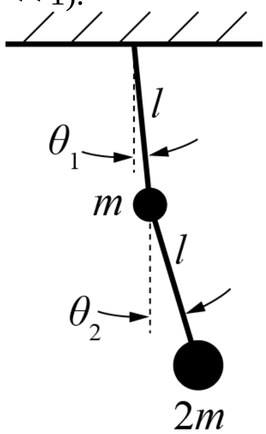
Name:

Physics & Astronomy Comprehensive Exam, UCLA, Fall 2013

1. Classical Mechanics

Consider the double pendulum shown below with equal string lengths *l* and unequal masses *m* and 2*m*, (constrained to move in the plane shown, gravitational acceleration *g*). Find the frequencies of the normal modes for small oscillations about the equilibrium position (θ_1 and θ_2 << 1).



Name:

2. Classical Mechanics

A photon of energy E_1 collides at an angle θ with another photon of energy E_2. Find the minimum value of E_1 (given E_2 and θ) permitting the formation of a pair of particles of mass m. With this expression, calculate the minimum energy a gamma ray must have to create an electron-positron pair by colliding with a typical photon from the cosmic microwave background (one significant figure is sufficient). [Treat the photon here as a relativistic quantum particle.]

Name:

Questions for the Comprehensive Exam Fall 2013

3. Quantum Mechanics: Scattering

Consider the scattering of a spinless particle of mass *m* from a diatomic molecule. The incoming particle travels along the z-axis. Assume that the molecule is much heavier than the scattering particle and that there is no recoil. The two atoms in the molecule are aligned along the y-axis and localized at y = +b and y = −b. The potential the particle feels in the presence of the molecule can be modeled by the following potential:

$$V(\vec{x}) = \alpha\big(\delta(y-b)\delta(x)\delta(z) + \delta(y+b)\delta(x)\delta(z)\big)$$

(a) Calculate the scattering amplitude in the first Born approximation.
(b) Calculate the differential cross section from (a) (Express the result in terms of the scattering angles).
(c) Calculate the total cross section. Do the integrals exactly. You might find the following integrals helpful:

$$\int_0^{2\pi} d\alpha \left|\cos(x\sin\alpha)\right|^2 = \pi\big(1 + J_0(2x)\big), \qquad \int_0^{\pi} d\alpha (\sin\alpha) J_0(x\sin\alpha) = \frac{a\sin x}{x}$$

Name:

4. Quantum Mechanics

Consider a system of two spin 1/2 particles, labeled a and b, with respective spin operators $\mathbf{S}_a$ and $\mathbf{S}_b$. We ignore all quantum numbers but those of spin. The particles are in the state $|\Psi\rangle$ of zero total angular momentum, which we consider normalized to $\langle\Psi|\Psi\rangle = 1$. Let $\mathbf{n}_a$ and $\mathbf{n}_b$ be two independent unit vectors. Compute the expectation value of the product of the spin operators projected onto the directions $\mathbf{n}_a$ and $\mathbf{n}_b$ respectively, namely, $\langle\Psi|(\mathbf{n}_a\cdot \mathbf{S}_a)(\mathbf{n}_b\cdot \mathbf{S}_b)|\Psi\rangle$.

Name:

5. Quantum Mechanics

Consider a one dimensional simple harmonic oscillator with the usual Hamiltonian,

$$H_0 = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 x^2$$

Now consider the effect of a perturbation,

$$V = \frac{1}{2}\varepsilon m\omega^2 x^2 \qquad \text{with} \quad \varepsilon << 1.$$

Using perturbation theory, find the new ground state key $|0\rangle$ to order ε and the ground state energy shift to order ε^2. Solve this problem exactly and compare with the results obtained using perturbation theory. You may assume without proof that

$$\langle u_{n'} | x | u_n \rangle = \sqrt{\frac{\hbar}{2m\omega}} \left(\sqrt{n+1}\,\delta_{n',n+1} + \sqrt{n}\,\delta_{n',n-1} \right)$$

where $|u_n\rangle$ is the nth eigenstate of H_0.

Name:

Questions for the Comprehensive Exam Fall 2013

6. *Quantum Mechanics: Perturbation theory time (in)dependent*

The Hamiltonian of a particle of mass m in a 1D finite well is given by

$$H_0 = \frac{p^2}{2m} + V(x)\,, \qquad V(x) = \begin{cases} 0 & 0 < x < L \\ \infty & otherwise \end{cases}$$

A time dependent perturbation is added

$$H_{total} = H_0 + H'\,, \qquad H' = \lambda\delta\left(x - L/2\right) f\left(t\right)$$

where λ is a constant and *f(t)* is a time dependent function.
(a) Calculate the matrix elements of *H'* with the eigenstates of the unperturbed Hamiltonian.
(b) In this part of the problem we consider a time independent perturbation *f(t)* = 1. First, calculate the first nonzero correction of the ground state energy. Second, at which order (if any) will the first excited state receive a nonzero correction ? Please back up your answer with an argument/calculation.
(c) Now take the following time dependent function

$$f(t) = \int_{-\infty}^{\infty} d\omega\, \rho(\omega)\left(e^{i\omega t} + e^{-i\omega t}\right) \quad \text{with} \quad \rho(\omega) = \sqrt{\frac{\alpha}{\pi}}\, e^{-\alpha\omega^2}$$

If the system is in its ground state at time t = 0, what are the possible transitions into excited states that the system can make at time t > 0 ? Use first order time-dependent perturbation theory.
(d) Find the transition rate into these excited states using Fermi's golden rule.

Name:

7. *Quantum Mechanics: Jaynes-Cummings Model*

A spin-½ particle (bare eigenstates $|\uparrow\rangle$ and $|\downarrow\rangle$) is constrained to move in a 1D harmonic potential (bare eigenstates $|n\rangle$ for $n \in \{0,1,2,...\}$). In the absence of coupling, the Hamiltonian is given by

$$H_0 = \hbar\omega_0\left(\hat{a}^+\hat{a} + 1/2\right) + \hbar\omega_q\hat{\sigma}_z/2$$

where $\hbar\omega_0$ is the harmonic oscillation energy spacing and $\hbar\omega_q$ is the energy difference between $|\uparrow\rangle$ and $|\downarrow\rangle$ in the absence of coupling. Consider the case where the particle's spin is coupled to its motion through the interaction Hamiltonian

$$H_{\text{int}} = \hbar\Omega\left(\hat{a}^+\hat{\sigma}_- + \hat{a}\hat{\sigma}_+\right)/2$$

where Ω is the coupling strength and the operators for the spin degree of freedom are defined by $\hat{\sigma}_z = |\uparrow\rangle\langle\uparrow| - |\downarrow\rangle\langle\downarrow|$, $\hat{\sigma}_+ = |\uparrow\rangle\langle\downarrow|$, and $\hat{\sigma}_- = |\downarrow\rangle\langle\uparrow|$.

(a) Let $\Delta \equiv \omega_0 - \omega_q > 0$. Consider the "dispersive regime" defined by $\Delta >> \Omega/2$. Find the energies and eigenstates of $H = H_0 + H_{\text{int}}$. You may treat H_{int} as a perturbation and keep only the first nonzero-order terms, and you may consider coupling between nearly-degenerate states only. Do not bother to normalize the eigenstates.

(b) Consider the "resonant regime" where $\Delta \to 0$ and ω_0 and ω_q are both equal to some frequency ω. Find the energies and eigenstates of $H = H_0 + H_{\text{int}}$. You may treat H_{int} as a perturbation and use first-order degenerate perturbation theory, considering coupling between nominally degenerate states only. Hint: consider the state $|\psi\rangle = \frac{1}{\sqrt{2}}\left(|\downarrow, n\rangle + e^{i\varphi}|\uparrow, n-1\rangle\right)$.

Name:

8. Statistical Mechanics

We propose to evaluate the **Richardson** effect, namely the electric current density of electrons which is produced by heating up a metal in the presence of an external electric potential. The potential energy of an electron just outside the metal is denoted $W > 0$. The potential energy for electrons inside the metal is taken to be 0. The electrons are considered otherwise non-interacting, and filled up to chemical potential μ with $\mu < W$. Since we consider the problem to be at sufficiently low temperature, μ may be identified with the Fermi energy.

(a) State the condition on the momentum of an electron that can escape from the metal to the outside as a function of W and μ.

(b) Derive a general expression for the current density I of electrons leaving the metal.

(c) Obtain an approximation of your result in (b) valid for sufficiently low temperatures.

Name:

9. Statistical Mechanics

The flexing modes of a thin plate at wave number k have an angular frequency ω such that: $\omega^2 = \gamma k^4$. Consider such waves propagating in one dimension around a thin ring of radius R. What is the contribution to the heat capacity of these azimuthal modes? The plate is in thermal equilibrium at a low temperature T. You may write your answer in terms of:

$Z(x) = \int_{-\infty}^{+\infty} \frac{y^x}{e^y - 1} dy$ where x is a pure number.

