Physics - Comprehensive Exam Name: ________________________________
Fall 2017

1. Quantum Mechanics

Consider a 1D harmonic oscillator with creation and annihilation operators $a^\dagger$ and $a$ with the usual commutation relation $[a, a^\dagger] = 1$. An eigenstate of the annihilation operator is called a "coherent state," and is denoted with its eigenvalue $\beta$, which is in general a complex number:

$$a|\beta\rangle = \beta|\beta\rangle.$$

(a) (14 points) Derive the expansion for a coherent state $|\beta\rangle$ in the basis of number (Fock) states $|n\rangle$ where $n$ is a non-negative integer. Choose a global phase such that $\langle n = 0|\beta\rangle$ is a positive, real number. Hint: Recall that $a|n\rangle = \sqrt{n}|n-1\rangle$ and $a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$.

(b) (6 points) Calculate the overlap integral between two coherent states, $\langle\beta|\alpha\rangle$. Are coherent states with $\alpha \neq \beta$ ever strictly orthogonal?

Name: ________________________________

2. Quantum Mechanics

Here we study a simple one-dimensional quantum-mechanical system that resembles the hydrogen molecular ion. A particle of mass $m$ is in a potential

$V(x) = -A_0\,[\,\delta(x-a) + \delta(x+a)\,]$, where $A_0 > 0$.

(a) (3 points) The eigenstates of the (non-relativistic) Hamiltonian for this potential will have definite parity. Without doing any calculation, sketch the amplitude of an even eigenstate as a function of $x$.

(b) (7 points) Find the expression that determines the bound-state energy for even-parity states, and determine graphically how many even-parity bound states exist.

(c) (10 points) Repeat parts (a) and (b) for odd parity. For what values of $A_0$ is there at least one such bound state?

Hints: If you take advantage of the symmetry of the problem to minimize the number of constants-to-be determined in your eigenfunctions (and you should), the constraints on the eigenfunctions at $x = a$ and $x = -a$ will be redundant. You can solve the problem without explicitly normalizing the eigenfunctions.

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3. Quantum Mechanics

Consider a spin-half electron in a $p$-state ( $\ell = 1$). The total angular momentum is **J**=**L** +**s**. Add explicitly the angular momenta and find the quartet and the doublet in the combined representation in terms of the original representation.

Hint: For any operators satisfying angular momentum algebra, the raising and the lowering operators satisfy

$$J_{\pm} |j, m\rangle = \sqrt{(j \mp m)(j \pm m + 1)}\, |j, m \pm 1\rangle \,,$$

where $m$ is the corresponding $z$-component, ($\hbar = 1$). Once you have worked out the quartet states you should be able to write down the doublet states by inspection.

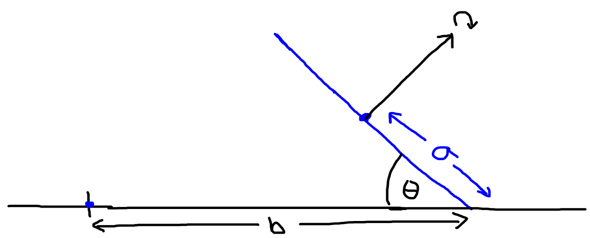
4. Quantum Mechanics

A one-dimensional harmonic oscillator of charge $e$ is perturbed by an electric field $E$ in the positive $x$-direction. Calculate the change in each energy level to second order in the perturbation, and calculate the induced electric dipole moment. Show that the problem can be solved exactly, and compare the result with perturbation approximation.

Hint: $a^\dagger |n\rangle = \sqrt{n+1}\,|n+1\rangle$ and $a|n\rangle = \sqrt{n}\,|n-1\rangle$ acting on harmonic oscillator states $|n\rangle$, where $a^\dagger$ and $a$ are the creation and the destruction operators. Note also the definitions: $a = \sqrt{\frac{m\omega}{2\hbar}}\left(x + \frac{ip}{m\omega}\right)$ and $a^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\left(x - \frac{ip}{m\omega}\right)$.

5. Classical Mechanics

A coin, idealized as a uniform disk of radius $a$ and negligible thickness, is rolling in a circle on a table. The point of contact describes a circle of radius $b$ on the table. The plane of the coin makes an angle $\theta$ with the plane of the table. Find the angular velocity $\omega$ of the motion of the center of mass of the coin. Hint: you don't need to use a Lagrangian for this problem, just Newton's laws.



6. Classical Mechanics

Consider a pendulum consisting of a point mass $m$ attached to a string of slowly increasing length $l(t)$. The motion is confined to a plane and we assume that $|l/\dot{l}|$ is much greater than the period of oscillation.

(a) (6 points) Find the Lagrangian $L(\theta, \dot{\theta}, t)$ and the Hamiltonian $H(\theta, p_\theta, t)$ of the system ($\theta$ is the angle of oscillation and $p_\theta$ is the conjugate momentum).

(b) (4 points) Is the Hamiltonian $H$ equal to the total energy $E$ of the pendulum? Are $E$ and $H$ conserved?

(c) (5 points) Derive the equation of motion for $\theta$ in the form of a second order ordinary differential equation. When $\dot{l} = 0$, what is the frequency of small oscillations?

(d) (5 points) Show that the amplitude of small oscillations is proportional to $l^{-3/4}$ as the length of the string $l(t)$ varies. (Hint: Consider using $\int d\theta p_\theta$ as an adiabatic invariant.)

