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1. Quantum Mechanics

A particle of mass M bounces elastically between two infinite plane walls separated by a distance D. The particle is in its lowest possible energy state.

a)What is the energy of this state?

b) The separation between the walls is very slowly increased to $2D$.

i) How does the energy change?

ii) Compare this energy change with the result obtained classically from the mean force exerted on a wall by the bouncing ball.

c) Now assume that the separation between the wall is increased very rapidly, with one wall moving at speed $v \gg \sqrt{E/M}$. Classically there is no change in the particle's energy since the wall is moving faster than the particle and cannot be struck by the particle while the wall is moving.

i) What happens to the energy quantum mechanically?

ii) Compute the probability that the particle is left in it lowest possible energy state.

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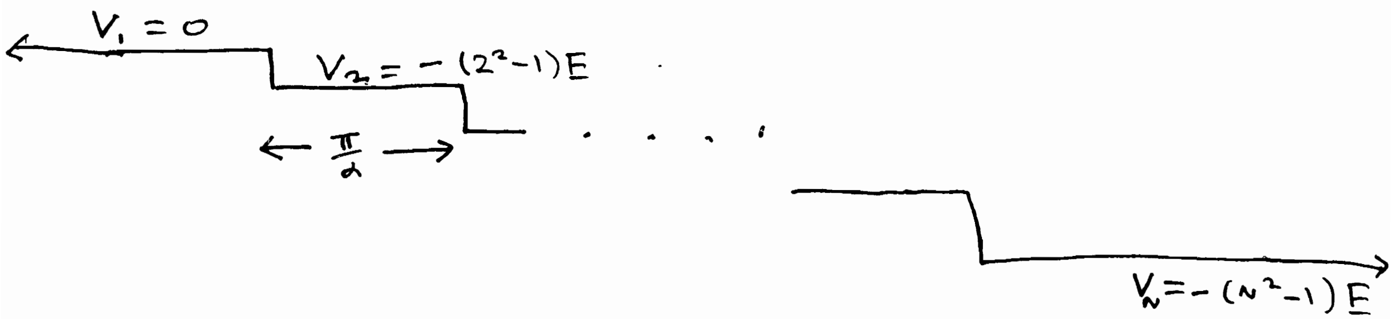
2. Quantum Mechanics

A particle of total energy $E = \hbar^2\alpha^2/2m$ moves in a series of N contiguous one-dimensional regions (see figure). The potential in the nth region is

$$V_n = -(n^2 - 1)E\ , \quad n = 1, 2, \ldots, N$$

All regions are of equal width π/α except for the first and last which are of effectively infinite extent. Calculate the two transmission coefficients for a particle incident from either end.



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3. Quantum Mechanics

Let $\mathbf{V}$ be a vector operator with components (V^1, V^2, V^3) or equivalently (V^+, V^-, V^0) where $V^\pm = V^1 \pm iV^2$ and $V^0 = V^3$. The states of the quantum system on which these operators act are denoted by $|\alpha, j, m\rangle$ where j, m are the customary total angular momentum quantum numbers, and α stands for any remaining quantum numbers.

a) Write the correct commutation relations in the $0, \pm$ basis between the angular momentum generators J^i and the components V^i of the vector operator.

b) Calculate the following ratios, for arbitrary values of j.

$$\begin{aligned}
R_1 &= \frac{\langle\alpha', j, j-1|V^-|\alpha, j, j\rangle}{\langle\alpha', j, j|V^0|\alpha, j, j\rangle} \\
R_2 &= \frac{\langle\alpha', j, j-1|V^0|\alpha, j, j-1\rangle}{\langle\alpha', j, j|V^0|\alpha, j, j\rangle} \\
R_3 &= \frac{\langle\alpha', j+1, j+1|V^+|\alpha, j, j\rangle}{\langle\alpha', j+1, j|V^0|\alpha, j, j\rangle}
\end{aligned} \tag{0.1}$$

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4. Quantum Mechanics

Consider a particle of mass m and electric charge e in a constant magnetic field B, which is in the z-direction, and subject to a harmonic potential. The dynamics of the particle in the z-direction decouples and may be ignored. The problem is effectively 2-dimensional, specified by the coordinates $x_j(t)$, $j = 1,2$ and governed by the Lagrangian,

$$L(x_j, \dot{x}_j) = \frac{1}{2}m(\dot{x}_1^2 + \dot{x}_2^2) - \frac{1}{2}m\omega^2(x_1^2 + x_2^2) + \frac{1}{2}eB(x_1\dot{x}_2 - x_2\dot{x}_1) \qquad (0.2)$$

(a) Compute the momenta p_j, canonically conjugate to x_j and the Hamiltonian H. Write down the canonical commutation relations.

(b) Show that the Lagrangian L is invariant under rotations of x_j in the x_1, x_2-plane and compute the associated angular momentum J.