Name:

10. Statistical Mechanics

A liquid is in equilibrium with its vapor at temperature and pressure: T_v; P_v. The surface between liquid and vapor is flat. The temperature of the vapor is increased to $T_v + \Delta T$ while keeping its pressure fixed. The liquid remains at temperature and pressure: T_v; P_v. Evaluate the net flux of gas to the liquid. You may treat the vapor as an ideal noble gas with atoms of mass *m*.

Name:

11. Electromagnetism: A spherical capacitor

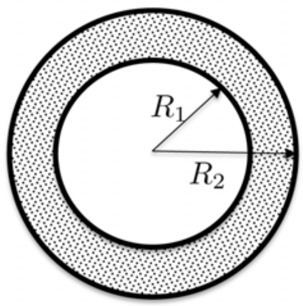
The gap between two spherical conducting shells of radii $R_1 < R_2$ is filled with a spatially inhomogeneous dielectric. As a result, the dielectric constant depends on the polar angle θ (measured from the north pole of the sphere) as:

$$\varepsilon(\theta) = \varepsilon_1 + \varepsilon_2 \cos^4 \theta$$

Both ε_1 and ε_2 are constants.

(a) When charged so that the inner and outer spheres have charges $+Q$ and $-Q$ respectively, show that the internal electric field is purely radial. Determine how that radial electric field E_r depends on the polar angles, θ and ϕ.

(b) Calculate the capacitance C of the system.



Name:

12. Electromagnetism

Assume the existence of magnetic charge related to the magnetic field by the local reaction $\vec{\nabla} \cdot \vec{B} = \mu_0 \rho_m$.

(a). Using the Gauss's theorem, obtain the magnetic field $\vec{B}$ of a point magnetic charge at the origin.

(b). In the absence of the magnetic charge, the curl of the electric field is given by the Faraday's law, $\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$. Show that this law is incompatible with the magnetic charge density that is a function of time.

(c). Assuming that magnetic charge is conserved, derive the local relation between the magnetic charge current density $\vec{J}_m$ and the magnetic density ρ_m.

(d). Modify Faraday's law as given in part (b) to obtain a law consistent with the presence of the magnetic charge density that is a function of position and time.

Name:

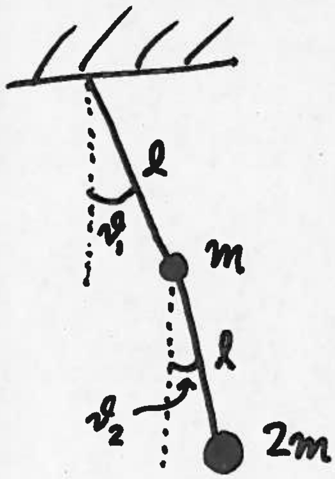
13. Electromagnetism

An un-polarized plane electromagnetic wave is scattered by a free electron. Derive the differential cross-section for scattering in the non-relativistic limit (Thompson scattering).

14. Electromagnetism

A plane electromagnetic wave of frequency ω is normally incident on an infinitesimally-thin planar sheet with a resistivity $Z_s = Z_0 = 377$ ohms per square ($4\pi/c$ in cgs units). The medium on both sides of the sheet is vacuum, and the sheet is non-magnetic.
(a) Calculate the power coefficients of reflection (R), absorption (A), and transmission (T).
(b) What value of the resistivity Z_m would maximize A? What is the maximal value of A, and why is this the limit?

Classical Mechanics Q1 : double pendulum



Consider the double pendulum with unequal masses shown at left. For small oscillations about the equilibrium position $\vartheta_1 \to 0$, $\vartheta_2 \to 0$, find the normal mode frequencies. You may assume $\vartheta_1 \ll 1$, $\vartheta_2 \ll 1$.

Solution 1 Lagrange differential equation method (double pendulum)

$$K = \frac{1}{2} m \left[l \frac{d}{dt} \sin(\vartheta_1) \right]^2 + \frac{1}{2} \cdot 2m \left[l \frac{d}{dt} \sin(\vartheta_1) + l \frac{d}{dt} \sin(\vartheta_2) \right]^2$$

①

$$\approx \frac{1}{2} m l^2 \dot{\vartheta}_1^2 + \frac{1}{2} m l^2 \left[2\dot{\vartheta}_1^2 + 4\dot{\vartheta}_1 \dot{\vartheta}_2 + 2\dot{\vartheta}_2^2 \right]$$

$$= \frac{1}{2} m l^2 \left(3\dot{\vartheta}_1^2 + 4\dot{\vartheta}_1 \dot{\vartheta}_2 + 2\dot{\vartheta}_2^2 \right)$$

$$V = (m + 2m) g \left(l - l\cos(\vartheta_1) \right) + 2mg \left(l - l\cos(\vartheta_2) \right)$$

$$\approx \frac{3}{2} mgl\vartheta_1^2 + mgl\vartheta_2^2$$

$$\therefore \quad L = \frac{1}{2} m l^2 \left(3\dot{\vartheta}_1^2 + 4\dot{\vartheta}_1 \dot{\vartheta}_2 + 2\dot{\vartheta}_2^2 \right) - \frac{3}{2} mgl\vartheta_1^2 - mgl\vartheta_2^2$$

The momenta are

$$p_1 = \frac{\partial L}{\partial \dot{\vartheta}_1} = 3ml^2 \dot{\vartheta}_1 + 2ml^2 \dot{\vartheta}_2$$

$$p_2 = \frac{\partial L}{\partial \dot{\vartheta}_2} = 2ml^2 \dot{\vartheta}_2 + 2ml^2 \dot{\vartheta}_1$$

we can write the equations of motion:

$$\frac{dp_1}{dt} = \frac{\partial L}{\partial \vartheta_1}$$

$$\frac{d}{dt} \left[3\cancel{m} l^2 \dot{\vartheta}_1 + 2\cancel{m} l^2 \dot{\vartheta}_2 \right] = -3\cancel{m} g \cancel{l} \vartheta_1$$

$$\text{(I)} \qquad 3\ddot{\vartheta}_1 + 2\ddot{\vartheta}_2 = -3\frac{g}{l}\vartheta_1$$

$$\text{and} \qquad \frac{dp_2}{dt} = \frac{\partial L}{\partial \vartheta_2}$$

②

$$\frac{d}{dt}\left[2m\ell^2\dot{\vartheta}_2 + 2m\ell^2\dot{\vartheta}_1\right] = -2mg\ell\vartheta_2$$

$$2\ddot{\vartheta}_2 + 2\ddot{\vartheta}_1 = -2\frac{g}{\ell}\vartheta_2$$

$$(\mathrm{II}) \qquad \ddot{\vartheta}_2 + \ddot{\vartheta}_1 = -\frac{g}{\ell}\vartheta_2$$

assume $\vartheta_1(t) = A_1 e^{i\omega t}$ $\qquad \vartheta_2(t) = A_2 e^{i\omega t}$

from (I) we have

$$-\omega^2 A_1 \cdot 3 - 2\omega^2 A_2 = -3A_1\frac{g}{\ell}$$

$$A_1\left(3\frac{g}{\ell} - 3\omega^2\right) = 2\omega^2 A_2$$

$$\therefore A_2 = A_1\left(\frac{3}{2}\frac{g}{\ell}\frac{1}{\omega^2} - \frac{3}{2}\right)$$

from (II) we have

$$-A_2\omega^2 - A_1\omega^2 = -A_2\frac{g}{\ell}$$

$$A_1\omega^2 = A_2\left(\frac{g}{\ell} - \omega^2\right)$$

$$= A_1\left(\frac{3}{2}\frac{g}{\ell}\frac{1}{\omega^2} - \frac{3}{2}\right)\left(\frac{g}{\ell} - \omega^2\right)$$

$$\therefore \omega^2 = \frac{3}{2}\left(\frac{g}{\ell}\right)^2\frac{1}{\omega^2} \underbrace{-\frac{3}{2}\frac{g}{\ell} - \frac{3}{2}\frac{g}{\ell}}_{-3\frac{g}{\ell}} + \frac{3}{2}\omega^2$$

$$0 = \frac{3}{2}\left(\frac{g}{\ell}\right)^2\frac{1}{\omega^2} - 3\frac{g}{\ell} + \frac{1}{2}\omega^2$$

$$0 = \frac{3}{2}\left(\frac{g}{\ell}\right)^2 - 3\frac{g}{\ell}\omega^2 + \frac{1}{2}\omega^4$$

$$0 = \frac{3}{2} - 3\frac{\ell}{g}\omega^2 + \frac{1}{2}\left(\frac{\ell}{g}\omega^2\right)^2$$