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1. Electromagnetism

An infinite straight wire carrying a current I is suspended parallel to the plane interface between vacuum and a medium with magnetic permeability $\mu \neq 1$, at a distance $a$ from the interface. Calculate the force per unit length on the wire, and state whether it is attractive or repulsive.

**Hint:** for this problem, it is helpful to introduce a scalar potential. Also, it is helpful to take coordinates in the complex plane perpendicular to the wire.

2. Electromagnetism

The possibility that photons have some small, nonzero rest mass $m$ can be introduced consistently into Maxwell's equations to transform them into the Proca equations:

$$\nabla \cdot \boldsymbol{E} = \frac{\rho}{\epsilon_0} - \frac{m^2c^2}{\hbar^2}\Phi$$
$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}$$
$$\nabla \cdot \boldsymbol{B} = 0$$
$$\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{J} + \frac{1}{c^2}\frac{\partial \boldsymbol{E}}{\partial t} - \frac{m^2c^2}{\hbar^2}\boldsymbol{A},$$

which are valid for the Lorentz gauge, with the familiar relations given by

$$\nabla \cdot \boldsymbol{A} = -\frac{1}{c^2}\frac{\partial \Phi}{\partial t}$$
$$\boldsymbol{B} = \nabla \times \boldsymbol{A}$$
$$\boldsymbol{E} = -\nabla\Phi - \frac{\partial \boldsymbol{A}}{\partial t}$$
$$c \equiv \frac{1}{\sqrt{\epsilon_0\mu_0}}.$$

(a) (13 points) Derive the modified wave equation obeyed by $\boldsymbol{E}$ in free space (*i.e.* no charge or current density).

(b) (7 points) Consider transverse electromagnetic plane waves that propagate through free space along the $x$-axis with electric field magnitude

$$E(x,t) = \frac{1}{2}\mathcal{E}_0\left(e^{i(kx-\omega t)} + e^{-i(kx-\omega t)}\right)$$

Derive the expression for the group velocity as a function of optical frequency $v_g(\omega) \equiv \frac{d\omega}{dk}$ of electromagnetic waves in free space.

Potentially useful vector math: $\nabla \times (\nabla \times \boldsymbol{G}) = \nabla(\nabla \cdot \boldsymbol{G}) - \nabla^2\boldsymbol{G}$.

3. Electromagnetism

(a) (5 points) An element of wire of oriented length $d\boldsymbol{\ell}$ is moving with velocity $\boldsymbol{v}$ in a magnetic field $\boldsymbol{B}$. Starting from the Lorentz force law, calculate the motional EMF $d\varepsilon$ developed in the element.

(b) (15 points) A large sheet of copper moves with constant velocity $\boldsymbol{v}$ through the narrow gap of a C-shaped permanent magnet. The copper has thickness $h$ and conductivity $\sigma$. The magnet's field may be considered to have a constant value $B_0$ inside, and be negligible outside, the rectangular area $w \times \ell$ determined by the magnet's pole pieces (take the sheet's velocity to be parallel to the $w$ dimension). The motion induces an EMF in the conducting sheet, which drives a two-dimensional pattern of eddy currents in the sheet. The portion of this current pattern flowing within the region of magnetic field experiences a Lorentz force. Calculate, to within a constant of proportionality $\alpha$, the resulting electromagnetic force on the moving sheet (state explicitly the direction of this force).

The dimensionless constant $\alpha$ is of order unity and is meant to save you the trouble of calculating the exact path of the eddy currents in the sheet. Take $\alpha$ to be the constant of proportionality relating the total resistance of the current path (which is difficult to calculate) to the resistance of a piece of the current path that you can calculate easily. However you decide to define $\alpha$, be sure to state your definition clearly.

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4. Electromagnetism

Consider the backscattering of laser photons from a counter-propagating relativistic electron. The electron is taken to be traveling in the *z*-direction with speed *v* giving a Lorentz facton $\gamma = \left[1-(v/c)^2\right]^{-1/2}$. The laser photons propagate in the opposite (-*z*) direction and the scattered photons in the positive *z*-direction. The laser photons have a free-space wavelength of 800 nm ( $\hbar\omega = 1.55$ eV), and the electron total energy is 100 MeV.

(a) Assuming the *Thomson* approximation, the frequency $\omega'$ of the scattered radiation in the *electron rest frame* obeys $\hbar\omega' << m_e c^2$, and the backscattered radiation has nearly the same frequency, but opposite wavenumber $k'$ (reversed propagation direction) as the laser in this frame. Write expressions for $k'$ and $\omega'$, and evaluate the adequacy of the Thomson approximation.

(b) What is the energy of the scattered photons in the laboratory frame?

5. Statistical Mechanics

Consider a closed LC circuit. It is to be used as a thermometer, by measuring the rms voltage across the capacitance (and inductance). Find an expression for the temperature dependence of the rms voltage, valid for all temperatures. Then find the limits for high and low temperature.

Hint: at low enough temperatures quantum effects may be important.

6. Statistical Mechanics

(a) (4 points) Consider a polymer chain comprising $N$ segments. Each segment of length $d$ can freely rotate relative to each other. The system is in 3D and at temperature $T$ . What is the mean displacement $\langle \mathrm{r}_{1N} \rangle$ between the chain ends?

(b) (8 points) Apply a stretching force $f$ to both ends of the polymer. Find the partition function as a function of $f$ and then use it to derive the mean distance between the ends.

(c) (8 points) Now, the polymer in (a) is suspended at one end in a gravitational field of strength $g$. The mass of each segment of the chain is $m$. Calculate the average length of the chain. (Hint: you can replace a sum by an integral when $N$ is large.)

