m\omega}} (a^+ + a)$

$\Rightarrow z|0\rangle = \sqrt{\frac{\hbar}{2m\omega}} |1\rangle$

$\langle n_x n_y n_z | V(t) | i \rangle = -qE_0 e^{-\frac{t^2}{\tau^2}} \delta_{n_x 0} \delta_{n_y 0} \cdot \sqrt{\frac{\hbar}{2m\omega}} \delta_{n_z 1}$

$\underset{\substack{\uparrow \\ n_z}}{C_{001}}(\infty) = \frac{-i}{\hbar} \cdot (-qE_0) \sqrt{\frac{\hbar}{2m\omega}} \int_{-\infty}^{\infty} e^{-\frac{t'^2}{\tau^2}} e^{i\omega t'} dt'$

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the integral $= \int_{-\infty}^{\infty} \exp\left[-\frac{1}{\tau^2}\left(t'^2 - \frac{i\omega}{2}\tau^2 t' - \frac{\omega^2 \tau^4}{4}\right) + \frac{\omega^2\tau^2}{4}\right] dt'$

$$= \int_{-\infty}^{\infty} e^{-\frac{1}{\tau^2}(t' - i\omega\tau^2)^2} e^{-\frac{\omega^2\tau^2}{4}} dt'$$

$$= e^{\frac{-\omega^2\tau^2}{4}} \cdot \sqrt{\pi}\,\tau$$

$\Rightarrow P_{i \to n_z = 1}(\infty) = |C_{001}(\infty)|^2$

$$= \frac{q^2 E_0^2}{\hbar^2} \cdot \frac{\hbar}{2m\omega} \cdot \pi \tau^2 \cdot e^{\frac{-\omega^2\tau^2}{2}}$$

$$= \frac{q^2 E_0^2 \pi \tau^2}{2m\omega\hbar} e^{\frac{-\omega^2\tau^2}{2}}$$

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a)

$$\langle n|\hat{V}|n'\rangle = \lambda \cdot \frac{\hbar}{2m\omega_0} \langle n|(a^+ + a)(a^+ + a)|n'\rangle$$

$$= \frac{\lambda\hbar}{2m\omega_0} \langle n|(a^+ + a)\left[\sqrt{n'+1}\,|n'+1\rangle + \sqrt{n'}\,|n'-1\rangle\right]$$

$$= \frac{\lambda\hbar}{2m\omega_0} \langle n| \left\{ \sqrt{(n'+1)(n'+2)}\,|n'+2\rangle + n'|n'\rangle + (n'+1)|n'\rangle + \sqrt{n'(n'-1)}\,|n'-2\rangle \right\}$$

$$= \frac{\lambda\hbar}{2m\omega_0} \left\{ (2n+1)\delta_{nn'} + \sqrt{(n'+1)(n'+2)}\,\delta_{n,n'+2} + \sqrt{n'(n'-1)}\,\delta_{n,n'-2} \right\}$$

b)

1st order:

$$E_n^{(1)} = \langle n|\hat{V}|n\rangle$$

$$= \frac{\lambda\hbar}{2m\omega_0} \cdot (2n+1)$$

2nd order:

$$E_n^2 = \sum_{m\neq n} \frac{|\langle n|V|m\rangle|^2}{E_n^0 - E_m'} = \left(\frac{\lambda\hbar}{2m\omega_0}\right)^2 \left\{ \frac{(n-1)n}{E_n^0 - E_{n-2}^0} + \frac{(n+1)(n+2)}{E_n^0 - E_{n+2}^0} \right\}$$

$$= \left(\frac{\lambda\hbar}{2m\omega_0}\right)^2 \left\{ \frac{(n^2-n)}{2\hbar\omega_0} - \left(\frac{n^2+3n+2}{2\hbar\omega_0}\right) \right\}$$

$$= \frac{\lambda^2\hbar^{\cancel{2}}}{2\cancel{\hbar}\omega_0 \cdot 4m^2\omega_0^2} \left[-\overset{2}{\cancel{4}}n - \overset{1}{\cancel{2}} \right]$$

$$= -\frac{\lambda^2\hbar}{4m^2\omega_0^3}(2n+1)$$

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c)

$$\hat{H} = \frac{p^2}{2m} + \frac{m\omega_0^2}{2} X^2 + \lambda X^2$$

$$= \frac{p^2}{2m} + \frac{m\omega_0^2}{2}\left(1 + \frac{2\lambda}{m\omega_0^2}\right) X^2$$

$$\therefore E_n' = (n+\tfrac{1}{2})\hbar\omega, \quad \text{with } \omega \equiv \omega_0\sqrt{1+\frac{2\lambda}{m\omega_0^2}}$$

when λ is small

$$\omega \simeq \omega_0\left[1 + \frac{\lambda}{m\omega_0^2} - \frac{1}{8}\left(\frac{2\lambda}{m\omega_0^2}\right)^2\right]$$

$$\Rightarrow E_n' \simeq (n+\tfrac{1}{2})\hbar\omega_0\left[1 + \frac{\lambda}{m\omega_0^2} - \frac{\lambda^2}{2m^2\omega_0^4}\right]$$

It's consistent with the result in b)

$$E_n^{(1)} = \frac{\lambda\hbar}{m\omega_0}(n+\tfrac{1}{2})$$

$$E_n^{(2)} = -\frac{\lambda^2\hbar^3}{2m^2\omega_0^3}(n+\tfrac{1}{2})$$

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$$\hat{H} = J\ S_1^z \otimes S_2^z + B\ (S_1^x \otimes I_2 + I_1 \otimes S_2^x)$$

$$= \frac{J\hbar^2}{4}\begin{pmatrix} 1 & 0 & & \\ 0 & -1 & & 0 \\ & & -1 & 0 \\ & 0 & 0 & 1 \end{pmatrix} + \frac{B\hbar}{2}\left[\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} + \begin{pmatrix} S^x & 0 \\ 0 & S^x \end{pmatrix}\right]$$

$$= \frac{J\hbar^2}{4}\left(\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ \hline 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right) + \frac{B\hbar}{2}\left(\begin{array}{cc|cc} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ \hline 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{array}\right)$$

$$= \frac{\hbar}{2}\begin{array}{c} \begin{array}{cccc} |\uparrow\uparrow\rangle & |\uparrow\downarrow\rangle & |\downarrow\uparrow\rangle & |\downarrow\downarrow\rangle \end{array} \\ \begin{pmatrix} \frac{J\hbar}{2} & B & B & 0 \\ B & -\frac{J\hbar}{2} & 0 & B \\ B & 0 & -\frac{J\hbar}{2} & B \\ 0 & B & B & \frac{J\hbar}{2} \end{pmatrix} \end{array} \begin{array}{c} |\uparrow\uparrow\rangle \\ |\uparrow\downarrow\rangle \\ |\downarrow\uparrow\rangle \\ |\downarrow\downarrow\rangle \end{array}$$

$$\det\begin{pmatrix} \frac{J\hbar}{2}-\lambda & B & B & 0 \\ B & -\frac{J\hbar}{2}-\lambda & 0 & B \\ B & 0 & -\frac{J\hbar}{2}-\lambda & B \\ 0 & B & B & \frac{J\hbar}{2}-\lambda \end{pmatrix} = 0$$

$$\Rightarrow \left(\frac{J\hbar}{2}-\lambda\right)\cdot\left[\left(\frac{J\hbar}{2}-\lambda\right)\left(\frac{J\hbar}{2}+\lambda\right)^2 + 2\left(\frac{J\hbar}{2}+\lambda\right)B^2\right] + B\left[B\left(+\frac{J\hbar}{2}+\lambda\right)\left(\frac{J\hbar}{2}-\lambda\right) + \cancel{B^3} - \cancel{B^3}\right]$$

$$+ B\left[\cancel{B^3} - \cancel{B^3} + \left(\frac{J\hbar}{2}+\lambda\right)\left(\frac{J\hbar}{2}-\lambda\right)B\right] = 0$$