so $\frac{l}{g}\omega^2 = \dfrac{3 \pm \sqrt{9-3}}{1} = 3 \pm \sqrt{6}$

$$\omega_+ = \sqrt{\frac{g}{l}}\sqrt{3+\sqrt{6}}$$

$$\omega_- = \sqrt{\frac{g}{l}}\sqrt{3-\sqrt{6}}$$

Solution 2 Matrix method

$$K = \frac{1}{2} m l^2 (3\dot{\vartheta}_1^2 + 4\dot{\vartheta}_1\dot{\vartheta}_2 + 2\dot{\vartheta}_2^2)$$

$$V = \frac{3}{2} mgl\vartheta_1^2 + mgl\vartheta_2^2$$
$$= \frac{1}{2}\cdot 3mgl\cdot\vartheta_1^2 + \frac{1}{2}\cdot 2mgl\cdot\vartheta_2^2$$

we identify

$$M_{11} = 3ml^2 \qquad M_{22} = 2ml^2 \qquad M_{12} = M_{21} = 2ml^2$$

$$\tilde{M} = \begin{pmatrix} 3ml^2 & 2ml^2 \\ 2ml^2 & 2ml^2 \end{pmatrix} = ml^2 \begin{pmatrix} 3 & 2 \\ 2 & 2 \end{pmatrix}$$

$$K_{11} = 3mgl \qquad K_{22} = 2mgl \qquad K_{12} = K_{21} = 0$$

$$\tilde{K} = mgl \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$$

we are looking for the solution to

$$(\tilde{K} - \lambda\tilde{M})\vec{a} = 0 \qquad \lambda \equiv \omega^2$$

we construct the characteristic equation by taking the determinant:

$$\begin{vmatrix} 3mgl - \omega^2\cdot 3ml^2 & -\omega^2 2ml^2 \\ -\omega^2 2ml^2 & 2mgl - \omega^2\cdot 2ml^2 \end{vmatrix} = 0$$

$$(3g - 3l\omega^2)(2g - 2l\omega^2) - 4l^2\omega^4 = 0$$

$$6g^2 - 6gl\omega^2 - 6gl\omega^2 + 6l^2\omega^4 - 4l^2\omega^4 = 0$$

$$2l^2\omega^4 - 12gl\omega^2 + 6g^2 = 0$$

$$\frac{l^2}{g^2}\omega^4 - 6\frac{l}{g}\omega^2 + 3 = 0$$

so $$\omega^2\frac{l}{g} = \frac{6 \pm \sqrt{36-12}}{2} = 3 \pm \sqrt{6}$$

$$\omega_+ = \sqrt{\frac{g}{l}}\sqrt{3+\sqrt{6}}$$

$$\omega_- = \sqrt{\frac{g}{l}}\sqrt{3-\sqrt{6}}$$

Q2

Without loss of generality we take photon 1 to be traveling in the $\hat{z}$ direction and photon 2 to have its momentum in the XZ plane. Then the 4-momenta are

$$p_1 = \left(p_x, p_y, p_z, \frac{E}{c}\right) = \left(0, 0, \frac{E_1}{c}, \frac{E_1}{c}\right)$$

$$p_2 = \left(\frac{\sin\theta E_2}{c}, 0, \frac{\cos\theta E_2}{c}, \frac{E_2}{c}\right)$$

$$p_t = p_1 + p_2 = \left(\frac{\sin\theta E_2}{c}, 0, \frac{\cos\theta E_2 + E_1}{c}, \frac{E_1+E_2}{c}\right)$$

The total $|\vec{p}| = \frac{1}{c}\sqrt{\sin^2\theta E_2^2 + \cos^2\theta E_2^2 + E_1^2 + 2E_1E_2\cos\theta}$

$$= \frac{1}{2}\sqrt{E_1^2 + E_2^2 + 2E_1E_2\cos\theta}$$

At threshold $m_3 = m_4$ are not moving relative to each other. $\Rightarrow$ They have the same momenta.

$p_3 = p_4$ such that $|\vec{p}| = p$ and $E = \sqrt{p^2c^2 + m^2c^4}$

Conservation of $\vec{p}$ implies $2p = \frac{1}{c}\sqrt{E_1^2 + E_2^2 + 2E_1E_2\cos\theta}$

conservation of E implies $2\sqrt{p^2c^2 + m^2c^4} = E_1 + E_2$

$$4p^2c^2 = E_1^2 + E_2^2 + 2E_1E_2\cos\theta$$

$$4(p^2c^2 + m^2c^4) = E_1^2 + E_2^2 + 2E_1E_2$$

$$4m^2c^4 = 2E_1E_2(1-\cos\theta) \Rightarrow E_1E_2 = \frac{2m^2c^4}{1-\cos\theta}$$

$$\boxed{E_1 = \frac{2m^2c^4}{E_2(1-\cos\theta)}}$$

E_1 will be minimized for a head-on collision

$$\cos\theta = -1 \iff \theta = \pi$$

In this case $E_1 = \dfrac{m^2c^4}{E_2} = \dfrac{(mc^2)^2}{E_2}$

$$m_e c^2 \cong 0.5 \times 10^6 \, eV$$

$$E_2 \simeq 3K \cdot \frac{\frac{1}{40} eV}{300K} = \frac{1}{4000} eV$$

$$E_1 = \frac{0.25 \times 10^{12} (eV)^2}{\frac{1}{4000} eV} = \boxed{10^{15} eV}$$

Question: Scattering

Consider the scattering of a spinless particle of mass m from a diatomic molecule. The incoming particle travels along the z-axis. Assume that the molecule is much heavier than the scattering particle and that there is no recoil. The two atoms in the molecule are aligned along the y-axis and localized at $y = b$ and $y = -b$. The potential the particle feels in the presence of the molecule can be modeled by the following potential:

$$V(\vec{x}) = \alpha\Big(\delta(y-b)\delta(x)\delta(z) + \delta(y+b)\delta(x)\delta(z)\Big)$$

a) Calculate the scattering amplitude in the first Born approximation.

b) Calculate the differential cross section from a) (Express the result in terms of the scattering angles).

c) Calculate the total cross section. Do the integrals exactly. You might find the following integrals helpful:

$$\int_0^{2\pi} \alpha \cos(x \sin\alpha)^2 = \pi(1 + J_0(2x)), \qquad \int_0^{\pi} d\alpha \sin\alpha J_0(x\sin\alpha) = \frac{2\sin x}{x}$$

Solution:

a) The scattering amplitude in the first Born approximation is given by

$$f^{(1)}(\vec{k}', \vec{k}) = -\frac{2m(2\pi)^{3/2}}{4\pi\hbar^2} \int d^3y e^{-i(\vec{k}'-\vec{k})\vec{y}} V(\vec{y}) \tag{0.1}$$

We have

$$\vec{k} = \hat{e}_z k, \qquad \vec{k'} = k(\hat{e}^z \cos\theta + \hat{e}_x \sin\theta\cos\phi + \hat{e}_y \sin\theta\sin\phi) \tag{0.2}$$

Plugging in the potential and evaluating the integral over y gives

$$\begin{aligned} f^{(1)}(\vec{k}', \vec{k}) &= -\frac{2m(2\pi)^{3/2}\alpha}{4\pi\hbar^2}\Big(e^{ikb\sin\theta\sin\phi} + e^{-ikb\sin\theta\sin\phi}\Big) && (0.3) \\ &= -\frac{m\alpha(2\pi)^{3/2}}{\hbar^2\pi}\cos(kb\sin\theta\sin\phi) && (0.4) \end{aligned}$$

b) The differential cross section can be calculated from the scattering amplitude by

$$\begin{aligned} \frac{d\sigma}{d\Omega} &= \mid f(\theta,\phi) \mid^2 \\ &= \frac{\alpha^2 m^2 8\pi}{\hbar^4} |\cos(kb\sin\theta\sin\phi)|^2 && (0.5) \end{aligned}$$

c) The total cross section is given by

$$\begin{aligned} \sigma_{tot} &= \int d\Omega \mid f(\theta,\phi) \mid^2 \\ &= \frac{\alpha^2 m^2 8\pi}{\hbar^4} \int_0^\pi d\theta \sin\theta \int_0^{2\pi} d\phi |\cos(kb\cos\theta\cos\phi)|^2 \\ &= \frac{\alpha^2 m^2 8\pi^2}{\hbar^4} \int_0^\pi d\theta \sin\theta \Big(1 + \pi J_0(2kb\sin\theta)\Big) \\ &= \frac{\alpha^2 m^2 8\pi^2}{\hbar^4}\Big(2 + \frac{\sin(2bk)}{bk}\Big) && (0.6) \end{aligned}$$

Q4.