**QM 1 SOLUTION**

(a) (14 points) Derive the expansion for a coherent state $|\beta\rangle$ in the basis of number (Fock) states $|n\rangle$ where $n$ is a non-negative integer. Choose a global phase such that $\langle n=0|\beta\rangle$ is a positive, real number. Hint: Recall that $a|n\rangle = \sqrt{n}|n-1\rangle$ and $a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$.

We can begin by writing

$$|\beta\rangle = \sum_{n=0}^{\infty} c_n\,|n\rangle$$

and apply the annihilation operator:

$$a|\beta\rangle = \sum_{k=0}^{\infty} \beta\, c_k\,|k\rangle = \sum_{m=0}^{\infty} c_m\,\sqrt{m}|m-1\rangle.$$

Operating on this from the left with $\langle n|$ yields the recursion relation

$$c_{n+1} = \frac{\beta}{\sqrt{n+1}}c_n.$$

Now all we have to do is find $c_0 \equiv \langle n=0|\beta\rangle$, which we can approach by looking at normalization:

$$1 = \langle\beta|\beta\rangle = \sum_{n=0}^{\infty}|c_n|^2 = \sum_{n=0}^{\infty}\frac{(|\beta|^2)^n}{n!}|c_0|^2 = e^{|\beta|^2}|c_0|^2.$$

Using our phase convention that $c_0 > 0$ gives us $c_0 = e^{-|\beta|^2/2}$, and we can see that

$$|\alpha\rangle = e^{-|\beta|^2/2}\sum_{n=0}^{\infty}\frac{\beta^n}{\sqrt{n!}}|n\rangle.$$

(b) Calculate the overlap integral between two coherent states $\langle\beta|\alpha\rangle$. Are coherent states with $\alpha \neq \beta$ ever strictly orthogonal?

$$\langle\beta|\alpha\rangle = \exp\left[-\frac{1}{2}(|\beta|^2 + |\alpha|^2)\right]\sum_{k,m=0}^{\infty}\frac{\beta^{*k}}{\sqrt{k!}}\frac{\alpha^m}{\sqrt{m!}}\langle k|m\rangle$$

$$= \exp\left[-\frac{1}{2}(|\beta|^2 + |\alpha|^2)\right]\sum_{n}^{\infty}\frac{(\beta^*\alpha)^n}{n!}$$

$$= \exp\left[-\frac{1}{2}(|\beta|^2 + |\alpha|^2 - 2\beta^*\alpha)\right],$$

from which we see that coherent states are never strictly orthogonal.

The potential $V(x) = -V_0\left[\delta(x-a)+\delta(x+a)\right]$ is inversion-symmetric, and bound-state solutions are non-degenerate (although the delta function is infinite). Thus the bound states have definite parity, even or odd. The Schroedinger equation can be written:

$$\psi'' - k^2\psi + U_0\left[\delta(x-a)+\delta(x+a)\right]\psi = 0 \tag{1}$$

where $k^2 = -\dfrac{2mE}{\hbar^2} = \dfrac{2m|E|}{\hbar^2}$, and $U_0 = \dfrac{2mV_0}{\hbar^2}$. For $x > a$, $\psi'' - k^2\psi = 0$, and we must have:

$$\psi = Ae^{-kx} \qquad (x>a) \tag{2}$$

Thus for $x < -a$ we have:

$$\psi = \pm Ae^{kx}, \quad (x < -a, \ \pm \text{ for even, odd parity}) \tag{3}$$

For $|x| < a$, $\psi = B(e^{kx} \pm e^{-kx})$. To determine $B$ we must use the boundary conditions at $x = \pm a$. At $x = a$, $\psi$ is continuous, $\psi'$ is discontinuous, and $\psi''$ is infinite. Since $\psi$ is continuous, we have:

$$B\left(e^{ka} \pm e^{-ka}\right) = Ae^{-ka}$$

and therefore

$$\frac{A}{B} = e^{2ka} \pm 1 \tag{4}$$

To find a second constraint, we integrate (1) from $x = a-\varepsilon$ to $a+\varepsilon$ where $\varepsilon$ is very small:

$$\int_{a-\varepsilon}^{a+\varepsilon} \psi'' dx - k^2 \int_{a-\varepsilon}^{a+\varepsilon} \psi dx + U_0 \int_{a-\varepsilon}^{a+\varepsilon} \delta(x-a) dx = 0$$

This yields:

$$\psi'(a+\varepsilon) - \psi'(a-\varepsilon) - 2k^2\varepsilon\psi(a) + U_0\psi(a) = 0$$

In the limit as $\varepsilon \to 0$, we obtain:

$$\psi'(a+) = \psi'(a-) - U_0\psi(a)$$

Hence:

$$-kAe^{-ka} = kB(e^{ka} \mp e^{-ka}) - U_0 Ae^{-ka} \tag{5}$$

which, for even parity, yields:

$$\frac{2ka}{U_0 a} = \frac{e^{2ka}+1}{e^{2ka}} = \left(1 + e^{-2ka}\right) \qquad \text{Even parity} \tag{6}$$

This equation can be solved graphically for given $U_0$ and $a$, as we see in Fig. 1.