$$\lambda = \pm\frac{J\hbar}{2} \quad \&$$

$$\left(\frac{J\hbar}{2}-\lambda\right)\left(\frac{J\hbar}{2}+\lambda\right)^2 + 2B^2\cancel{\left(\frac{J\hbar}{2}+\lambda\right)} + 2B\cancel{\left(\frac{J\hbar}{2}+\lambda\right)} = 0$$

$$-\lambda^2 + \frac{J^2\hbar^2}{4} + 4B^2 = 0$$

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$$\Rightarrow \quad \lambda = \pm\sqrt{\frac{J^2\hbar^2}{4} + 4B^2}$$

$$\because \quad E = \frac{\lambda\hbar}{2}$$

$$\therefore \quad E = \pm\frac{J\hbar^2}{4}, \quad \pm\frac{\hbar}{2}\sqrt{\frac{J^2\hbar^2}{4} + 4B^2}$$

eigenvalue

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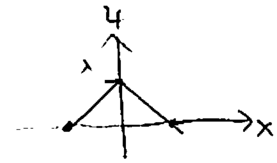
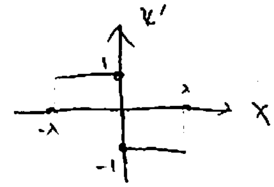
a) $\langle\psi|T|\psi\rangle$:

$\psi(x;\lambda)$:



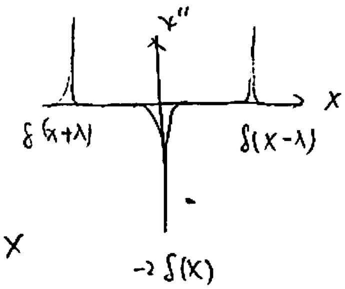
$T|\psi\rangle = \frac{-\hbar^2}{2\mu}\frac{\partial^2}{\partial x^2}(\lambda - |x|)$

$\frac{d\psi(x;\lambda)}{dx}$:



$= \frac{-\hbar^2}{2\mu}\left[\delta(x+\lambda)+\delta(x-\lambda) - 2\delta(x)\right]$

$\Rightarrow \frac{d^2\psi}{dx^2}(x;\lambda)$ =



$\langle\psi|T|\psi\rangle = \int_{-\lambda}^{\lambda}\frac{-\hbar^2}{2\mu}(\lambda-|x|)\left[\delta(x+\lambda)+\delta(x-\lambda)-2\delta(x)\right]dx$

$= +\frac{\cancel{2}\hbar^2}{\cancel{2}\mu}\cdot\lambda$

$= \frac{\hbar^2}{\mu}\lambda$

$\langle\psi|V|\psi\rangle = \int_{-\lambda}^{\lambda}(\lambda-|x|)^2\cdot\frac{a^3}{2\mu}|x|\,dx$

$= \cancel{2}\int_0^{\lambda}(\lambda-x)^2\, x\,\frac{a^3}{\cancel{2}\mu}dx$

$= \frac{a^3}{\mu}\int_0^{\lambda}(x^3-2\lambda x^2+\lambda^2 x)\,dx$

$= \frac{a^3}{\mu}\left[\frac{\lambda^4}{4}-2\lambda\frac{\lambda^3}{3}+\frac{\lambda^4}{2}\right]$

$= \frac{a^3\lambda^4}{\mu}\left(\frac{3-8+6}{12}\right)$

$= \frac{a^3\lambda^4}{12\mu}$

$\langle\psi|\psi\rangle = \int_{-\lambda}^{\lambda}(\lambda-|x|)^2\,dx = 2\int_0^{\lambda}(x^2-2\lambda x+\lambda^2)\,dx$

$= 2\lambda^3\left[\frac{1}{3}-\cancel{1}+\cancel{1}\right] = \frac{2}{3}\lambda^3$

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$$\therefore E(\lambda) = \frac{\frac{\hbar^2}{\mu}\lambda + \frac{a^3\lambda^4}{12\mu}}{\frac{2}{3}\lambda^3}$$

$$= \frac{3\hbar^2}{2\mu\lambda^2} + \frac{a^3\lambda}{8\mu}$$

$$\left.\frac{dE}{d\lambda}\right|_{\lambda_0} = 0 = -\frac{3\hbar^2}{\mu\lambda_0^3} + \frac{a^3}{8\mu}$$

$$\lambda_0^3 = \frac{8\times 3\hbar^2}{a^3}$$

$$\lambda_0 = \frac{2}{a}(3\hbar^2)^{1/3}$$

$$\Rightarrow E(\lambda_0) = \frac{3\hbar^2}{2\mu\left(\frac{2}{a}\right)^2(3\hbar^2)^{2/3}} + \frac{a^{\cancel{3}\,2}}{\cancel{8}_4\mu} \times \frac{\cancel{8}^1}{\cancel{a}}(3\hbar^2)^{1/3}$$

$\geq \hat{E}_G$

$$= \frac{3\hbar^2 a^2}{8\mu(3\hbar^2)^{2/3}} + \frac{a^2}{4\mu}(3\hbar^2)^{1/3}$$

$$= \frac{a^2}{8\mu}\left(\frac{27\hbar^6}{9\hbar^4}\right)^{1/3} + \frac{a^2}{4\mu}(3\hbar^2)^{1/3} = \frac{3a^2}{8\mu}(3\hbar^2)^{1/3}$$

Check virial theorem.

$2\langle T\rangle = \langle V\rangle$

$\langle E\rangle = \langle T\rangle + \langle V\rangle$

$= \frac{3}{2}\langle V\rangle$.

$\langle T\rangle = \frac{1}{3}\langle E\rangle$, $\langle V\rangle = \frac{2}{3}\langle E\rangle$

✓

b)

* Since The Hamiltonian is symmetric under parity, so the wave functions must be odd or even states.

For ground state, it is even and has 0 node;

For 1st excited state, it is odd and has 1 node, (at origin) so we can use $\psi_{t,1}(x) = x(\lambda - |x|)$ as our trial wave function for 1st excited state.

likewise, for nth excited state, $\psi_{t,n}(x) = x^n(\lambda - |x|)$

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c)

$$\frac{\partial}{\partial x}\left[x(\lambda-|x|)\right] = \begin{cases} \lambda-2x & x>0 \\ \lambda+2x & x<0 \end{cases}$$

$$\frac{\partial^2}{\partial x^2}\left[x(\lambda-|x|)\right] = \begin{cases} -2 & x>0 \\ 2 & x<0 \end{cases}$$

$$\langle\psi_1|T|\psi_1\rangle = \frac{-\hbar^2}{2\mu}\int_{-\lambda}^{\lambda} x(\lambda-|x|)\frac{\partial^2}{\partial x^2}\left[x(\lambda-|x|)\right]dx$$

$$= \frac{-\hbar^2}{2\mu}\left\{\int_0^\lambda x(\lambda-x)(-2)\,dx + \int_{-\lambda}^0 x(\lambda+x)\,2\,dx\right\}$$

$$= \frac{-\hbar^2}{\mu}\left\{\int_0^\lambda (x^2-\lambda x)\,dx + \int_{-\lambda}^0 (x^2+\lambda x)\,dx\right\}$$

$$= \frac{-\hbar^2}{\mu}\lambda^3\left\{-\frac{1}{6} + \frac{1}{3}-\frac{1}{2}\right\}$$

$$= \frac{\hbar^2}{3\mu}\lambda^3$$

$$\langle\psi_1|V|\psi_1\rangle = \frac{a^3}{2\mu}\left\{\int_{-\lambda}^{\lambda} x^2(\lambda-|x|)^2|x|\,dx\right\}$$

$$= \frac{a^3}{\mu}\cdot\int_0^\lambda (x^5-2\lambda x^4+\lambda^2x^3)\,dx$$

$$= \frac{a^3}{\mu}\lambda^6\left(\frac{1}{6}-\frac{2}{5}+\frac{1}{4}\right)$$

$$= \frac{a^3}{60\mu}\lambda^6$$

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$$\langle\psi_1|\psi_1\rangle = \int_0^{\lambda} x^2 (x-\lambda)^2 \, dx$$