Introduce the harmonic oscillator operators a_j (and their adjoints $a_j^\dagger$)

$$a_j = \frac{1}{\sqrt{2m\hbar\omega_B}}(ip_j + m\omega_B x_j) \qquad \omega_B^2 = \omega^2 + \frac{e^2B^2}{4m^2} \qquad (0.3)$$

whose canonical commutation relations are $[a_j, a_k] = 0$ and $[a_j, a_k^\dagger] = \delta_{jk}$.

(c) Express H and J in terms of a_i and $a_i^\dagger$; compute their energy and angular momentum eigenvalues, and their degeneracies.

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5. Quantum Mechanics

A system of three distinguishable particles, whose spin $\frac{1}{2}$ operators are $\mathbf{S}_1$, $\mathbf{S}_2$, $\mathbf{S}_3$, is governed by the Hamiltonian

$$H = \frac{A}{\hbar^2}\mathbf{S}_1 \cdot \mathbf{S}_2 + \frac{B}{\hbar^2}(\mathbf{S}_1 + \mathbf{S}_2) \cdot \mathbf{S}_3$$

a) Express the Hamiltonian in terms of the complete set of operators $\mathbf{S}_{12}^2$, $\mathbf{S}^2$, and $\mathbf{S}_3^2$, where $\mathbf{S}_{12} = \mathbf{S}_1 + \mathbf{S}_2$ and $\mathbf{S} = \mathbf{S}_1 + \mathbf{S}_2 + \mathbf{S}_3$.

b) Now let the Hamiltonian act on the kets $|S_{12}, S_3, S, m_S\rangle$, where m_S is the z-component of the total spin $\mathbf{S}$, find all the energy eigenvalues and the degeneracies.

c) Check that the total number of states works out correctly.

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6. Thermo./Stat Mech

a) Considering an atmosphere in equilibrium in a uniform gravitational field, derive the Einstein relation between the mobility μ and the diffusion constant D of an atmospheric particle.

b) Consider a 1 μm diameter oil droplet in water at temperature $T = 300K$. What is the rms displacement of the droplet after $10s$?

The viscosity of water is $\eta = 10^{-2} g/(cm\ s)$ and the Boltzmann constant is $k = 1.4 \times 10^{-14} ergs/K$.

The mobility of a particle in a medium is defined by $u = \mu F$ where u is the velocity and F the applied force.

Also, you may recall that the drag force on a sphere moving through a medium of viscosity η is $F = 6\pi\eta R u$ where R is the radius of the sphere and u is the velocity (Stokes formula).

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7. Thermo/Stat Mech

Atomic hydrogen in a strong magnetic field can be in two states: in state a, the spin of the electron equals $-1/2$ and the spin of the proton equals $+1/2$ while in state b the electron spin is $-1/2$ but the proton spin is $-1/2$. The energy difference between states a and b is denoted by $\delta\epsilon$.

a) Show that hydrogen atoms in both state a and state b must be treated as *bosons*.

b) A gas of atomic hydrogen can undergo Bose-Einstein condensation. Borrow general arguments from statistical mechanics and quantum mechanics to demonstrate that the critical temperature for Bose-Einstein condensation must increase with $\delta\epsilon$.

c) An ideal gas of N atomic hydrogen particles with mass m is in a volume V. Derive an expression for the critical temperature $T_c(\delta\epsilon)$. Express your results in terms of the function $F_{32} = \sum_{n=1} \frac{1}{n^{3/2}} e^{-nx}$. Demonstrate that your argument in b) is valid

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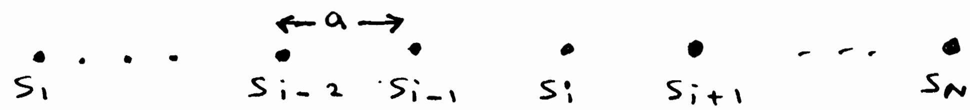
8. Thermo/Stat Mech

Consider a chain of N Ising spins ($S_i = \pm 1$) arranged in a one-dimensional lattice of spacing a. It is assumed that the only interactions are between nearest neighbors and of ferromagnetic sign, that is, the exchange constant $J < 0$. We assume periodic boundary conditions, that is, the spin $S_{N+1} = S_1$, and will eventually want the limit $N \to \infty$.



a) Show that the partition function

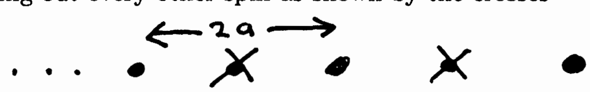
$$Z = \sum_{\{S_i = \pm 1\}} e^{-\frac{J}{k_B T} \sum_i S_i S_{i+1}}$$

can be alternatively written as

$$Z = \sum_{\{S_i = \pm 1\}} (\cosh K)^N \prod_i (1 + \tanh K S_i S_{i+1})$$

where $K = |J|/k_B T$ (k_B the Boltzmann constant and T the temperature).

b) By integrating out every other spin as shown by the crosses



Show that the same partition function can be exactly written as

$$Z = (2\cosh^2 K)^{N/2} \sum_{\{S'_i = \pm 1\}} \prod_i (1 + \tanh K' S'_i S'_{i+1})$$

where S'_i are the remaining spins separated by a lattice spacing $2a$. Show that the new couplings between the remaining spins S'_i are given by

$$\tanh K' = \tanh^2 K$$

c) By iterating the above procedure you can obtain exactly the partition function of the Ising model with $K \to K' \to K'' \to \ldots$. What can you conclude from the recursion relation at each step ($\tanh K' = \tanh^2 K$)?