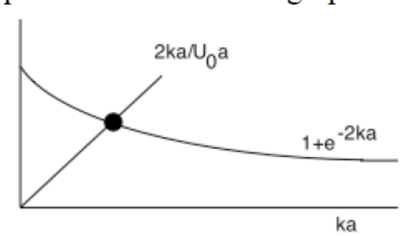



Fig.1

The figure shows that for any $U_0$ there is one and only one even-parity solution. For odd parity, (5) results in the transcendental equation:

$$\frac{2ka}{U_0 a} = 1 - e^{-2ka} \qquad \text{Odd parity} \qquad (7)$$

Here we have the graph:

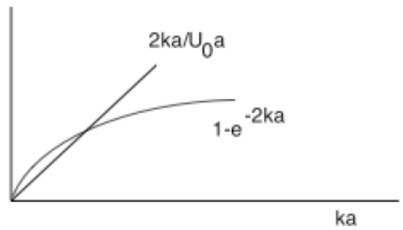
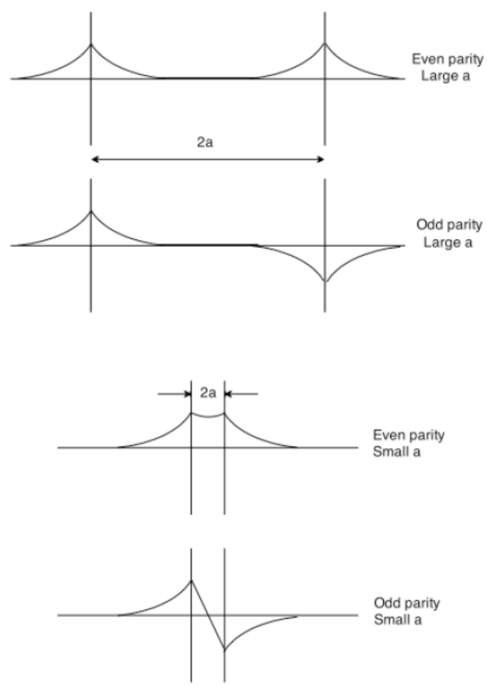



Fig.2

A solution exists if and only if the slope of $1 - e^{-2y}$ is greater than that of $2y / U_0 a$ at y=0. Thus we require that $U_0 a > 1$ for an odd-parity bound state. The following schematic figures reveal the general nature of the even and odd parity bound state wave functions as a function of the separation *2a* between the delta functions.




For sufficiently large a, the even and odd parity solutions are determined by the asymptotic form of equations (6, 7):

$$\frac{2k}{U_0} \to 1 \qquad (8)$$

Hence both energies become the same (we have degeneracy). Physically this happens because the probability of tunneling between the regions x ≈ -a and x ≈ a goes to zero.

**Problem 1**

There are six states in all. In the "old" basis these are three spin up states

$$|\psi_{1,1/2}\rangle\ ,\ |\psi_{0,1/2}\rangle\ ,\ |\psi_{-1,1/2}\rangle$$

and three spin down states

$$|\psi_{1,-1/2}\rangle\ ,\ |\psi_{0,-1/2}\rangle\ ,\ |\psi_{-1,-1/2}\rangle$$

The state with highest $m = m_1 + m_2$ has $m = 3/2$. The six states must divide into four $j = 3/2$ states and two $j = 1/2$ states. Here the top state must be

$$|\phi_{3/2,3/2}\rangle = |\psi_{1,1/2}\rangle$$

The other states with $j = 3/2$ can be found by applying $J_- = L_- + s_-$.

$$J_- |\phi_{3/2,3/2}\rangle = \sqrt{3}\,|\phi_{3/2,1/2}\rangle$$
$$(L_- + s_-)\,|\psi_{1,1/2}\rangle = \sqrt{2}\,|\psi_{0,1/2}\rangle + |\psi_{1,-1/2}\rangle$$

or

$$|\phi_{3/2,1/2}\rangle = \frac{1}{\sqrt{3}}\left[\sqrt{2}\,|\psi_{0,1/2}\rangle + |\psi_{1,-1/2}\rangle\right]$$

continuing

$$|\phi_{3/2,-1/2}\rangle = \frac{1}{\sqrt{3}}\left[|\psi_{-1,1/2}\rangle + \sqrt{2}\,|\psi_{0,-1/2}\rangle\right]$$

and finally

$$|\phi_{3/2,-3/2}\rangle = |\psi_{-1,-1/2}\rangle$$

The states come out automatically normalized.

The remaining two states must be orthogonal to those in the $j = 3/2$ ladder, since they are eigenstates of the Hermitian operator $\mathbf{J^2}$ with a different eigenvalue $j = 1/2$ Thus, simply from inspection

$$|\phi_{1/2,1/2}\rangle = \frac{1}{\sqrt{3}}\left[\sqrt{2}\,|\psi_{1,-1/2}\rangle - |\psi_{0,1/2}\rangle\right]$$

similarly

$$|\phi_{1/2,-1/2}\rangle = \frac{1}{\sqrt{3}}\left[|\psi_{0,-1/2}\rangle - \sqrt{2}\,|\psi_{-1,1/2}\rangle\right]$$

The Hamiltonian for this problem is

$$H = (n + 1/2)\hbar\omega - eEx.$$

We will first treat $eE$ as a perturbation. Recall that $x = \sqrt{\frac{\hbar}{2m\omega}}(a + a^\dagger)$. Since the energy levels of the unperturbed harmonic oscillator are non-degenerate, the first-order energy shifts are given by

$$E_n^{(1)} = \langle n|eEx|n\rangle = \sqrt{\frac{\hbar}{2m\omega}}eE\langle n|(a + a^\dagger)|n\rangle = 0.$$