$$= \int_0^{\lambda} (x^4 - 2\lambda x^3 + \lambda^2 x^2) \, dx$$

$$= \lambda^5 \left(\frac{1}{5} - \frac{\cancel{2}}{\cancel{4}_2} + \frac{1}{3} \right)$$

$$= \frac{\lambda^5}{30}$$

$$\therefore E_1(\lambda) = \frac{\frac{\hbar^2}{3\mu}\lambda^3 + \frac{a^3}{60\mu}\lambda^6}{\frac{\lambda^5}{30}}$$

$$= \frac{10\hbar^2}{\mu}\frac{1}{\lambda^2} + \frac{a^3\lambda}{2\mu}$$

$$\left.\frac{\partial E_1}{\partial \lambda}\right|_{\lambda_0} = 0 = \frac{-20\hbar^2}{\mu\lambda_0^3} + \frac{a^3}{2\mu} \Rightarrow \lambda_0 = \left(\frac{20\hbar^2}{\cancel{\mu}} \times \frac{2\cancel{\mu}}{a^3}\right)^{1/3} = \left(\frac{8}{a^3}\right)^{1/3} \cdot (5\hbar^2)^{1/3} = \frac{2(5\hbar^2)^{1/3}}{a}$$

$$\therefore E_1(\lambda_0) = \frac{10\hbar^2}{\mu}\frac{1}{\left(\frac{2}{a}\right)^2 (5\hbar^2)^{2/3}} + \frac{a^{\cancel{3}2}}{\cancel{2}\mu} \cdot \frac{\cancel{2}}{\cancel{a}} (5\hbar^2)^{1/3}$$

$$= \frac{5\hbar^2 a^2}{2\mu (5\hbar^2)^{2/3}} + \frac{a^2}{\mu} (5\hbar^2)^{1/3}$$

$$= \frac{1}{2}\frac{a^2}{\mu}(5\hbar^2)^{1/3} + \frac{a^2}{\mu}(5\hbar^2)^{1/3}$$

$$= \frac{3}{2}\frac{a^2}{\mu}(5\hbar^2)^{1/3}$$

Question # 7 Name ______________________________

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$$Q_k = \sum_{n=0}^{n_{max}} e^{\frac{n(\mu-\varepsilon_k)}{T}} \quad , \quad \bar{n}_k = \sum_n n P(n) = \frac{\sum_{n=0}^{n_{max}} n e^{n\left(\frac{\mu-\varepsilon_k}{T}\right)}}{Q_k}$$

$$\langle n_k \rangle = \bar{n}_k = \frac{T}{Q_k}\frac{\partial Q_k}{\partial \mu}$$ ✓

$$\langle n_k^2 \rangle = \sum_{n=0}^{n_{max}} n^2 P(n) = \frac{\sum_{n=0} n^2 e^{\frac{n(\mu-\varepsilon_k)}{T}}}{Q_k}$$

$$= \frac{T^2}{Q_k}\frac{\partial^2 Q_k}{\partial \mu^2}$$ ✓

~~15~~

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$$\therefore \langle (\Delta n_k)^2 \rangle = \langle (n_k - \bar{n}_k)^2 \rangle$$

$$= \langle n_k^2 - 2 n_k \bar{n}_k + \bar{n}_k^2 \rangle$$

$$= \langle n_k^2 \rangle - 2\langle n_k \rangle^2 + \langle n_k \rangle^2$$

$$= \langle n_k^2 \rangle - \langle n_k \rangle^2$$

$$= \frac{T^2}{Q_k}\frac{\partial^2 Q_k}{\partial \mu^2} - \left(\frac{T}{Q_k}\frac{\partial Q_k}{\partial \mu}\right)^2$$

$$= T \cdot \frac{\partial}{\partial \mu}\left[\frac{T}{Q_k}\frac{\partial Q_k}{\partial \mu}\right]$$

$$= T\frac{\partial \bar{n}_k}{\partial \mu}$$ ✓ , the equation is for any kind of statistics ✓

for Boltzmann stat. $\bar{n}_k = e^{\frac{\mu-\varepsilon_k}{T}}$

Fermi " $\bar{n}_k = \frac{1}{e^{\frac{\varepsilon_k-\mu}{T}}+1}$

Bose " $\bar{n}_k = \frac{1}{e^{(\varepsilon_k-\mu)/T}-1}$

Question # 8

Page # 1

Name William Farmer

OK TO CONTACT

a) $Z_0 = 1 + 3e^{-\Delta/\tau}$ $\Rightarrow$ w/out spin.

With Spin, $U = -\vec{\mu}\cdot\vec{B}$, $\vec{\mu} = 0$ for the singlet,

$U = \begin{cases} -\mu B \\ 0 \\ \mu B \end{cases}$ for the triplet depending on the z-th component of Spin.

Thus, $Z_0 = 1 + e^{-\frac{(\Delta+\mu B)}{\tau}} + e^{-\frac{\Delta}{\tau}} + e^{-\frac{(\Delta-\mu B)}{\tau}} = 1 + e^{-\Delta/\tau}\left(2\cosh\frac{\mu B}{\tau} + 1\right)$

The magnetization is given by

$$M = \tau \frac{\partial}{\partial B} \ln Z = \tau \frac{\partial}{\partial B} \ln\left[\frac{1}{N!} Z_0^N\right]$$

$$= N\tau \frac{2e^{-\frac{\Delta}{\tau}} \sinh\frac{\mu B}{\tau}}{1 + e^{-\frac{\Delta}{\tau}}\left(1 + 2\cosh\frac{\mu B}{\tau}\right)} \cdot \frac{\mu}{\tau} = N\mu \frac{2\sinh\left(\frac{\mu B}{\tau}\right)}{e^{\frac{\Delta}{\tau}} + 1 + 2\cosh\left(\frac{\mu B}{\tau}\right)}$$

$$\chi = \left(\frac{\partial M}{\partial B}\right)_\tau = N\mu\left[\frac{2\cosh\left(\frac{\mu B}{\tau}\right)}{1 + e^{\Delta/\tau} + 2\cosh\left(\frac{\mu B}{\tau}\right)} - \frac{2\sinh\left(\frac{\mu B}{\tau}\right)\cdot 2\sinh\left(\frac{\mu B}{\tau}\right)}{\left(e^{\frac{\Delta}{\tau}} + 1 + 2\cosh\left(\frac{\mu B}{\tau}\right)\right)^2}\right]\frac{\mu}{\tau}$$

$$= \frac{N\mu^2}{\tau}\left[\frac{\left(1 + e^{\frac{\Delta}{\tau}}\right)2\cosh\left(\frac{\mu B}{\tau}\right) + 4\left(\cosh^2\frac{\mu B}{\tau} - \sinh^2\frac{\mu B}{\tau}\right)}{\left(1 + e^{\Delta/\tau} + 2\cosh\frac{\mu B}{\tau}\right)^2}\right]$$

$$\chi = \frac{N\mu^2}{\tau} \frac{\left(1 + e^{\Delta/\tau}\right)2\cosh\left(\frac{\mu B}{\tau}\right) + 4}{\left(1 + e^{\Delta/\tau} + 2\cosh\frac{\mu B}{\tau}\right)^2}$$

in $B \Rightarrow 0$

In the high temperature limit,

$$\chi \simeq \frac{N\mu^2}{\tau}\frac{8}{16} = \frac{N\mu^2}{2\tau}$$

In the low temperature limit

$$\chi \simeq \frac{N\mu^2}{\tau}\frac{2\left(e^{\frac{\Delta}{\tau} + \frac{\mu B}{\tau}}\right)}{\left(e^{\Delta/\tau} + 2e^{\mu B/\tau}\right)^2} = \frac{2N\mu^2}{\tau}\frac{e^{\Delta/\tau + \frac{\mu B}{\tau}}}{e^{\frac{2\Delta}{\tau}} + 4e^{\frac{2\mu B}{\tau}} + 4e^{\frac{\Delta}{\tau} + \frac{\mu B}{\tau}}}$$

$$\chi = \frac{2N\mu^2}{\tau}\frac{1}{e^{\Delta/\tau - \mu B/\tau} + 4e^{\mu B/\tau - \Delta/\tau} + 4} \rightarrow 0$$

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Question # 8 Name William Farmer

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b) Here $Z = 1 + e^{-\Delta/\tau}(2 + 2\cosh\frac{\mu B}{\tau})$

As there is an additional excited state.