Fall 2013 Comprehensive Exam Questions

Question 4: Quantum Mechanics

Consider a system of two spin 1/2 particles, labelled a and b, with respective spin operators $\mathbf{S}_a$ and $\mathbf{S}_b$. We ignore all quantum numbers but those of spin. The particles are in the state $|\Psi\rangle$ of zero total angular momentum, which we consider normalized to $\langle\Psi|\Psi\rangle = 1$. Let $\mathbf{n}_a$ and $\mathbf{n}_b$ be two independent unit vectors. Compute the expectation value of the product of the spin operators projected onto the directions $\mathbf{n}_a$ and $\mathbf{n}_b$ respectively, namely,

$$\langle\Psi|(\mathbf{n}_a \cdot \mathbf{S}_a)(\mathbf{n}_b \cdot \mathbf{S}_b)|\Psi\rangle \tag{0.1}$$

Solution to Question 4:

Recall that the normalized spin 0 state $|\Psi\rangle$ of the two-spin system, in a basis where the spin operators S_a^z and S_b^z are diagonal with eigenstates $|\pm\rangle_a$ and $|\pm\rangle_b$, is given by,

$$|\Psi\rangle = \frac{1}{2}\Big(|+\rangle_a \otimes |-\rangle_b - |-\rangle_a \otimes |+\rangle_b\Big) \tag{0.2}$$

Also recall that the spin operators in this basis are given by $S_a^i = \hbar\sigma_a^i/2$ and $S_b^i = \hbar\sigma_b^i/2$, where σ_a^i and σ_b^i are the standard Pauli matrices acting in the Hilbert spaces for a and b.

The expectation value to be calculated is linear in $\mathbf{n}_a$ and linear in $\mathbf{n}_b$. It is also invariant under rotations, since the state $|\Psi\rangle$ is invariant under rotations. Thus, the outcome must be proportional to the only possible rotation invariant bilinear in $\mathbf{n}_a$ and $\mathbf{n}_b$,

$$\langle\Psi|(\mathbf{n}_a \cdot \mathbf{S}_a)(\mathbf{n}_b \cdot \mathbf{S}_b)|\Psi\rangle = C\, \mathbf{n}_a \cdot \mathbf{n}_b \tag{0.3}$$

The coefficient C is independent of $\mathbf{n}_a$ and $\mathbf{n}_b$. To evaluate it, we may choose any convenient assignment of $\mathbf{n}_a$ and $\mathbf{n}_b$ such that their inner product is non-zero. We take $\mathbf{n}_a = \mathbf{n}_b = (0,0,1)$. It is now straightforward to evaluate C by first computing, by inspection,

$$S_a^z S_b^z|\Psi\rangle = -\left(\frac{\hbar}{2}\right)^2 |\Psi\rangle \tag{0.4}$$

Since $|\Psi\rangle$ is normalized, we thus find,

$$C = -\left(\frac{\hbar}{2}\right)^2 \tag{0.5}$$

By the way, it is also easy to double check that the expectation value vanishes when $\mathbf{n}_a \cdot \mathbf{n}_b = 0$, by taking for example $\mathbf{n}_a = (1,0,0)$ and $\mathbf{n}_b = (0,0,1)$. We then find,

$$S_a^x S_b^z|\Psi\rangle = -\left(\frac{\hbar}{2}\right)^2 \frac{1}{2}\Big(|-\rangle_a \otimes |-\rangle_b + |+\rangle_a \otimes |+\rangle_b\Big) \tag{0.6}$$

whose inner product with $|\Psi\rangle$ vanishes term by term.

Problem 3

Consider a one-dimensional simple harmonic oscillator with the usual Hamiltonian,

$$H_0 = \frac{p^2}{2m} + \frac{1}{2} m \omega^2 x^2$$

Now, consider the effect of a perturbation,

$$V = \frac{1}{2} \epsilon m \omega^2 x^2 \quad \text{with} \quad \epsilon \ll 1$$

Find using perturbation theory, the new ground state ket $|0\rangle$ to order ϵ and the ground state energy shift to order ϵ^2. Solve this problem exactly and compare with the results obtained using perturbation theory.

You may assume without proof that

$$\langle u_{n'} | x | u_n \rangle = \sqrt{\frac{\hbar}{2m\omega}} \left(\sqrt{n+1}\, \delta_{n', n+1} + \sqrt{n}\, \delta_{n', n-1} \right)$$

where $|u_n\rangle$ is the nth eigenstate of H_0.

Soln:-

The new ground state, $|0\rangle$ is given by

$$|0\rangle = |0^{(0)}\rangle + \sum_{\vec{k} \neq 0} |k^{0}\rangle \frac{V_{k0}}{E_0^{(0)} - E_k^{(0)}} + \ldots$$

& the ground state energy shift Δ_0 by

$$\Delta_0 = V_{00} + \sum_{k \neq 0} \frac{|V_{k0}|^2}{E_0^{(0)} - E_k^{(0)}} + \ldots$$

Now

$$V_{00} = \frac{\epsilon m \omega^2}{2} \langle 0^{(0)} | x^2 | 0^{(0)} \rangle$$

$$= \sum_n \frac{\epsilon m \omega^2}{2} \langle 0^{(0)} | x | n^{(0)} \rangle \langle n^{(0)} | x | 0^{(0)} \rangle$$

$$= \sum_n \frac{\epsilon m \omega^2}{2} \left(\sqrt{\frac{\hbar}{2m\omega}} \, \delta_{n,1} \times \sqrt{\frac{\hbar}{2m\omega}} \, \delta_{1,n} \right)$$

$$= \frac{\epsilon \hbar \omega}{4}$$

For $k \neq 0$,

$$V_{k0} = \frac{\epsilon m \omega^2}{2} \sum_n \langle k^{(0)} | x | n^{(0)} \rangle \langle n^{(0)} | x | 0 \rangle$$

$$= \frac{\epsilon m \omega^2}{2} \times \left(\frac{\hbar}{2m\omega} \right) \times \sum_n \sqrt{2} \, \delta_{k,2} \times \delta_{1,n}$$

$$= \frac{\epsilon \hbar \omega}{2\sqrt{2}} \delta_{k,2}$$

Thus, $|0\rangle = |0^{(0)}\rangle - \frac{\epsilon}{4\sqrt{2}} |2^{(0)}\rangle + O(\epsilon^2)$

$$\Delta_0 = E_0 - E_0^{(0)} = \hbar\omega \left[\frac{\epsilon}{4} - \frac{\epsilon^2}{16} + O(\epsilon^3) \right]$$

The exact solution for $H_0 + V$ is obtained by substitution $\omega \to \omega\sqrt{1+\epsilon}$ in the solution for H_0

Thus ground state energy

$$= \frac{\hbar\omega}{2}\sqrt{1+\epsilon} = \frac{\hbar\omega}{2} \left[1 + \frac{\epsilon}{2} - \frac{\epsilon^2}{8} + \cdots \right]$$

The ground state wave function of H_0 is

$$\langle x|0\rangle = \frac{1}{\pi^{1/4}} \frac{1}{\sqrt{x_0}} e^{-x^2/2x_0^2}$$

where $x_0 = \sqrt{\frac{\hbar}{m\omega}}$

Hence the G.S wave fn of $H_0 + V$ is obtained by $x_0 \to \frac{x_0}{(1+\epsilon)^{1/4}}$.

Hence $\langle x|0\rangle \to \frac{1}{\pi^{1/4}\sqrt{x_0}} (1+\epsilon)^{1/8} \exp\left[-\left(\frac{x^2}{2x_0^2}\right) (1+\epsilon)^{1/2} \right]$

$$\simeq \frac{1}{\pi^{1/4}} \frac{1}{\sqrt{x_0}} e^{-\frac{x^2}{2x_0^2}} + \frac{\epsilon}{\pi^{1/4}\sqrt{x_0}} e^{-x^2/2x_0^2} \left[\frac{1}{8} - \frac{1}{4}\frac{x^2}{x_0^2} \right]$$

Now, $\langle x | 2^{(0)} \rangle = \frac{1}{\sqrt{2}} (a^+)^2 |0\rangle$, with

$$a^+ = \sqrt{\frac{m\omega}{2\hbar}} \left(x - \frac{ip}{m\omega} \right) = \frac{1}{\sqrt{2}\, x_0} \left(x - x_0^2 \frac{d}{dx} \right)$$

Thus, $$\langle x | 2^{(0)} \rangle = \frac{1}{\sqrt{2}} \left(\frac{1}{\sqrt{2}\, x_0} \right)^2 \left(x - x_0^2 \frac{d}{dx} \right)^2 \langle x | 0 \rangle$$

$$= \frac{1}{2\sqrt{2}} \frac{1}{\pi^{1/4}} \frac{1}{\sqrt{x_0}} e^{-x^2/2x_0^2} \left[-2 + 4 \left(\frac{x}{x_0} \right)^2 \right]$$

$$\therefore \langle x | 0 \rangle = \langle x | 0^{(0)} \rangle - \frac{\epsilon}{4\sqrt{2}} \langle x | 2^{(0)} \rangle$$