The second-order shifts are

$$\begin{aligned}
E_n^{(2)} &= \frac{e^2E^2\hbar}{2m\omega}\sum_{m\neq n}\frac{|\langle m|(a + a^\dagger)|n\rangle|^2}{\hbar\omega(n-m)} \\
&= \frac{e^2E^2\hbar}{2m\omega}\sum_{m\neq n}\frac{|\sqrt{n}\delta_{m,n-1} + \sqrt{n+1}\delta_{m,n+1}|^2}{\hbar\omega(n-m)} \\
&= \frac{e^2E^2\hbar}{2m\omega}\sum_{m\neq n}\frac{n\delta_{m,n-1} + (n+1)\delta_{m,n+1}}{\hbar\omega(n-m)} \\
&= \frac{e^2E^2}{2m\omega^2}([n] - [n+1]) \\
&= -\frac{e^2E^2}{2m\omega^2};
\end{aligned}$$

independent of $n$. The transformed eigenkets are

$$\begin{aligned}
|n\rangle &= |n^{(0)}\rangle - eE\sqrt{\frac{\hbar}{2m\omega}}\sum_{m\neq n}|m^{(0)}\rangle\frac{\langle m^{(0)}|(a + a^\dagger)|n^{(0)}\rangle}{\hbar\omega(n-m)} + \mathcal{O}((eE)^2) \\
&= |n^{(0)}\rangle - \frac{eE}{\hbar\omega}\sqrt{\frac{\hbar}{2m\omega}}\sum_{m\neq n}|m^{(0)}\rangle\frac{\sqrt{n}\delta_{m,n-1} + \sqrt{n+1}\delta_{m,n+1}}{n-m} + \mathcal{O}((eE)^2) \\
&= |n^{(0)}\rangle - \frac{eE}{\hbar\omega}\sqrt{\frac{\hbar}{2m\omega}}(\sqrt{n}|(n-1)^0\rangle - \sqrt{n+1}|(n+1)^{(0)}\rangle) + \mathcal{O}((eE)^2).
\end{aligned}$$

Then the induced dipole moment is

$$e\langle n|x|n\rangle = e\sqrt{\frac{\hbar}{2m\omega}}\left[\langle n^{(0)}| - \frac{eE}{\hbar\omega}\sqrt{\frac{\hbar}{2m\omega}}(\sqrt{n}\langle (n-1)^0| - \sqrt{n+1}\langle (n+1)^{(0)}|)\right](a+a^\dagger)\left[|n^{(0)}\rangle - \frac{eE}{\hbar\omega}\sqrt{\frac{\hbar}{2m\omega}}(\sqrt{n}|(n-1)^0\rangle -\right.$$

$$= e\sqrt{\frac{\hbar}{2m\omega}}\left[-\frac{eE}{\hbar\omega}\sqrt{\frac{\hbar}{2m\omega}}(n-n-1) - \frac{eE}{\hbar\omega}\sqrt{\frac{\hbar}{2m\omega}}(n-n-1)\right]$$

$$= 2\frac{e^2E}{\hbar\omega}\frac{\hbar}{2m\omega}$$

$$= \frac{e^2E}{m\omega^2} + \mathcal{O}(E^2).$$

The problem can be solved exactly by completing the square. The Hamiltonian is

$$\begin{aligned} H &= \frac{p^2}{2m} + \frac{1}{2}m\omega^2x^2 - eEx \\ &= \frac{p^2}{2m} + \frac{1}{2}m\omega^2(x - \frac{eE}{m\omega^2})^2 - \frac{e^2E^2}{2m\omega^2}. \end{aligned}$$

If we define $y = x - eE/m\omega^2$, we have

$$H = \frac{p_y^2}{2m} + \frac{1}{2}m\omega^2y^2 - \frac{e^2E^2}{2m\omega^2}.$$

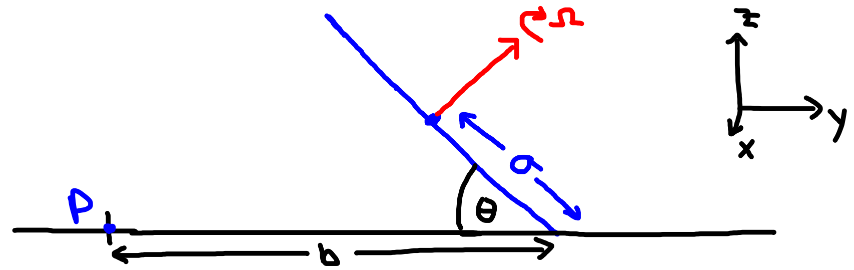
This is simply a harmonic oscillator of the same frequency again, with shifted ground-state energy, so that we have

$$E_n = (n+1/2)\hbar\omega - \frac{e^2E^2}{2m\omega^2}.$$

This shift is exactly what we saw to second-order, a shift independent of $n$. The induced dipole moment is calculated as

$$\langle n|ex|n\rangle = e\langle n|(y + \frac{eE}{m\omega^2})|n\rangle = \frac{e^2E}{m\omega^2},$$

exactly the result we found before. So for this problem, second-order perturbation theory yields the exact results.




Let $\omega$ be the angular velocity of the center of mass. If $\Omega$ denotes the angular velocity of the disk about its center of mass then by equating the distance travelled by the point of contact we have

$$a\Omega = b\omega$$

We solve the problem by $\vec{\tau} = \frac{d\vec{L}}{dt}$

We choose the origin to be at P.