The derivation is the same

$$M = N\mu \frac{2e^{-\Delta/\tau}\sinh(\frac{\mu B}{\tau})}{1 + e^{-\Delta/\tau}(2 + 2\cosh\frac{\mu B}{\tau})} = N\mu \frac{2\sinh\frac{\mu B}{\tau}}{e^{\Delta/\tau} + 2 + 2\cosh\frac{\mu B}{\tau}}$$

$$\chi = \frac{N\mu^2}{\tau} \frac{(2 + e^{\Delta/\tau})2\cosh\frac{\mu B}{\tau} + 4}{(e^{\Delta/\tau} + 2 + 2\cosh\frac{\mu B}{\tau})^2}$$

lim $B \to 0$

In the high temp limit,

$$\chi = \frac{N\mu^2}{\tau}\frac{10}{25} = \frac{N\mu^2}{\tau}\frac{2}{5}$$

B

c) So there is a difference in the high temperature limit.

This is assuming I interpreted what you said correctly. I didn't really understand what you said to the class.

B

Name:

9. *Statistical Mechanics*

A rubber band can be modeled by a one-dimensional chain of linked segments, as shown in the figure. The segments cannot come apart. Any given element has two states: the "long" state, contributing a distance a to the total length of the band, and a "short" one contributing a distance b. The chain is in contact with a thermal reservoir of temperature T. The only thermal motion of the elements is flipping between the long and short states. Let us denote the number of segments in the long and short states by N_l and N_s respectively. The total number of segments, $N = N_l + N_s$ is constant.

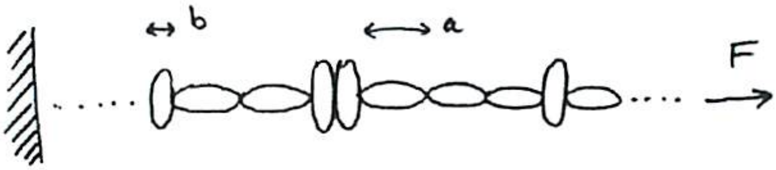

Figure 1:

a) What is the total length of the chain, if there is no energy difference between the long and short states?

b) One end of the chain is fixed, and the other end is pulled by an external force F. What is the length L of the chain in the limit of $T = 0$? What is the length if the temperature is very high? What is the sign of the thermal expansion coefficient? (Justify your answers with short statements; no calculation is necessary.)

c) We characterize the state of the system by a single parameter, $x = (N_l - N_s)/N$. For a given value of x, determine the entropy of the chain.

d) Determine the length of the chain at arbitrary temperature. Discuss the low and high temperature limits.

e) In what temperature range is Hookes's law ($F = \kappa \Delta L/L$, where ΔL is the change of length due to the application of force) valid? What is the spring constant κ?

19 points

$$S = \cancel{\ln N_L} - N_L \ln(N_L) - (N - N_L)\ln(N - N_L)$$

$$= \ln \quad \frac{\partial S}{\partial N_L} = 0 = \ln(N_L) + 1 + \frac{N - N_L}{N - N_L} - \ln(N - N_L)$$

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$$\Gamma = \frac{N!}{N_L!\,N_S!} = \frac{N!}{N_L!\,(N-N_L)!}$$

$$S = \ln(\Gamma) = \ln(N!) - \ln(N_L!) - \ln(N-N_L)!$$

$$\approx N\ln(N) - N_L\ln(N_L) - N\ln(N-N_L) + N_L\ln(N-N_L)$$

$$= N\ln\left(\frac{N}{N-N_L}\right) - \cancel{N_L\left(\frac{N_L}{N-N_L}\right)}\ N_L\ln\left(\frac{N_L}{N-N_L}\right)$$

$$\frac{\partial S}{\partial N_L} = -\ln(N_L) - 1 + \frac{N}{N-N_L} + \ln(N-N_L) + \frac{-N_L}{N-N_L} = 0$$

$$= (N-N_L)\ln\left(\frac{N-N_L}{N_L}\right) - N + N_L + N - N_L = \cancel{\ldots}$$

$$\cancel{= 0 = \ln\left(\frac{N-N_L}{N_L}\right) = 2}$$

$$= (N-N_L)\ln\left(\frac{N-N_L}{N_L}\right) = 0 \Rightarrow \ln\left(\frac{N-N_L}{N_L}\right) = 0$$

$$N - N_L = N_L \qquad N_L = \frac{N}{2}, \quad N_S = \frac{N}{2}$$

b) ~~$dE = TdS - Fdl$~~ $\boxed{L = aN_L + bN_S = \frac{N}{2}(a+b)}$

~~At $T=0$~~

b) At $T=0$ band will be in ground state which will be a fully extended chain ($dE = Tds - Fd\ell$

at $T=0$ $dE = -Fd\ell$ so energy ~~minimi~~ lowered as ℓ grows

Thermal expansion is negative as length shrinks as temperture increases

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$$L = Na$$

At high temp $dE = TdS - Fdl \approx TdS$ so will again be where S is minimized at $N_L = N_S = \frac{N}{2}$

$$L = \frac{N}{2}(a+b) \quad \text{like in a)}$$

c) $N = N_L + N_S \qquad N_S = N - N_L$

$$x = \frac{N_L - (N - N_L)}{N} = \frac{2N_L - N}{N} = 2\frac{N_L}{N} - 1$$

is $k_B = 1$?

$$2\frac{N_L}{N} = 1+x \qquad N_L = \frac{N}{2}(1+x)$$

From result in a) $\quad S = N \ln\left(\frac{N}{N-N_L}\right) - N_L \ln\left(\frac{N_L}{N-N_L}\right)$

$$= N \ln\left[\frac{N}{N - \frac{N}{2}(1+x)}\right] - \frac{N}{2}(1+x)\ln\left[\frac{\frac{N}{2}(1+x)}{N - \frac{N}{2}(1+x)}\right]$$

$$= N \ln\left[\frac{1}{1 - \frac{1+x}{2}}\right] - \frac{N}{2}(1+x)\ln\left[\frac{\frac{1+x}{2}}{1 - \frac{1}{2}(1+x)}\right]$$

$$S = -N \ln\left[\frac{1-x}{2}\right] - \frac{N}{2}(1+x)\ln\left[\frac{1+x}{1-x}\right]$$

6

d) At equilibrium $dE = 0 = TdS - Fdl$

$Fdl = TdS \qquad \frac{dS}{dl} = \frac{-F}{T}$ because F is force band exerts

?

