

# Fall 2007 Q1

$N$  spin- $\frac{1}{2}$ , mass  $m$  particles in common  $V(r) = \frac{1}{2} m \omega^2 r^2$  potential.

a)  $E_{\text{ground}}(N) \Big|_{N=19} = ?$ ,

b)  $\lim_{N \rightarrow \infty} \frac{E_{\text{ground}}(N)}{AN^\alpha} = 1$ , find  $\alpha = \text{const.}$

a)  $E_{\text{g.s.}}(1) = \hbar\omega \left( \frac{1}{2} + 0 \right)$

$E_{\text{g.s.}}(3) = \hbar\omega \left( \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + 1 \right)$

:

$E_{\text{g.s.}}(N) = \hbar\omega \left( \frac{N}{2} + 2 \sum_{n=0}^M n + \text{mod} \left\lfloor \frac{N}{2} \right\rfloor \cdot \frac{N-1}{2} \right)$ , where  $M \begin{cases} \frac{N_{\text{even}}-2}{2} \\ \frac{N_{\text{odd}}-3}{2} \end{cases}$

$\Rightarrow E_{\text{g.s.}}(19) = \hbar\omega \left( \frac{19}{2} + 2 \cdot (0+1+2+3+\dots+8) + 9 \right) =$   
 $= \hbar\omega (18.5 + 72) = \underline{\underline{(90.5)\hbar\omega}}$

b)  $\lim_{N \rightarrow \infty} \frac{E_{\text{g.s.}}(N)}{AN^\alpha} = 1 \approx \frac{\hbar\omega}{A} \left( \frac{N}{2N^\alpha} + 2 \sum_{n=0}^M \frac{n}{N^\alpha} \right) \Big|_{N \rightarrow \infty}$

where we took  $\frac{N_{\text{even}}-2}{2} \approx \frac{N_{\text{odd}}-3}{2} \approx \frac{N_{\text{even}}}{2}$

Now,  $2 \sum_{n=0}^M \frac{n}{N^\alpha} \Big|_{N \rightarrow \infty} \approx \frac{1}{N^\alpha} \int_0^{\frac{N}{2}} 2n \, dn = \frac{N^2}{4N^\alpha}$  (see Stirling's approx.  $\rightarrow$  [\*])

$\therefore \lim_{N \rightarrow \infty} \frac{\hbar\omega}{A} \left( \frac{N}{2N^\alpha} + \frac{N^2}{4N^\alpha} \right) = 1$  if  $\alpha=2$  and  $A = \frac{\hbar\omega}{4}$ .

[\*]  $\ln(N!) \approx N \ln(N) - N$

Proof:  $\ln(N!) = \sum_{n=1}^N \ln(n) \Big|_{N \rightarrow \infty} \approx \int_1^N \ln(u) \, du = \int_1^N (\ln(u) + 1) \, du - \int_1^N 1 \, du = N \ln(N) - N + 1$   
 $= N \ln(N) - N$