Torque

$$\vec{\tau}_{tot} = \vec{\tau}_{grav} + \vec{\tau}_{table}$$

Since the gravitational force is applied symmetrically about the center of mass we have

$$\vec{\tau}_{grav} = Mg(b - a\cos\theta) \otimes$$

Also: $\vec{\tau}_{table} = Mgb \odot$

$$\Rightarrow \vec{\tau}_{tot} = Mga\cos\theta \odot$$

Angular momentum

$\vec{L}$ is clearly in the plane of the page, and is precessing with frequency $\omega$ about the z-axis. Hence

$$\frac{d\vec{L}}{dt} = L_y \omega \odot$$

Use: $\vec{L}_{tot} = \vec{L}_{cm} + \vec{L}_{rel}$

$$(L_{cm})_y = a\sin\theta\, p_\phi = a\sin\theta\, M\omega(b - a\cos\theta)$$

$$(L_{rel})_y = I\Omega\sin\theta = \frac{1}{2}Ma^2\Omega\sin\theta = \frac{1}{2}Mab\omega\sin\theta$$

$$\Rightarrow (L_{tot})_y = Ma\sin\theta\left(\frac{3}{2}b - a\cos\theta\right)\omega$$

So: $\vec{\tau} = \frac{d\vec{L}}{dt} \Rightarrow Mga\cos\theta = Ma\sin\theta\left(\frac{3}{2}b - a\cos\theta\right)\omega^2$

$$\Rightarrow \omega^2 = \frac{g}{\left(\frac{3}{2}b - a\cos\theta\right)\tan\theta}$$

CM 2 SOLUTION

Classical mechanics

(a)

$L = T - V = \frac{m}{2}(\dot{l}^2 + l^2\dot{\theta}^2) + mgl\cos\theta$.

Using $p_\theta = \frac{\partial L}{\partial \dot{\theta}} = ml^2\dot{\theta}$, we have $H = \dot{\theta}p_\theta - L = \frac{p^2}{2ml^2} - \frac{m\dot{l}^2}{2} - mgl\cos\theta$.

(b)

$E = T + V = \frac{m}{2}(\dot{l}^2 + l^2\dot{\theta}^2) - mgl\cos\theta = \frac{p^2}{2ml^2} + \frac{m}{2}\dot{l}^2 - mgl\cos\theta$, so $E \neq H$.

In general, neither $H$ nor $E$ is conserved.

(c)

Using the Lagrange's equation $\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = 0$,

we get $\frac{d}{dt}(ml^2\dot{\theta}) + mgl\sin\theta = 0$, and therefore, $\ddot{\theta} + \frac{2\dot{l}}{l}\dot{\theta} + \frac{g}{l}\sin\theta = 0$.

When $\dot{l} = 0$ and for small oscillations $\sin\theta \approx \theta$, the equation of motion becomes $\ddot{\theta} + \frac{g}{l}\theta = 0$. Hence, we have a solution $\theta(t) \propto \cos(\omega t + \phi)$ with a frequency $\omega = \sqrt{g/l}$.

(d)

Given the adiabatic assumption $\dot{l}/l \ll \omega$, $I = \int d\theta p_\theta$ is an adiabatic invariant. Using the trial solution $\theta(t) = A(t)\cos(\omega t + \phi)$, $I = \int d\theta(ml^2\dot{\theta}) = \int_0^{2\pi/\omega} dt(ml^2\dot{\theta}^2) \approx ml^2\int_0^{2\pi/\omega} dtA^2\omega^2\sin(\omega t + \phi)^2 = ml^2A^2\omega\pi \approx \pi ml^2A^2\sqrt{g/l}$.

Since $I \propto A(t)^2 l(t)^{3/2}$, we find the amplitude $A \propto l^{-3/4}$.

# Comp exam 2017 Solutions

## GZ EM:

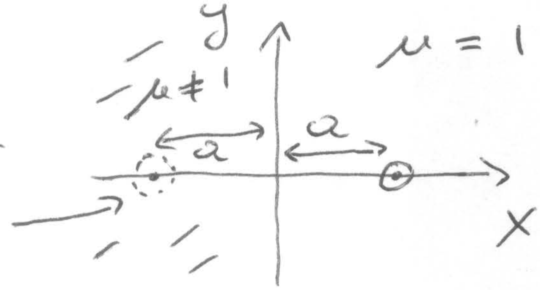



image wire

$x = 0$ is the interface

for $x > 0$, $\mu = 1$

for $x < 0$, $\mu \neq 1$

~~Firm~~ Everywhere outside the wire, $\vec{\nabla} \times \vec{H} = \vec{0}$

$\Rightarrow \quad \vec{H} = \vec{\nabla} \phi$ . First consider a wire along the z axis $(x = y = 0)$; cylindrical coord. $(r, \varphi, z)$ but also $\vec{r} = r e^{i\varphi} = x + iy$

$r = |\vec{r}|$ . The field from the wire is

$$B_\varphi = \frac{2I}{cr} \qquad \left( \vec{\nabla} \times \vec{B} = \frac{4\pi}{c} \vec{J} \Rightarrow B\, 2\pi r = \frac{4\pi}{c} I \right)$$

therefore $\phi \sim \ln(x + iy)$ and you see that you must take

$$\phi = \cancel{\frac{2I}{c}} \frac{2I}{c} \operatorname{Im}\{\ln(x + iy)\}$$

i.e. you must take the imaginary part, so that

$$\begin{cases} \dfrac{\partial \phi}{\partial x} \sim \operatorname{Im}\left\{\dfrac{1}{x + iy}\right\} = \operatorname{Im}\left\{\dfrac{x - iy}{x^2 + y^2}\right\} = \dfrac{-y}{r^2} \\[2ex] \dfrac{\partial \phi}{\partial y} \sim \operatorname{Im}\left\{\dfrac{i}{x + iy}\right\} = \operatorname{Im}\left(\dfrac{ix + y}{x^2 + y^2}\right) = \dfrac{x}{r^2} \end{cases}$$

i.e. this potential gives the correct field circulating around the wire.