Question # 9 Name

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$$L = N_L a + N_s b = N_L a + (N-N_L)b = Nb + N_L(a-b)$$

$$N_L(a-b) = L - Nb \qquad N_L = \frac{L-Nb}{a-b}$$

$$S = N \ln\left[\frac{N}{N+\frac{Nb-L}{a-b}}\right] - \frac{L-Nb}{a-b}\ln\left[\frac{\frac{L-Nb}{a-b}}{N-\frac{L-Nb}{a-b}}\right]$$

$$\frac{\partial S}{\partial L} = -N\frac{\partial}{\partial L}\ln\left[N+\frac{Nb-L}{a-b}\right] - \frac{1}{a-b}\ln\left[\frac{\frac{L-Nb}{a-b}}{N-\frac{L-Nb}{a-b}}\right]$$

$$- \frac{L-Nb}{a-b}\frac{\partial}{\partial L}\ln\left[L-Nb\right] + \frac{L-Nb}{a-b}\frac{\partial}{\partial L}\ln\left[N-\frac{L-Nb}{a-b}\right]$$

$$= \frac{N\cdot\frac{1}{a-b}}{N+\frac{Nb-L}{a-b}} - \frac{1}{a-b}\ln\left[\frac{\frac{L-Nb}{a-b}}{N-\frac{L-Nb}{a-b}}\right] - \frac{L-Nb}{a-b}\cdot\frac{1}{(L-Nb)}$$

$$+ \frac{L-Nb}{a-b}\cdot\frac{(-1)}{a-b}\cdot\frac{1}{N-\frac{L-Nb}{a-b}} = -\frac{F}{T}$$

$$= \frac{N}{N(a-b)+Nb-L} - \frac{L-Nb}{(a-b)\left[N(a-b)+Nb-L\right]} - \frac{1}{a-b}$$

$$- \ln\left[\frac{L-Nb}{N(a-b)-L+Nb}\right]\cdot\frac{1}{a-b}$$

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$$= \frac{1}{Na-L}\left(N - \frac{L-Nb}{a-b}\right) - \frac{1}{a-b} - \ln\left[\frac{L-Nb}{Na-L}\right]\frac{1}{a-b}$$

$$= \frac{1}{Na-L}\left(\frac{Na-Nb-L+Nb}{a-b}\right) - \frac{1}{a-b} - \ln\left[\frac{L-Nb}{Na-L}\right]\frac{1}{a-b}$$

$$= -\ln\left[\frac{L-Nb}{Na-L}\right]\frac{1}{a-b} = \ln\left[\frac{Na-L}{L-Nb}\right] = \frac{F}{T}$$

$$\frac{Na-L}{L-Nb} = e^{F/T} \Rightarrow Na-L = Le^{F/T} - Nbe^{F/T}$$

$$L(e^{F/T}+1) = Na+Nbe^{F/T}$$

$$L = \frac{Na+Nbe^{F/T}}{1+e^{F/T}}$$

As $T \to \infty$ $\quad L = \frac{Na+Nb}{2} = \frac{N}{2}(a+b)$

As $T \to 0$ $\quad L = \frac{Nae^{-F/T}+Nb}{e^{-F/T}+1} = Nb$

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$$= \frac{1}{a-b} \ln\left[\frac{Na-L}{L-Nb}\right] = \frac{-F}{T}$$

$$\frac{Na-L}{L-Nb} = e^{\frac{-F(a-b)}{T}}$$

$$Na - L = Le^{\frac{-F(a-b)}{T}} - Nbe^{\frac{-F(a-b)}{T}}$$

$$L\left(1+e^{\frac{-F(a-b)}{T}}\right) = Na + Nbe^{\frac{-F(a-b)}{T}}$$

$$L = \frac{Na + Nbe^{\frac{-F(a-b)}{T}}}{1+e^{\frac{-F(a-b)}{T}}}$$

$k_B = 1?$

6

As $T \to \infty$ $\quad L = \frac{Na + Nb}{2} = \frac{N}{2}(a+b)$

As $T \to 0$ and $a > b$ $\quad L = \frac{Na}{1} = Na$

Same as b

e) $F = \frac{-T}{a-b} \ln\left[\frac{Na-L}{L-Nb}\right]$

$\Delta L = L - \frac{N}{2}(a+b) = \cancel{\frac{Na}{b}} \cancel{\ \ } \frac{1}{2}(L - Na + L - Nb)$

$\cancel{F = \frac{T}{a-b} \ln \frac{Na-L}{\Delta L + \frac{1}{2}(L-Na)}} = \cancel{\ \ } \frac{T}{a-b} \ln\left[\frac{Na-L}{\Delta L} - 2\right]$

$= \frac{T}{a-b}\left[\cancel{\frac{Na-L}{2\Delta L} - \frac{2}{\Delta L}} + \cancel{\frac{T\ln(2)}{a-b}} = \cancel{\frac{T}{a+b}} \ln\right.$

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$$\cancel{F = \frac{T}{a-b} \ln\left[\frac{Na-L}{\Delta L} + 2x\right]}$$

$$\Delta L = \frac{1}{2}(L-Na) + \frac{1}{2}(L-Nb)$$

$$Na - L = L - Nb - 2\Delta L$$

$$F = \frac{-T}{a-b} \ln\left[\frac{L-Nb-2\Delta L}{L-Nb}\right]$$

$$= \frac{-T}{a-b} \ln\left[1 - \frac{2\Delta L}{L-Nb}\right]$$

$$\approx \frac{-T}{a-b} \cdot \frac{-2\Delta L}{L-Nb} \approx \frac{2T}{(a-b)} \frac{\Delta L}{L} = k \frac{\Delta L}{L}$$

(3)

$k = \frac{2T}{a-b}$? is valid if $L \gg Nb$

~~Since $L_{MIN} = \frac{Na}{2} + \frac{Nb}{2}$~~

and $L \gg \Delta L$

$\cancel{L = \frac{Na + Nb\,e^{-F(a-b)/T}}{1+e^{-F(a-b)/T}}}$ ~~$L \gg \frac{Nb}{2}$ is equivalent to saying~~

~~$a \gg b$, $\Delta L = L - \frac{Na}{2} - \frac{Nb}{2}$~~

~~$L \gg \Delta L$ requires $\frac{N}{2}(a+b)$~~ $\frac{\Delta L}{L} \ll 1$

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~~$\Delta L = L - \frac{Na}{2} - \frac{Nb}{2}$~~

~~$\frac{\Delta L}{L} = 1 - \frac{N}{2L}(a+b) \ll 1$~~

Requirement of $L \gg Nb$ is met as long as $a \gg b$ so equilibrium position with no force is much longer than shortest position

$\frac{\Delta L}{L} \ll 1$ means ΔL should be small as happens in ~~high~~ ~~low~~ high T so Hooke's law is valid at high temperture

3

Question # 10 Name ______________________________

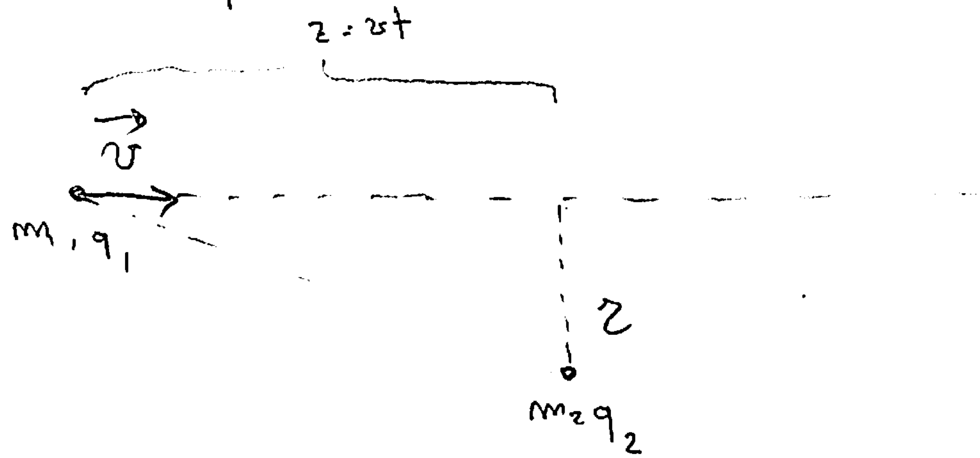
Page # 1

SINCE $m_1 v^2,\ m_2 v^2 >> \frac{q_1 q_2}{r}$

WE CAN TREAT THE PROBLEM PERTURBATIVELY I.E. WE CAN FIRST ASSUME THAT THE TRAJECTORY OF THE TWO PARTICLES IS NOT PERTURBED BY THE COULOMB INTERACTION.