QUESTION: Perturbation theory time (in)dependent

The Hamiltonian of a particle of mass m in a one dimensional infinite well is given by

$$H_0 = \frac{p^2}{2m} + V(x), \qquad V(x) = \begin{cases} 0 & 0 < x < L \\ \infty & \text{otherwise} \end{cases}$$

A time dependent perturbation is added

$$H_{\text{total}} = H_0 + H', \qquad H' = \lambda\, \delta(x - L/2)\, f(t)$$

Where λ is constant and $f(t)$ is a time dependent function.

a) Calculate the matrix elements of H' with the eigenstates of the unperturbed Hamiltonian.

b) In this part of the problem we consider a time independent perturbation $f(t) = 1$.
First, calculate the first nonzero correction of the ground state energy. Second, at which order (if any) will the first excited state receive a nonzero correction ? Please back up your answer with an argument/calculation.

c) Now take the following time dependent function

$$f(t) = \int_{-\infty}^{\infty} d\omega \rho(\omega) \Big(e^{i\omega t} + e^{-i\omega t} \Big)$$

with

$$\rho(\omega) = \sqrt{\frac{\alpha}{\pi}} e^{-\alpha\omega^2}$$

If the system is in it's ground state at time $t = 0$, what are the possible transitions into excited states that the system can make at time $t > 0$? Use first order time-dependent perturbation theory.

d) Find the transition rate into these excited states using Fermi's golden rule.

Solution:

The free particle in a box has eigenfunctions

$$\psi_n(x) = \sqrt{\frac{2}{L}} \sin \frac{n\pi x}{L}, \quad n \in N \tag{0.7}$$

With eigenvalues of H_0 given by

$$E_n = \frac{\hbar^2}{2m} \frac{n^2\pi^2}{L^2} \tag{0.8}$$

a) The matrix elements are

$$\begin{aligned}
\langle n \mid H' \mid m \rangle &= \frac{2}{L}\lambda f(t) \int_0^L dx \sin \frac{n\pi x}{L} \sin \frac{m\pi x}{L} \delta(x - L/2) \\
&= \frac{2\lambda f(t)}{L} \sin \frac{n\pi}{2} \sin \frac{m\pi}{2} \\
&= \begin{cases} \frac{2\lambda f(t)}{L} & n, m \text{ odd} \\ 0 & \text{otherwise} \end{cases}
\end{aligned} \tag{0.9}$$

b) For the ground state one has $n = 1$ which is odd, hence the first nonzero correction occurs at first order in perturbation theory and is given by

$$\Delta E_1^{(1)} = \langle 1 \mid H' \mid 1 \rangle = \frac{2\lambda}{L} \tag{0.10}$$

For the first excited state one has $n = 2$, in the formal power series of time independent non degenerate perturbation theory the m-th order contribution to the energy E_2 is given by

$$\Delta E_2^{(m)} = \langle 2_{(0)} \mid H' \mid 2^{(m-1)} \rangle \tag{0.11}$$

Where $\mid 2^{(m-1)} \rangle$ is the wave function correction to order $m - 1$. No matter which form $\mid 2^{(m-1)} \rangle$ takes, the inner product always vanishes since $\psi_2(x)$ has a zero at $x = L/2$. Hence the first excited state is not corrected to all orders in perturbation theory.

c) The generate time dependent wave function van be expanded as

$$\mid \psi(t) \rangle = \sum_n c_n(t) e^{-i\frac{1}{\hbar}E_n t} \mid n \rangle \tag{0.12}$$

With

$$c_n(t) = c_n(0) - \frac{i}{\hbar} \sum_{m \neq n} \int_0^t dt' \langle n \mid H' \mid m \rangle e^{-i\frac{(E_m - E_n)t'}{\hbar}} c_m(t') \tag{0.13}$$

In first order time dependent perturbation theory $c_m(t')$ gets replaced by $c_m(0)$. Since the initial condition sets $c_m(0) = 0$ for $m \neq 1$ the only nonzero matrix element occurs for odd$n = 1, 3, 5, \cdots$, because of (0.23). Hence the only transitions can occur to states with odd n.

d) For an interaction of the form

$$H' = V(e^{i\omega t} + e^{-i\omega t}) \tag{0.14}$$

Fermi's Golden rule gives the transition rate

$$\Gamma(\omega)_{1\to 2k+1} = \frac{2\pi}{\hbar} \mid \langle 1|V \mid 2k+1\rangle|^2 \Big(\delta(E_{2k+1} - E_1 - \hbar\omega) + \delta(E_{2k+1} - E_1 + \hbar\omega)\Big) \tag{0.15}$$

With

$$V = \lambda\delta(x - L/2) \tag{0.16}$$

One finds from part a)

$$\mid \langle 1|V \mid 2k+1\rangle|^2 = \frac{4\lambda^2}{L^2} \tag{0.17}$$

The total rate then becomes

$$\begin{aligned}
\Gamma_{tot} &= \int_{-\infty}^{\infty} d\omega \rho(\omega)\Gamma(\omega)_{1\to 2k+1} \\
&= \frac{2\pi}{\hbar}\Big\{\rho\Big(\frac{1}{\hbar}(E_{2k+1} - E_1)\Big) + \rho\Big(-\frac{1}{\hbar}(E_{2k+1} - E_1)\Big)\Big\} \mid \langle 0|V \mid 2k+1\rangle|^2 \\
&= \frac{2\pi}{\hbar}\frac{4\lambda^2}{L^2}\sqrt{\frac{\alpha}{\pi}} 2e^{-\frac{\alpha}{\hbar^2}(E_{2k+1}-E_1)^2}
\end{aligned} \tag{0.18}$$

Quantum Mechanics Q1 Jaynes-Cummings Model

A two-level system with bare eigenstates $|\uparrow\rangle$ and $|\downarrow\rangle$ ("a qubit") has an energy splitting ω_q ($\hbar = 1$). Consider the coupling between the quibit and a harmonic oscillator (bare eigenstates $|n\rangle$ for $n \in \{0,1,2,\dots\}$) with energy splitting ω_0, ~~which~~ described by the Hamiltonian

$$H = H_0 + H_{int}$$

$$H_0 = \omega_0 (\hat{a}^\dagger \hat{a} + \tfrac{1}{2}) + \tfrac{1}{2}\omega_q \hat{\sigma}_z$$

$$H_{int} = \frac{\Omega}{2}(\hat{a}^\dagger \hat{\sigma}_- + \hat{a}\hat{\sigma}_+)$$

where Ω is the coupling strength, $\hat{a}^\dagger$ and $\hat{a}$ are the harmonic oscillator raising and lowering operators, and the spin-$\frac{1}{2}$ operators are defined as $\hat{\sigma}_z \equiv |\uparrow\rangle\langle\uparrow| - |\downarrow\rangle\langle\downarrow|$, $\hat{\sigma}_+ \equiv |\uparrow\rangle\langle\downarrow|$, $\hat{\sigma}_- \equiv |\downarrow\rangle\langle\uparrow|$.

a.) Consider the "dispersive regime," defined by $\omega_0 - \omega_q \gg \frac{\Omega}{2}$. Find the energies and eigenstates of H. You may treat H_{int} as a perturbation and use 1st-nonzero-order terms. Consider coupling between nearly-degenerate states only. Let $\Delta \equiv \omega_0 - \omega_q > 0$. Don't bother to normalize the eigenstates.

b.) Consider the "resonant regime," defined by $\omega_q \approx \omega_0 \equiv \omega$. Find the energies and eigenstates of H. You may treat H_{int} as a perturbation and use 1st-order degenerate perturbation theory. Hint: consider the state $\frac{1}{\sqrt{2}}(|\downarrow, n\rangle + e^{i\varphi}|\uparrow, n-1\rangle)$ consider coupling between degenerate states only.