Back to the original problem: you put an image wire at $X = -a$, carrying a current $I_2$.

The potential in the vacuum half space $(x > 0)$ is:

$$\phi = \frac{2}{c} \left\{ I \ln[(x-a)+iy] + I_2 \ln[(x+a)+iy] \right\} \qquad x > 0$$

(we agree to take Im )

The potential in the material half space $(x < 0)$ is

$$\phi = \frac{2}{c} I_3 \ln[(x-a)+iy] \qquad x < 0$$

notice $I_3 \neq I$.

The b.c. are: normal component of $\vec{B}$ cont., tangential component of $\vec{H}$ cont.:

$$\boxed{\vec{B} = \mu \vec{H}}$$

$\mu \frac{\partial \phi}{\partial x}$ cont. at $x = 0$:

$$\frac{2}{c} \left\{ \frac{I}{-a+iy} + \frac{I_2}{a+iy} \right\} = \frac{2}{c} \mu \frac{I_3}{-a+iy}$$

$$\Rightarrow \quad (-a-iy) I + (a-iy) I_2 = (-a-iy) I_3$$

taking imag. part: $\quad I_1 + I_2 = \mu I_3 \qquad (1)$

$\frac{\partial \phi}{\partial y}$ cont. at $x=0$ :

$$i \frac{I}{-a+iy} + i \frac{I_2}{a+iy} = i \frac{I_3}{-a+iy}$$

$$\Rightarrow \quad i(-a-iy) I + i(a-iy) I_2 = i(-a-iy) I_3$$

taking Im{ } : $\quad I - I_2 = I_3 \quad (2)$

From (1) and (2) : $\quad I_2 = \frac{\mu-1}{\mu+1} I \,, \quad I_3 = \frac{2}{\mu+1} I$

The force on a wire in a field $\vec{B}$ is

$$d\vec{F} = \frac{I}{c} (d\vec{\ell} \times \vec{B}) \quad \text{(Biot-Savart)} \quad (3)$$

The B field at the physical wire due to the image wire $I_2$ is $\quad B = \frac{2}{c} \frac{I_2}{2a} \quad$ so :

$$\frac{\text{force}}{\text{length}} = \frac{I}{c} \frac{2}{c} \frac{I_2}{2a} = \frac{I^2}{c^2} \frac{1}{a} \frac{\mu-1}{\mu+1} .$$

Using (3) you figure out that if $I_2 \parallel I$ (i.e. $\mu > 1$) the force is attractive, whereas if $I_2 \parallel (-I)$ (i.e. $\mu < 1$) it is repulsive. So :

diamagnetic ($\mu > 1$) $\rightarrow$ attractive

paramagnetic ($\mu < 1$) $\rightarrow$ repulsive.

**<u>E&M</u> 2 SOLUTION**

(a) Derive the modified wave equation obeyed by $\boldsymbol{E}$ in free space (i.e. no charge or current density).

We can essentially follow the classic procedure of taking the curl of the Maxwell-Faraday law,

$$\nabla \times (\nabla \times \boldsymbol{E}) = -\frac{\partial}{\partial t}(\nabla \times \boldsymbol{B}) = -\frac{\partial}{\partial t}\left(\frac{1}{c^2}\frac{\partial \boldsymbol{E}}{\partial t} - \left(\frac{mc}{\hbar}\right)^2 \boldsymbol{A}\right) = -\frac{1}{c^2}\frac{\partial^2 \boldsymbol{E}}{\partial t^2} + \left(\frac{mc}{\hbar}\right)^2 \frac{\partial \boldsymbol{A}}{\partial t}.$$

but $\frac{\partial \boldsymbol{A}}{\partial t} = -\boldsymbol{E} - \nabla\Phi$, so

$$\nabla \times (\nabla \times \boldsymbol{E}) = -\frac{1}{c^2}\frac{\partial^2 \boldsymbol{E}}{\partial t^2} - \left(\frac{mc}{\hbar}\right)^2 \boldsymbol{E} - \left(\frac{mc}{\hbar}\right)^2 \nabla\Phi.$$

Now we apply vector maths: $\nabla \times (\nabla \times \boldsymbol{E}) = \nabla(\nabla \cdot \boldsymbol{E}) - \nabla^2 \boldsymbol{E} = -\left(\frac{mc}{\hbar}\right)^2 \nabla\Phi - \nabla^2 \boldsymbol{E}$, which gets rid of that $\nabla\Phi$ term to leaves us with the Klein-Gordon equation (for the electric field):

$$\left(\nabla^2 - \left(\frac{mc}{\hbar}\right)^2\right)\boldsymbol{E} = \frac{1}{c^2}\frac{\partial^2 \boldsymbol{E}}{\partial t^2}.$$

(b) Consider transverse electromagnetic plane waves that propagate through free space along the $x$-axis with electric field magnitude

$$E(x,t) = \frac{1}{2}\mathcal{E}_0\left(e^{i(kx-\omega t)} + e^{-i(kx-\omega t)}\right).$$

Derive the expression for the group velocity as a function of optical frequency $v_g(\omega) \equiv \frac{d\omega}{dk}$ of electromagnetic waves in free space.