WITH THE GIVEN TRAJECTORY WE CAN INTEGRATE THE INSTANTANEOUS POWER RADIATED KNOWING THE ACCELERATION:

$$m\vec{a} = q\vec{E}$$



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IN THE DIPOLE APPROXIMATION THE POWER PER UNIT SOLID ANGLE IS GIVEN BY

$$\frac{dP}{d\Omega} = \frac{1}{32\pi^2} \frac{Z_0}{c^2} (\omega^2 p)^2 \sin^2\vartheta$$

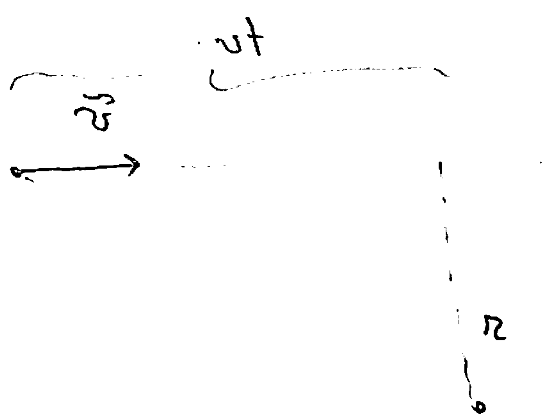
WITH $p = \vec{x} q \Rightarrow \omega^2 p = a \cdot e$

THE TOTAL POWER RADIATED IS THEN

$$\frac{dP}{dt} = \frac{1}{6\pi} \frac{Z_0 e^2 a^2(t)}{c^2}$$

WHICH IS THE USUAL LARMOR FORMULA

THE RECIPROCAL COULOMB FORCE IS



$$F \cong \frac{q_1 q_2}{4\pi\varepsilon_0} \cdot \frac{1}{r^2 + v^2 t^2} = m_1 a_1 = m_2 a_2$$

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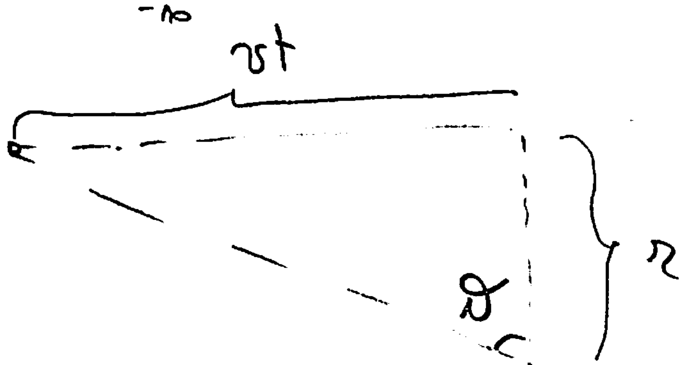
$$\Rightarrow \quad a_1(t) \cong \frac{q_1 q_2}{4\pi\varepsilon_0}\cdot\frac{1}{m_1}\cdot\frac{1}{r^2+v^2t^2}$$

$$a_2(t) \cong \frac{q_1 q_2}{4\pi\varepsilon_0}\cdot\frac{1}{m_2}\cdot\frac{1}{r^2+v^2t^2}$$

$$P_{TOT} = P_1+P_2 = \left(\frac{q_1 q_2}{4\pi\varepsilon_0}\right)^2\cdot\left(\frac{1}{m_1^2}+\frac{1}{m_2^2}\right)\cdot\frac{1}{(r^2+v^2t^2)^2}\cdot\frac{Z_0 e^2}{6\pi e^2}$$

$$\mathcal{E}_{TOT} = \int_{-\infty}^{+\infty} P_{TOT}(t)\, dt$$



$$dt = \frac{r}{v}\cdot\frac{1}{\cos^2\vartheta}\, d\vartheta$$

$$\frac{1}{r^2+v^2t^2} = \frac{\cos^4\vartheta}{r^4}$$

$$\Rightarrow \quad \int_{-\infty}^{+\infty}\frac{1}{r^2+v^2t^2}\, dt = \frac{1}{v r^3}\cdot\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos^2\vartheta\, d\vartheta = \frac{\pi}{2}\cdot\frac{1}{r^3 v}$$

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$$\Rightarrow \boxed{\mathcal{E}^{RAD}_{TOT} = \left(\frac{q_1 q_2}{4\pi\varepsilon_0}\right)^2 \cdot \left(\frac{1}{m_1^2} + \frac{1}{m_2^2}\right) \cdot \frac{Z_0 e^2}{6\pi c^2} \cdot \frac{\pi}{2} \cdot \frac{1}{b^3 v}}$$

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$$\vec{\nabla} \times \vec{E} = \frac{-\partial \vec{B}}{\partial t}$$

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$$

In conductor, we have

$$\vec{\nabla} \times \vec{B} = \mu \vec{J}_f + \mu\varepsilon \frac{\partial \vec{E}}{\partial t}$$ (assume linear behavior)

$$\vec{\nabla} \times \vec{B} = \mu \sigma \vec{E} + \mu\varepsilon \frac{\partial \vec{E}}{\partial t}$$

deriving wave equation:

$$-\vec{\nabla} \times \vec{\nabla} \times \vec{E} = \mu \sigma \frac{\partial \vec{E}}{\partial t} + \mu\varepsilon \frac{\partial^2 \vec{E}}{\partial t^2}$$

$$\vec{\nabla}^2 \vec{E} = \mu \sigma \frac{\partial \vec{E}}{\partial t} + \mu\varepsilon \frac{\partial^2 \vec{E}}{\partial t^2}$$

assume incident plane waves, $\vec{\nabla} \cdot \vec{E} = \vec{k} \cdot \vec{E} = 0$

dispersion relation

$$+k^2 = +i\omega\mu\sigma + \omega^2\mu\varepsilon$$

$$k = \sqrt{\left((\omega\mu\sigma)^2 + (\omega^2\mu\varepsilon)^2\right) e^{i\delta}}$$

$$\sigma >> \mu\varepsilon$$

$$k = e^{\frac{i\delta}{2}} \sqrt{\omega\mu\sigma} \Rightarrow$$

$$\delta = \arctan\left(\frac{\omega\mu\sigma}{\omega^2\mu\varepsilon}\right)$$

$$\delta \approx \frac{\pi}{2}$$

$$k = e^{\frac{i\pi}{4}} \sqrt{\omega\mu\sigma} \quad \frac{1+i}{\sqrt{2}} \sqrt{\omega\mu\sigma}$$

+1

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The electric field takes the form.

$$\vec{E} = E_0 e^{i(\vec{k}\cdot\vec{r} - \omega t)} = E_0 e^{i\sqrt{\frac{\omega\mu\sigma}{2}}z} e^{-\sqrt{\frac{\omega\mu\sigma}{2}}z} e^{-i\omega t}$$

$$\vec{\nabla}\times\vec{E} = \vec{k}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} = i\omega\vec{B} \qquad \vec{B} = -\frac{i\,\vec{k}\times\vec{E}}{\omega}$$

$$\vec{S}_{inst} = \frac{1}{\mu_0}\vec{E}\times\vec{B}^* = \frac{+i}{\omega\mu_0}\left[\vec{E}\times\vec{k}^*\times\vec{E}^*\right]$$

$$= \frac{i}{\omega\mu}\left[\vec{k}^*(\vec{E}\cdot\vec{E}^*)\right] = \frac{i}{\omega\mu}$$

$$\vec{S} = \frac{i}{\omega\mu}\left(\frac{1-i}{\sqrt{2}}\sqrt{\omega\mu\sigma}\right)\hat{z}\left[E_0^2 e^{-2\sqrt{\frac{\omega\mu\sigma}{2}}z}\right]$$

$$\langle\vec{S}_{time\text{-}avg}\rangle = Re\left(\frac{1}{2\mu}\vec{E}\times\vec{B}^*\right)$$

$$= \frac{1}{2\omega\mu}\sqrt{\frac{\omega\mu\sigma}{2}}\,E_0^2 e^{-2\sqrt{\frac{\omega\mu\sigma}{2}}z}\,\hat{z}$$

$$\boxed{\langle\vec{S}_{time\ averaged}\rangle = \frac{1}{2\omega\mu}K|E_0|^2 e^{-2Kz}\,\hat{z} \qquad K = \sqrt{\frac{\omega\mu\sigma}{2}}} \quad \boxed{\frac{dP}{da}}$$

+3

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Name [illegible]