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9. Thermo/Stat Mech

N indistinguishable, quasi-classical particles move in one dimension. Their coordinates q_i are confined to a one-dimensional box of length L. The Hamiltonian is:

$$H(\{p_i, q_i\}) = \sum_{i=1}^{N} c|p_i|$$

You will use the microcanonical ensemble with total energy E.

a) Compute the number of states $\Omega(E, L, N)$ with energy less than E. Compute the number of states $\Omega'(E, L, N, \Delta E)$ with energy between E and $E + \Delta E$.
Hint: The volume of the hyper-pyramid in D dimensions defined by $\sum_{i=1}^{D} x_i = R$ with $x_i \geq 0$ equals $R^D/D!$.

b) Compute the entropy $S(E, L, N)$. Show that in the large N limit the entropy is an *extensive* quantity.

c) Give a relation between the temperature T and the energy E. Show that this violates the Equipartition Theorem. Using the *canonical ensemble*, explain why.

d)Find the Equation of State $P(N, L, T)$ and compute the Heat Capacities at constant L and constant P.

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10. Electromagnetism

A circular conducting loop is moving in the $x - y$ plane with constant, non-relativistic velocity $\vec{v} = v\hat{e}_x$. A homogenous magnetic field $B_0\hat{e}_z$ is present in the area $x > 0$, indicated with dots in the figure below.

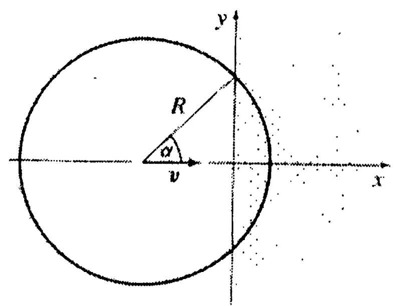

Figure 1:

a) Compute the induced voltage $V(t)$. Discuss the direction of the induced current in the conducting loop.

b) Draw a sketch of the time dependence of $V(t)$. What is the largest value of $|V(t)|$?

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11. Electromagnetism

A sphere of radius a made of linear magnetic material with permeability μ is placed in an otherwise uniform magnetic field $\vec{H}_0 = \vec{e}_z\ H_0$ in vacuum.

a) Find the new fields, $\vec{H}$, inside and outside the sphere.

b) Find the induced magnetic moment $\vec{m}$ and magnetization $\vec{M}$.

c) Find the bound currents inside the sphere, $\vec{J}_b$, and on its surface, $\vec{K}_b$.

d) What is the field $\vec{H}_{M_0}$ created by a uniformly magnetized permanent magnet of spherical shape? The radius of the sphere is a, and magnetization is $\vec{M}_0$.

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12. Electromagnetism

A plane wave refracts and follows a ray given by the equation $y = y_0 \sin(x/y_0)$ where y_0 is a constant. Find the refractive index $n(y)$ and plot it.

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13. Electromagnetism

A linearly polarized laser pulse acts on an electron, initially at rest. The pulse propagates along the z-axis. The electromagnetic wave is given by the vector potential

$$A_x(ct - z) = A_0(ct - z)\sin[k(ct - z)] \ , \quad A_y(ct - z) = 0 \ , \quad A_z(ct - z) = 0$$

The pulse amplitude, $A_0(ct - z)$, is a slowly varying function and k is the wavenumber.

a) Find the conserved quantities of the electron motion.

b) Determine the x and y components of the electron momentum

c) Show that the electron proper time, τ, and the "pulse coordinate" $\zeta \equiv ct - z$ are proportional to each other, $\tau = \zeta = ct - z$.

d) Evaluate the displacement of the electron in the z-direction due to its interaction with the laser pulse.

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14. Electromagnetism

Particles with negative charge $-|e|$ and mass m are emitted from a cathode, at potential zero, and accelerated across a gap of length a to the anode, which is held at positive potential V_0. Particles leave the surface of the cathode with zero velocity and form a steady space charge distribution in the gap which reduces the field at the cathode surface to zero. A steady current density j flows between the plates. Suppose that the plates of the electrodes are large relative to the separation between them so that edge effects can be neglected and the problem can be considered as one-dimensional.

a) Find the potential, $V(x)$, and space charge density, $\rho(x)$, assuming that cathode is at $x = 0$ and anode is at $x = a$.

b) Derive, the so-called Child-Langmuir law relating steady current density j and anode potential V_0.