Solution to Quantum Mechanics: Jaynes-Cummings Model

a.) Solution #1: plain-old perturbation theory

$$E^0_{\downarrow,n} = n\omega_0 + \frac{\omega_0}{2} - \frac{\omega_a}{2} = n\omega_0 + \frac{\Delta}{2}$$

$$E^0_{\uparrow,n-1} = n\omega_0 - \frac{\Delta}{2}$$

Apply Merzbacher 8.37 or Sakurai 5.1.11

$$E_{+,n} = \frac{1}{2}\left[2n\omega_0 + \sqrt{(-\Delta)^2 + 4\left|\frac{\Omega}{2}\sqrt{n}\right|^2}\,\right]$$

$$= n\omega_0 + \frac{1}{2}\Delta\underbrace{\sqrt{1 + \frac{4\left|\frac{\Omega}{2}\sqrt{n}\right|^2}{\Delta^2}}}_{\approx\, 1 + \frac{n\Omega^2}{2\Delta^2}}$$

$$\approx \boxed{n\omega_0 + \frac{1}{2}\Delta + \frac{n\Omega^2}{4\Delta}}$$

$$E_{-,n} = \frac{1}{2}\left[2n\omega_0 - \frac{\Delta}{2}\sqrt{1 + \frac{\Omega^2 n}{\Delta^2}}\,\right]$$

$$\approx \boxed{n\omega_0 - \frac{1}{2}\Delta - \frac{n\Omega^2}{4\Delta}}$$

Un-normalized eigenstates:

$$|+,n\rangle = |\downarrow,n\rangle + \frac{\langle\uparrow,n-1|H_{int}|\downarrow,n\rangle}{E^0_{\downarrow,n} - E^0_{\uparrow,n-1}} = \boxed{|\downarrow,n\rangle + \frac{g\sqrt{n}}{\Delta}|\uparrow,n-1\rangle}$$

$$\boxed{|-,n\rangle = |\uparrow,n-1\rangle - \frac{g\sqrt{n}}{\Delta}|\downarrow,n\rangle}$$

Solution to QM1 Jaynes-Cummings Model

- Solution #1: eigenstates by inspection

b)

Since we are instructed to consider coupling between degenerate states only (1^{st} order), we have a series of two-level systems described by $|\downarrow, n\rangle$ and $|\uparrow, n-1\rangle$.

These have the same energy under H_0 $(n\omega)$, so we will subtract this constant from the Hamiltonian. We note that H_{int} is completely off-diagonal in this basis

$$\langle\downarrow,n|H_{int}|\downarrow,n\rangle = \langle\uparrow,n-1|H_{int}|\uparrow,n-1\rangle = 0, \quad \langle\downarrow,n|H_{int}|\uparrow,n-1\rangle = \frac{\Omega}{2}\sqrt{n}$$

so we are asked to diagonalize a 2x2 matrix that is proportional to $\hat{\sigma}_x$:

$$H_n = \frac{\Omega}{2}\sqrt{n}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

The eigenstates are therefore given by

b)

$$\boxed{\begin{aligned} |+,n\rangle &\equiv \left(|\downarrow,n\rangle + |\uparrow,n-1\rangle\right)\frac{1}{\sqrt{2}} \\ |-,n\rangle &\equiv \left(|\downarrow,n\rangle - |\uparrow,n-1\rangle\right)\frac{1}{\sqrt{2}} \end{aligned}}$$

The energies can be found as follows

$$H_n|+,n\rangle = E_{+n}|+,n\rangle = \frac{\Omega}{2}\sqrt{n}\,|+,n\rangle$$

$$H_n|-,n\rangle = E_{-n}|-,n\rangle = -\frac{\Omega}{2}\sqrt{n}\,|-,n\rangle$$

b) so the energies are given by $\boxed{n\omega \pm \frac{\Omega}{2}\sqrt{n}}$

Solution #2 : degenerate perturbation theory

b.)

Using, e.g., Merzbacher 8.37 we can write

$$E_{\pm} = \frac{1}{2}\left[n\omega + n\omega \pm \sqrt{\left(\cancelto{0}{n\omega - n\omega}\right)^2 + 4\left|\langle\downarrow, n|\, H_{int}\, |\uparrow, n-1\rangle\right|^2}\,\right]$$

Since $\langle\downarrow, n|\, H_{int}\, |\uparrow, n-1\rangle = \frac{\Omega}{2}\sqrt{n}$, we have

b.)
$$\boxed{\begin{aligned} E_+ &= n\omega + \frac{\Omega}{2}\sqrt{n} \\ E_- &= n\omega - \frac{\Omega}{2}\sqrt{n} \end{aligned}}$$

Another way to get this is to notice the hint about the states $|\pm\rangle \equiv \frac{1}{\sqrt{2}}\left(|n,\downarrow\rangle \pm |n-1,\uparrow\rangle\right)$. Do these states diagonalize H_{int}? Well, $H_{int}|+,n\rangle = \frac{\Omega}{2}\sqrt{n}\,|+,n\rangle$, so yes, they do. We can now use them to apply Sakurai 5.2.11:

$$\Delta_l^+ = \langle +,n|\, H_{int}\, |+,n\rangle = \frac{\Omega}{2}\sqrt{n}$$
$$\Delta^- = \langle -,n|\, H_{int}\, |-,n\rangle = -\frac{\Omega}{2}\sqrt{n}$$

b.) therefore,
$$E_+ = \langle +,n|\, H_0\, |+,n\rangle + \frac{\Omega}{2}\sqrt{n} = \boxed{n\omega + \frac{\Omega}{2}\sqrt{n}}$$
$$E_- = \langle -,n|\, H_0\, |-,n\rangle - \frac{\Omega}{2}\sqrt{n} = \boxed{n\omega - \frac{\Omega}{2}\sqrt{n}}$$

Since we know that $|\pm,n\rangle$ diagonalize H_{int}, we can check to see if they're also eigenstates of H_0:

$H_0|\pm,n\rangle = n\omega$, so we have

$$\boxed{\begin{aligned} |+,n\rangle &= \frac{1}{\sqrt{2}}\left(|\downarrow,n\rangle + |\uparrow,n-1\rangle\right) \\ |-,n\rangle &= \frac{1}{\sqrt{2}}\left(|\downarrow,n\rangle - |\uparrow,n-1\rangle\right) \end{aligned}}$$

Q 8

Question 8: Statistical Mechanics

We propose to evaluate the *Richardson effect*, namely the electric current density of electrons which is produced by heating up a metal in the presence of an external electric potential. The potential energy of an electron just outside the metal is denoted $W > 0$.

The potential energy for electrons inside the metal is taken to be 0. The electrons are considered otherwise non-interacting, and filled up to chemical potential μ with $\mu < W$. Since we consider the problem to be at sufficiently low temperature, μ may be identified with the Fermi energy.

(a) State the condition on the momentum of an electron that can escape from the metal to the outside as a function of W and μ.

(b) Derive a general expression for the current density I of electrons leaving the metal.

(c) Obtain an approximation of your result in (b) valid for sufficiently low temperatures.

Solution to Question 8

(a) We take the edge of the metal where the electrons are being emitted to be orthogonal to the z-direction. The condition for an electron to be able to escape the metal to the outside is that its kinetic energy in the z-direction can overcome the potential energy outside the metal, so that we then must have,

$$\frac{p_z^2}{2m} > W \tag{0.19}$$

where m is the electron mass, and p_z is the electron momentum in the z-direction.

(b) The density of electrons inside the metal in an infinitesimal phase space volume dVd^3p (where dV is the spacial volume element) is given by,

$$2\frac{dV\,d^3p}{(2\pi\hbar)^3}\frac{1}{e^{\beta(\mathbf{p}^2/2m-\mu)}+1} \tag{0.20}$$

The factor of 2 arises from the two spin states of the electron, and we have $\mathbf{p}^2 = p_x^2+p_y^2+p_z^2$. The electric current density is then given by the thermal expectation value of the observable,

$$\frac{e\,p_z}{m} \tag{0.21}$$

per unit volume, restricted to the range $p_z > \sqrt{2mW}$. Thus the current density $I = I_z$ is given by the following integral,

$$I_z = 2\frac{e}{m}\frac{1}{(2\pi\hbar)^3}\int_{\sqrt{2mW}}^{\infty} dp_z \int_{-\infty}^{\infty} dp_x \int_{-\infty}^{\infty} dp_y \frac{p_z}{e^{\beta(\mathbf{p}^2/2m-\mu)}+1} \tag{0.22}$$

Changing variables to the following dimensionless combinations s, t defined by,

$$s = \beta\left(\frac{p_z^2}{2m} - W\right) \qquad\qquad t = \beta\frac{p_x^2+p_y^2}{2m} \tag{0.23}$$

The integral for I_z reduces to,

$$I_z = \frac{e\,m}{2\pi^2\hbar^3} (kT)^2 \int_0^\infty ds \int_0^\infty dt \, \frac{1}{e^{s+t+\beta(W-\mu)} + 1} \tag{0.24}$$

(c) For sufficiently low temperatures, namely $T \ll W - \mu$, we may drop the 1 in the denominator, and carry out the integrals over s and t explicitly. We are then left with the following approximate formula,

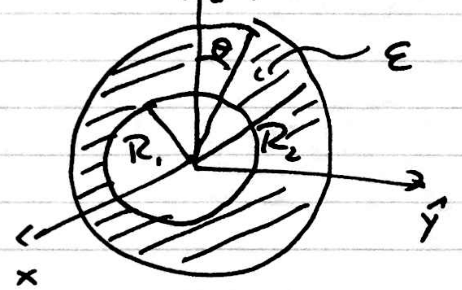
$$I_z = \frac{e\,m}{2\pi^2\hbar^3} (kT)^2 \exp\left\{ -\frac{W-\mu}{kT} \right\} \tag{0.25}$$

1. The gap between two spherical conducting shells — see the figure below — is filled with a spatially inhomogeneous dielectric so that the dielectric constant depends on the polar angle θ as: ①

$$\varepsilon(\theta) = \varepsilon_1 + \varepsilon_2 \cos^4\theta$$

a) when charged so that the inner and outer spheres have charges $+Q$ and $-Q$ respectively, show that the internal electric field is purely radial i.e. $E = \hat{r} E_r$ and is independent of the angles φ, θ.

b) Calculate the capacitance C of the system.