Instantaneous $\vec{S}$:

$$\vec{S} = \frac{1}{\omega\mu} K E_0^2 e^{-2kz}\hat{z} = \frac{dP}{da} \quad k \equiv \sqrt{\frac{\omega\mu\sigma}{2}}$$

Therefore,

$$\frac{d^4 W}{dt\,dV} \; \cancel{=} \; \frac{dP}{da\,dl} = \frac{1}{2}\frac{1}{\omega\mu} K E_0^2 e^{-2kz}\hat{z}$$

$$\frac{d\langle S_{time\text{-}avg}\rangle}{\cancel{dz}} = \frac{1}{2\omega\mu} K E_0^2 \cdot (-2K)\, e^{-2KZ} dz$$

(Intensity lost per unit length)

Therefore

$$\frac{d^4 W}{dt\,dV} = -\frac{K^2 E_0^2}{\omega\mu} e^{-2KZ} \quad \text{(time-averaged)} \quad +1$$

$$\frac{d^4 W}{dt\,dV} = \frac{-2K^2 E_0^2}{\omega\mu} e^{-2KZ} \cdot e^{-i2\omega t} \quad \text{(instantaneous)} \quad +0$$

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b) total power loss per unit area:

$$\frac{d^3W}{dt\,dx\,dy} = \int_0^\infty \frac{d^4W}{dt\,dV}\,dz$$

$$= -\frac{k^2E_0^2}{\omega\mu_0}\cdot\int_0^\infty e^{-2kz}\,dz = \frac{-k^2E_0^2}{\omega\mu_0}\cdot\frac{e^{-2kz}}{-2k}\Bigg]_0^\infty$$

$$= -\frac{k^2E_0^2}{\omega\mu_0}\cdot\frac{1}{2k}$$

$$\boxed{\frac{d^3W}{dt\,dx\,dy} = \frac{-kE_0^2}{2\omega\mu_0}}$$

c) We already found the poynting vector to be
d)

$$\frac{1}{2\omega\mu}k|E_0|^2 e^{-2kz}\hat{z}$$ (time averaged). + 3

We see that the result at $z=0$ agrees perfectly with the result of part b)

(namely $\frac{-kE_0^2}{2\omega\mu}$).

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d) ~~[illegible]~~

part b): $\frac{d^3W}{dt\,dx\,dy} = \frac{-kE_0^2}{2\omega\mu}$

part c) $\langle S_{\substack{\text{time}\\\text{avg}}}\rangle = \frac{1}{2\omega\mu} kE_0^2 e^{-2kz} = \frac{1}{2\omega\mu} kE_0^2$ at $z=0$

The two agree and this is very reasonable. Over $z=0$ to $z=\infty$, the wave looses all of its energy. Therefore, the initial power per unit area should equal the total power lost per unit area from $z=0$ to ∞.

+2

9

18

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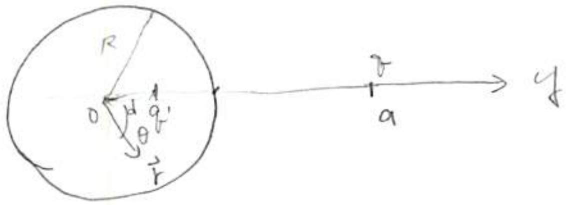

first, let the sphere grounded, assume the image charge on y-axis

$$\Rightarrow \Phi_{out} = \frac{1}{4\pi\epsilon_0}\frac{q}{|\vec{r}-a\hat{y}|} + \frac{1}{4\pi\epsilon_0}\frac{q'}{|\vec{r}-d\hat{y}|}$$

$$\Phi_{in} = \frac{1}{4\pi\epsilon_0}\frac{q}{|\vec{r}-a\hat{y}|}$$

Using b.c @ $y = \pm R$

$$\Rightarrow q' = -\frac{R}{a}q$$

$$d = R\cdot\frac{R}{a}$$

$$\therefore \Phi_{out} = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{|\vec{r}-a\hat{y}|} - \frac{Rq}{a|\vec{r}-\frac{R^2}{a}\hat{y}|}\right)$$

a) know let the potential of the sphere $= V$ $\Rightarrow$

$$\Phi_{out} = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{|\vec{r}-a\hat{y}|} - \frac{R}{a|\vec{r}-\frac{R^2}{a}\hat{y}|}\right) + \frac{VR}{|\vec{r}|}, \quad \Phi(R) = V$$

+3

$$\sigma = -\epsilon_0 \frac{\partial\Phi}{\partial r}\Big|_R$$

$$\Phi = \frac{q}{4\pi\epsilon_0}\sum_{l=0}^{\infty}\left(\frac{r^l}{a^{l+1}}P_l(\cos\theta) - \frac{R}{a}\cdot\left(\frac{(\frac{R^2}{a})^l}{r^{l+1}}\right)P_l(\cos\theta)\right) + \frac{VR}{r}$$

, where I use $r<a$ ∵ we are intersted the region on sphere,

$$\therefore \sigma = \frac{-q}{4\pi}\sum_{l=0}^{\infty}\left(l\frac{R^{l-1}}{a^{l+1}}P_l(\cos\theta) + \frac{R}{a}(l+1)\frac{R^{2l}}{a^l R^{l+2}}P_l(\cos\theta)\right) - \epsilon_0\frac{V}{R}$$

+2

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b) If we have total charge $Q \Rightarrow Q = q' + \Delta Q$, $\Delta Q = Q - q' = Q + \frac{Rq}{a}$.

ΔQ will distribute uniformly on the sphere.

$$\Phi(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{|\vec{r}-a\hat{y}|} - \frac{Rq}{a|\vec{r}-\frac{R^2}{a}\hat{y}|} + \frac{Q+\frac{Rq}{a}}{|\vec{r}|}\right) \quad +2.$$

$$\therefore \Phi(R) = \frac{1}{4\pi\varepsilon_0}\left(\frac{Q+\frac{Rq}{a}}{R}\right)$$

$$\sigma(R) = \frac{-q}{4\pi}\sum_{l=0}^{\infty}\left(l\frac{R^{l-1}}{a^{l+1}} + \frac{R}{a}(l+1)\frac{R^{2l}}{a^l R^{l+2}}\right)P_l(\cos\theta) + \frac{Q+\frac{Rq}{a}}{4\pi R^2}$$

simplify

c)

$\therefore$ The potential is a super position of several charge:

a) q @ $a\hat{y}$,

$-\frac{R}{a}q$ @ $\frac{R^2}{a}\hat{y}$

$4\pi\varepsilon_0 VR$ @ origin

b) q @ $a\hat{y}$ +2

$-\frac{Rq}{a}$ @ $\frac{R^2}{a}\hat{y}$

$\Delta Q = \left(Q+\frac{Rq}{a}\right)$ @ Origin

20

10

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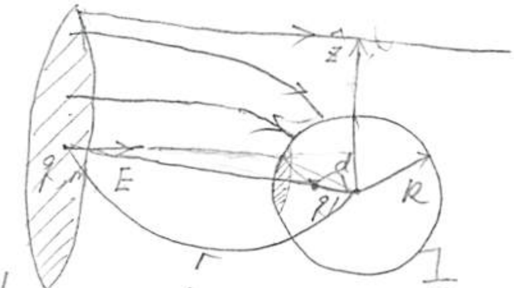


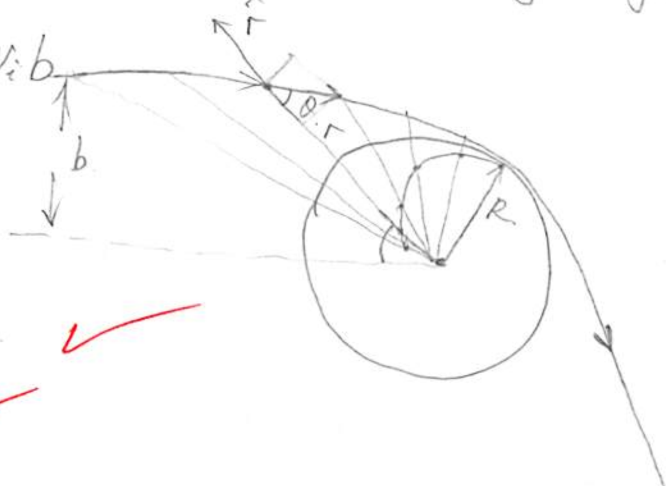
13> $E = \frac{1}{2}mv^2$ (non-relativistic)

cross section, in this case, is the real area that if the particle shoot in will fall onto the sphere.

method of image: $\begin{cases} q' = -\frac{R}{r}q \\ d = \frac{R^2}{r} \end{cases}$

Outside the sphere: $\Phi = \frac{1}{4\pi\epsilon_0}\frac{q'}{|\vec{r}-\vec{d}|}$ due to the induced image charge.