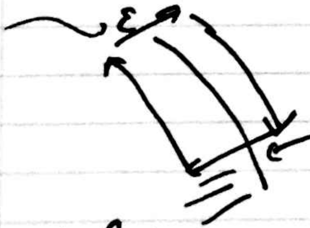


Ans.

a) no φ-dependence by azimuthal symmetry: $E_\varphi = 0$ and 5

Consider the line-integral shown below.

$\lim_{\varepsilon \to 0}$



$\vec{\nabla} \times \vec{E} = 0 \Rightarrow \oint \vec{E} \cdot d\vec{\ell} = 0$ and since $\vec{E}$ vanishes in the conductor, E_θ must vanish in the dielectric.

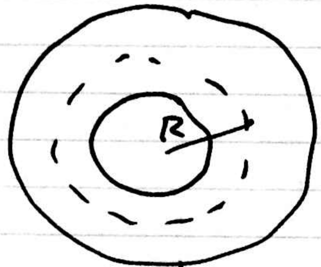
$\Rightarrow E_\theta = 0$ the same argument can be extended throughout the gap.

$\Rightarrow \vec{E} = \hat{r} E_r(r)$

any other dependence $\Rightarrow \vec{\nabla} \times \vec{E} \neq 0$.

②

b) Using Gauss's law inside.



$R_1 < R < R_2$

$$\oint \bar{D} \cdot d\vec{A} = 4\pi Q \quad \text{or}$$

$$2\pi R^2 E \int_{-1}^{+1} d(\cos\theta) \left\{ \varepsilon_1 + \varepsilon_2 \cos^4\theta \right\} = 4\pi Q$$

$$4\pi R^2 E \,\frac{(\varepsilon_2 + 5\varepsilon_1)}{5} = 4\pi Q \Rightarrow E = \frac{5Q}{R^2(\varepsilon_2 + 5\varepsilon_1)}$$

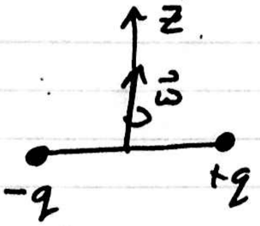
$$\Delta V = -\int_{R_1}^{R_2} d\vec{r} \cdot \vec{E} = \frac{5Q}{(\varepsilon_2 + 5\varepsilon_1)}\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$$

$$C' = \frac{Q}{\Delta V} = \frac{(5\varepsilon_1 + \varepsilon_2)}{5}\left[\frac{R_2 R_1}{R_2 - R_1}\right]$$

$$\int_{-1}^{+1} x^4 dx = \frac{1}{5}x^5 \Big|_{-1}^{+1} = 2/5$$

(5)

3. A small stick of length l is found rotating in the xy plane. This stick has charge $\pm q$ embedded on either end – see figure below.



$\vec{\omega} = \hat{z}\omega_0$ initial angular velocity. (moment of inertia I.)

You observe that its angular velocity is slowly decreasing: $\dot{\omega} < 0$ and $\frac{|\dot{\omega}|}{\omega} \ll 1$. Predict $\omega(t)$.

<u>Ans.</u> Slowing due to radiation.

Rotating dipole $\vec{p} = (\hat{x} + i\hat{y}) q l e^{-i\omega t}$ +5

Power radiated $\frac{dP}{d\Omega} = \frac{c k^4}{8\pi} |(\hat{r}\times\hat{p})\times\hat{r}|^2$; $k = \omega/c$ +5

$(\hat{r}\times\hat{p})\times\hat{r} = \vec{p} - (\hat{r}\cdot\hat{p})\hat{r}$

$\hat{r}\cdot\vec{p} = 2ql\,[\sin\theta\cos\varphi + i\sin\theta\sin\varphi] = 2ql\sin\theta\, e^{i\varphi}$

$|(\hat{r}\times\hat{p})\circ\hat{r}|^2 = 4q^2l^2[\hat{x} + i\hat{y} - \sin\theta\, e^{i\varphi}\hat{r}]\cdot[\hat{x} - i\hat{y} - \sin\theta\, e^{-i\varphi}\hat{r}]$

$= (2 - \sin^2\theta)\cdot 4q^2l^2$ After some algebra...

$\frac{dP}{d\Omega} = \frac{\omega^4}{2c^3\pi}[1 + \cos^2\theta]$; Integrate over the unit sphere to get total power radiated: +5

$$P = \oint \frac{dP}{d\Omega}d\Omega = \frac{\cancel{2}\cancel{\pi}\omega^4}{\cancel{2}c^3\cancel{\pi}}\left[x + x^3\right]_{-1}^{1} = \frac{8}{3}\frac{\omega^4}{c^3}(ql)^2$$

(6)

Now $\frac{d}{dt}\left(\frac{1}{2}I\omega^2\right) = -P$;

$$\Rightarrow \quad I\omega\dot{\omega} = -P = -A\omega^4$$

$$I\dot{\omega} = -A\omega^3 \quad \text{or} \quad \int_{\omega_0 \leftarrow \text{initial}}^{\omega} \frac{d\omega}{\omega^3} = -\frac{A}{I}\int_0^t dt$$

$$-\frac{1}{2}\left.\frac{1}{\omega^2}\right|_{\omega_0}^{\omega} = -\frac{A}{I}t$$

$$\frac{1}{\omega^2} - \frac{1}{\omega_0^2} = \frac{2At}{I} \Rightarrow \frac{1}{\omega^2} = \frac{I}{\omega_0^2 I} + \frac{2At\omega_0^2}{I\omega_0^2}$$

$$\omega(t) = \sqrt{\frac{\omega_0^2 I}{I + 2At\omega_0^2}} \quad ; \quad A = \frac{8}{3c^3}(q\ell)^2$$

+5

Q12

Assume the existence of magnetic charge related to the magnetic field by the local reaction $\vec{\nabla}\cdot\vec{B}=\mu_0\rho_m$.

(a). Using the Gauss's theorem, obtain the magnetic field $\vec{B}$ of a point magnetic charge at the origin.
(b). In the absence of the magnetic charge, the curl of the electric field is given by the Faraday's law, $\vec{\nabla}\times\vec{E}=-\frac{\partial\vec{B}}{\partial t}$. Show that this law is incompatible with the magnetic charge density that is a function of time.
(c). Assuming that magnetic charge is conserved, derive the local relation between the magnetic charge current density $\vec{J}_m$ and the magnetic density ρ_m.
(d). Modify Faraday's law as given in part (b) to obtain a law consistent with the presence of the magnetic charge density that is a function of position and time.

Solution

(a) $\int_V \vec{\nabla}\cdot\vec{B}dV=\oint_S \vec{B}\cdot d\vec{S}=4\pi r^2 B(r)=\mu_0 q_m$

$$\vec{B}(r)=\frac{\mu_0 q_m}{4\pi r^2}\hat{r}$$

(b) $\frac{\partial}{\partial t}\vec{\nabla}\cdot\vec{B}=\vec{\nabla}\cdot\frac{\partial\vec{B}}{\partial t}=-\vec{\nabla}\cdot\left(\vec{\nabla}\times\vec{E}\right)=0$

On the other hand, $\frac{\partial}{\partial t}\vec{\nabla}\cdot\vec{B}=\mu_0\frac{\partial\rho_m}{\partial t}$

Thus the Faraday's law is incompatible with the magnetic charge density that is a function of time.

(c) $\frac{\partial}{\partial t}\int_V \rho_m dV=-\oint_S \vec{J}_m\cdot d\vec{S}=-\oint_S \vec{\nabla}\cdot\vec{J}_m dV$

$$\frac{\partial\rho_m}{\partial t}+\vec{\nabla}\cdot\vec{J}_m=0$$

This is the continuity equation for magnetic charge.

(d) If we modify Faraday's law, $\vec{\nabla}\times\vec{E}=-\mu_0 J_m-\frac{\partial\vec{B}}{\partial t}$

and $-\mu_0\vec{\nabla}\cdot J_m-\frac{\partial\vec{\nabla}\cdot\vec{B}}{\partial t}=-\mu_0\left(\vec{\nabla}\cdot J_m+\frac{\partial\rho_m}{\partial t}\right)=0$

Hence $\frac{\partial}{\partial t}\vec{\nabla}\cdot\vec{B}=-\mu_0\vec{\nabla}\cdot J_m=\mu_0\frac{\partial\rho_m}{\partial t}$ which is consistent with the second equation in (b).

Q13

An un-polarized plane electromagnetic wave is scattered by a free electron. Derive the differential cross-section for scattering in the non-relativistic limit (Thompson scattering).

Solution

Consider an incident plane wave

$\vec{E}_i = E_0 e^{-i(\omega t - \vec{k}\cdot\vec{x})}\hat{e}_0$

The force on the free electron is

$\vec{F} = -e\vec{E}_i \sim -eE_0 e^{-i\omega t}\hat{e}_0 = m\ddot{\vec{x}} = -m\omega^2\vec{x} \qquad \vec{x} = \dfrac{eE_0\hat{e}_0}{m\omega^2}e^{-i\omega t}$

The induced dipole moment is $\vec{p} = -e\vec{x} = -\dfrac{e^2E_0\hat{e}_0}{m\omega^2}e^{-i\omega t}$

The scattered electric field is $\vec{E}_s = \dfrac{k^2}{4\pi\varepsilon_0}\dfrac{e^{ikr}}{r}\left[(\hat{n}\times\vec{p})\times\hat{n}\right]$

The differential cross section is $\dfrac{d\sigma}{d\Omega} = \dfrac{k^4}{(4\pi\varepsilon_0)^2}\left(\dfrac{q^2}{m\omega^2}\right)^2\left|\hat{n}\times\hat{e}_0\right|^2$ where $\omega = ck$.

$$\left|\hat{n}\times\hat{e}_0\right|^2 = \begin{vmatrix}\hat{x} & \hat{y} & \hat{z}\\ n_x & n_y & n_z\\ \sin\phi & \cos\phi & 0\end{vmatrix}^2 = \left|-\hat{x}n_z\cos\phi + \hat{y}n_z\sin\phi + \hat{z}(n_x\cos\phi - n_y\sin\phi)\right|^2$$

$$= n_z^2\left(\cos^2\phi + \sin^2\phi\right) + n_x^2\cos^2\phi + n_y^2\sin^2\phi - 2n_xn_y\sin\phi\cos\phi$$

Using $<\cos^2\phi> = <\sin^2\phi> = 1/2$ and $<\cos\phi\sin\phi> = 0$, we have

$$\left|\hat{n}\times\hat{e}_0\right|^2 = n_z^2 + \frac{1}{2}\left(n_x^2 + n_y^2\right) = \cos^2\theta + \frac{1}{2}\sin^2\theta = \frac{1}{2}(1+\cos^2\theta)$$

Thus we obtain $\dfrac{d\sigma}{d\Omega} = \dfrac{1}{(4\pi\varepsilon_0)^2}\left(\dfrac{e^2}{mc^2}\right)^2\dfrac{1+\cos^2\theta}{2} = \dfrac{1+\cos^2\theta}{2}r_e^2$

Where $r_e \equiv \dfrac{e^2}{4\pi\varepsilon_0}\dfrac{1}{mc^2}$ is the classical electron radius.

Q 14

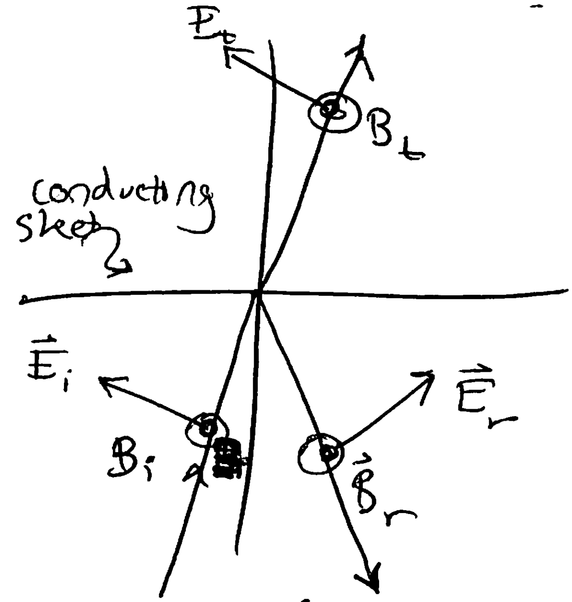
In cgs units

① $\nabla \cdot \vec{D} = 4\pi\rho$ $\qquad \vec{D} = \varepsilon \vec{E}$

② $\nabla \cdot \vec{B} = 0$ $\qquad \vec{B} = \mu \vec{H}$

③ $\nabla \times \vec{H} = \frac{4\pi}{c}\vec{J} + \frac{1}{c}\frac{\partial \vec{D}}{\partial t}$

④ $\nabla \times \vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t}$



Drawn with finite angle of incidence for clarity

assume $\vec{E} = \vec{E}_0 e^{i(\vec{k}\cdot\vec{r} - \omega t)}$

Then ④ gives $\vec{k} \times \vec{E} = \frac{\omega}{c}\vec{B}$

$$\Rightarrow \vec{B} = \hat{k} \times \vec{E}$$

Normally $\vec{J} = \sigma \vec{E}$. Here we have a surface current

$$\vec{K} = \sigma_{2D}\vec{E}$$

where $\sigma_{2D} = \frac{1}{Z_0} = \left.\frac{1}{377\Omega}\right|_{MKS} = \left.\frac{c}{4\pi}\right|_{cgs}$

The boundary conditions come from ④ and ③.

From ④ $\oint \vec{E}\cdot d\vec{l} = -\frac{1}{c}\frac{\partial}{\partial t}\int \vec{B}\cdot d\vec{a} = 0$ $\quad \vec{E}$ is continuous

$\vec{E}_i + \vec{E}_r = \vec{E}_t$ as drawn $\Rightarrow \boxed{E_i - E_r = E_t}$

From ③ $\oint \vec{H}\cdot d\vec{l} = \oint \vec{B}\cdot d\vec{l} = \frac{4\pi}{c}\int \vec{J}\cdot d\vec{a} + \frac{1}{c}\frac{\partial}{\partial t}\cancel{\int \vec{D}\cdot d\vec{a}}$

$B_i + B_r - B_t = \frac{4\pi}{c}K = \frac{4\pi}{c}\sigma_{2D}E_t$ $\qquad$ 0 loop has no area

using $|\vec{B}| = |\vec{E}|$

$$\boxed{E_i + E_r - E_t = S\,E_t}$$

where $S \equiv \frac{4\pi}{c}\sigma_{2D}$

$= Z_0 \sigma_{2D}$

With the boxed equations

Eliminate E_r

$$E_i + (E_i - E_t) - E_t = s E_t$$

$$2(E_i - E_t) = s E_t$$

$$E_i = \left(\frac{s}{2} + 1\right) E_t$$

$$\frac{E_t}{E_i} = \frac{1}{\frac{s}{2} + 1}$$

Eliminate E_t

$$E_i + E_r - (E_i - E_r) = s(E_i - E_r)$$

$$2E_r = s(E_i - E_r)$$

$$E_r\left(1 + \frac{s}{2}\right) = \frac{s}{2} E_i$$

$$\frac{E_r}{E_i} = \frac{s/2}{\frac{s}{2} + 1}$$

$$T = \left(\frac{E_t}{E_i}\right)^2 = \frac{1}{\left(1 + \frac{s}{2}\right)^2} \qquad R = \left(\frac{E_r}{E_i}\right)^2 = \left(\frac{\frac{s}{2}}{1 + \frac{s}{2}}\right)^2$$

$$T + R + A = 1 \quad \therefore \quad A = \frac{s}{\left(1 + \frac{s}{2}\right)^2}$$

For a sheet impedance of $377\,\Omega$, $s = 1$ and

$$\boxed{T = \frac{4}{9} \qquad R = \frac{1}{9} \qquad A = \frac{4}{9}}$$

To maximize A

$$\frac{dA}{ds} = \frac{1}{\left(1 + \frac{s}{2}\right)^2} - \frac{2s \cdot \frac{1}{2}}{\left(1 + \frac{s}{2}\right)^3} = 0 \qquad 1 = \frac{s}{1 + \frac{s}{2}}$$

$$1 + \frac{s}{2} = s \qquad \frac{s}{2} = 1 \qquad s = 2$$

So we need $\sigma_{2D} = \frac{2}{Z_0}$, or a sheet resistivity of $\frac{377}{2}\,\Omega = \boxed{189\,\Omega}$, this gives $\boxed{A = \frac{1}{2}}$

Half of the electromagnetic wave's energy is carried in the $\vec{B}$ field, while the thin sheet only couples dissipatively to the $\vec{E}$ field. Thus even with the best impedance match it can only achieve 50% absorption.