Angular moment is conserved: $L_i = mv_i b$



energy is conserved:

$E + q\Phi = E_i$

$$\Rightarrow \begin{cases} \frac{1}{2}mv^2 + q\cdot\frac{1}{4\pi\epsilon_0}\frac{\left(-\frac{R}{r}q\right)}{\left(r-\frac{R^2}{r}\right)} = \frac{1}{2}mv_i^2 \\ mv_i b = mv\cdot r\sin\theta \end{cases}$$

critical case: when $\theta = \pi/2$, $v > 0$. $r \geq R$

$$\Rightarrow \begin{cases} \frac{1}{2}mv^2 + \frac{1}{4\pi\epsilon_0}\frac{-\frac{R}{r}q^2}{r-\frac{R^2}{r}} = \frac{1}{2}mv_i^2 \\ mv_i b = mvr \end{cases} \Rightarrow b^2 = \frac{v^2r^2}{v_i^2} = \left.\frac{\frac{1}{2}mv^2r^2}{\frac{1}{2}mv_i^2}\right|_{r=R+\delta} =$$

$$= \left.\frac{\left(\frac{1}{2}mv_i^2 + \frac{1}{4\pi\epsilon_0}\frac{Rq^2}{r^2-R^2}\right)r^2}{\frac{1}{2}mv_i^2}\right|_{r=R+\delta} = (R+\delta)^2\left(1+\frac{1}{E}\cdot\frac{q^2}{4\pi\epsilon_0 R}\cdot\left(1-\frac{\delta}{2R}\right)\right)$$

$$\underset{\text{keep } \delta^0 \text{ order}}{=} (\alpha+1)R^2$$

$\Rightarrow$ Cross section: $\pi b^2 = \pi\cdot(\alpha+1)R^2$.

Sane check: $E\to 0$. $\alpha\to\infty$. $\pi b^2\to\infty$ ✓

$E\to\infty$. $\alpha\to 0$ $\pi b^2\to\pi R^2$ ✓

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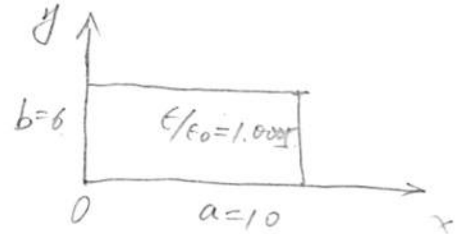

a) TE_{10}-mode:

B.C.: $\vec{n} \times \Delta \vec{E} = 0$

$\Rightarrow E_x = 0 \ @ \ y = 0, b$

$E_y = 0 \ @ \ x = 0, a$

$$\Rightarrow \begin{cases} E_x = E_{x_0} \cdot \cos\left(\frac{n\pi}{a} x\right) \cdot \sin\left(\frac{m\pi}{b} y\right) e^{i(k_z z - \omega t)} \overset{n=1, m=0}{=} 0 \\ E_y = E_{y_0} \sin\left(\frac{n\pi}{a} x\right) \cdot \cos\left(\frac{m\pi}{b} y\right) e^{i(k_z z - \omega t)} \end{cases}, \quad k^2 = k_z^2 + \left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2$$

10

$n=1, m=0 \Rightarrow \omega_c^2 = \frac{1}{\mu_0 \epsilon}\left[\left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2\right] = \frac{1}{\mu_0 \epsilon} \frac{\pi^2}{a^2} = \frac{1}{\mu_0 \epsilon_0} \frac{\epsilon_0}{\epsilon} \frac{\pi^2}{a^2} = c^2 \frac{\pi^2}{a^2} \cdot \frac{1}{1.0005}$

$\Rightarrow \omega_c = c \frac{\pi}{a} (1.0005)^{-1/2} = \frac{\pi}{a} \frac{1}{\sqrt{\mu_0 \epsilon}}$

$\boxed{\omega = 1.25 \omega_c = \frac{5\pi c}{4a} (1.0005)^{-1/2}} \Rightarrow \omega^2 = \frac{1}{\mu_0 \epsilon}\left(k_z^2 + \frac{\pi^2}{a^2}\right) \Rightarrow k_z = \sqrt{\omega^2 \mu_0 \epsilon - \left(\frac{\pi}{a}\right)^2} = \frac{3}{4} \frac{\pi}{a}$

$\boxed{\lambda = \frac{2\pi}{k_z} = \frac{2\pi}{\frac{3}{4}\frac{\pi}{a}} = \frac{8}{3} a}$ is the propagating wave length.

b)
$$\begin{cases} \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \\ \nabla \times \vec{B} = \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} \\ \vec{J} = n_0 e \vec{v} \end{cases} \Rightarrow \vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = -\frac{\partial}{\partial t} (\vec{\nabla} \times \vec{B})$$

#

$= \vec{\nabla}(\vec{\nabla} \cdot \vec{E}) - \nabla^2 \vec{E}$

$= \mu_0 \left(-\frac{\partial}{\partial t} \vec{J}\right) + \mu_0 \epsilon_0 \left(-\frac{\partial^2}{\partial t^2} \vec{E}\right)$

$= -\mu_0 n_0 e \cdot \dot{\vec{v}} + \mu_0 \epsilon_0 \left(-\frac{\partial^2}{\partial t^2} \vec{E}\right)$

plane-wave assumption: $\vec{E} \sim e^{i(\vec{k} \cdot \vec{r} - \omega t)}$ $= -\mu_0 n_0 \frac{e^2}{m} \vec{E} + \mu_0 \epsilon_0 \left(-\frac{\partial^2}{\partial t^2} \vec{E}\right)$

$\Rightarrow$ dispersion relation: $k^2 = -\mu_0 n_0 \frac{e^2}{m} + \frac{\omega^2}{c^2}$

(As the problem say. permittivity only depends on plasma $\Rightarrow \epsilon \to \epsilon_0$)

$\Rightarrow \frac{c^2 k^2}{\omega^2} = 1 - \frac{\omega_p^2}{\omega^2}, \quad \omega_p^2 = \frac{n_0 e^2}{\epsilon_0 m}$

$= \frac{\epsilon}{\epsilon_0} = n^2$, n: refractive index.

To cut off ω: $\Rightarrow \omega = \frac{c}{n} \frac{\pi}{a} = \frac{5}{4} \omega_c = \frac{5}{4} \cdot \frac{c}{\sqrt{1.0005}} \frac{\pi}{a}$

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b) $\Rightarrow n = \frac{4}{5} \times (1.0005)^{1/2}$

$\Rightarrow n^2 = \frac{16}{25} \times 1.0005 = 1 - \frac{n_0 e^2}{\epsilon_0 m \cdot \omega^2}$

$\Rightarrow n_0 = \left(1 - \frac{16}{25} \times 1.0005\right) \cdot \frac{\epsilon_0 m_e \omega^2}{e^2}$

$= \left(1 - \frac{16}{25} \times 1.0005\right) \cdot \frac{8.9 \times 10^{-12} \times 9.1 \times 10^{-31} \times \left(\frac{5\pi \times c}{4a}\right)^2 \frac{1}{1.0005}}{(1.6 \times 10^{-19})^2}$

$\doteq 1.5 \times 10^{16} \, (m^{-3})$ 10

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