Using the form provided for $E(x,t)$, we see that

$$\frac{\partial^2 E}{\partial x^2} = -k^2 E$$

$$\frac{\partial^2 E}{\partial t^2} = -\omega^2 E,$$

so the wave equation reduces to the following dispersion relation

$$k^2 + \left(\frac{mc}{\hbar}\right)^2 = \frac{\omega^2}{c^2}.$$

Solving for frequency allows us to take the derivative

$$\omega = c\sqrt{k^2 + \left(\frac{mc}{\hbar}\right)^2}$$

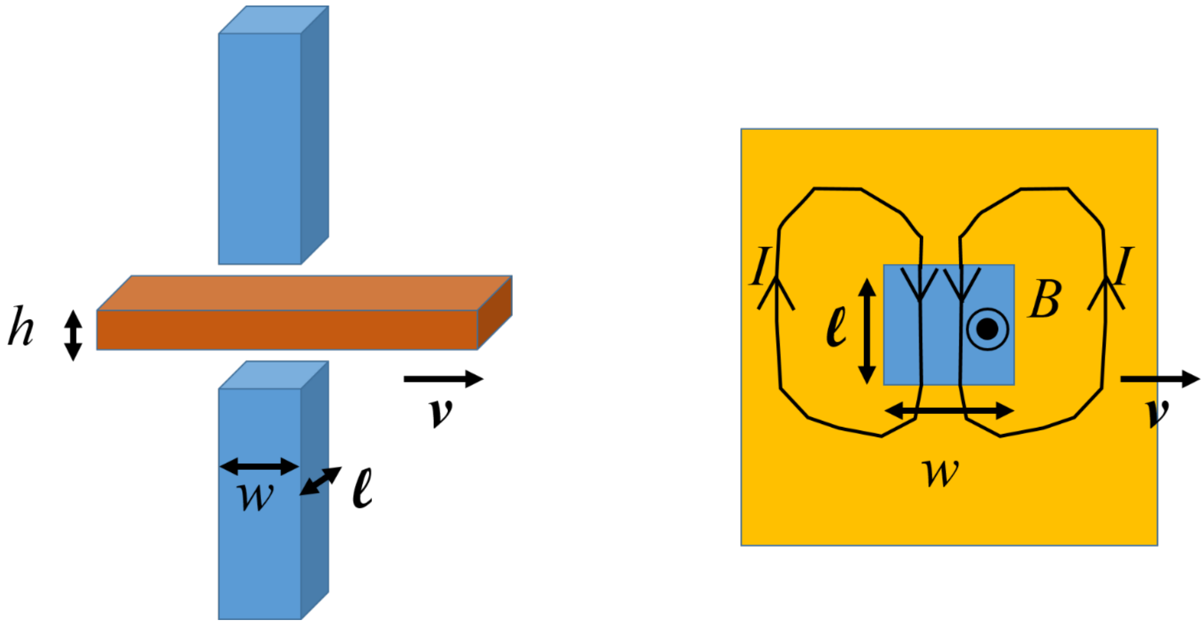
$$\frac{d\omega}{dk} = \frac{ck}{\sqrt{k^2 + \left(\frac{mc}{\hbar}\right)^2}} = \frac{ck}{\left(\frac{\omega}{c}\right)} = \frac{c^2}{\omega}k = \frac{c^2}{\omega}\sqrt{\frac{\omega^2}{c^2} - \left(\frac{mc}{\hbar}\right)^2},$$

which simplifies nicely to

$$v_g(\omega) = c\sqrt{1 - \left(\frac{mc^2}{\hbar\omega}\right)^2}.$$

a) The force on a free charge in the wire element is $\boldsymbol{F} = q(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}) = q\,\boldsymbol{v} \times \boldsymbol{B}$. The work to move the charge from one end of the element to the other is $W = \boldsymbol{F} \cdot d\boldsymbol{\ell}$, and the EMF is defined as the work per charge. Thus $d\varepsilon = \boldsymbol{v} \times \boldsymbol{B} \cdot d\boldsymbol{\ell}$.

b)




We integrate across the section of the plate in the magnetic field to find the EMF across this section, EMF = $\boldsymbol{v} \times \boldsymbol{B} \cdot \boldsymbol{\ell}$. The resistance of this section $R_{\text{section}} = \boldsymbol{\ell}/(w\,h\,\sigma)$. The total resistance for the current paths draw as shown will be of this order, $R_{\text{total}} = \alpha\ R_{\text{section}}$, where $\alpha$ is a constant greater than one but of order unity. Thus the total current $I$ = EMF/ $R_{\text{total}} = v\,B\,w\,h\,\sigma/\alpha$, with the sign as shown in the figure. Again from the Lorentz force law we have $d\boldsymbol{F} = dq\ \boldsymbol{v} \times \boldsymbol{B} = I\,d\boldsymbol{\ell} \times \boldsymbol{B}$. Integrating we find

$$\boldsymbol{F} = I\,\boldsymbol{\ell} \times \boldsymbol{B} = -\frac{B^2\,w\,h\,\ell\,\sigma}{\alpha}\boldsymbol{v},$$

where the minus sign in front of the (vector) velocity $\boldsymbol{v}$ indicates that this is a drag force, opposing the motion of the plate through the magnetic field. For $\boldsymbol{\ell} = w$ it is possible to show that $\alpha$=2.

**EM 4 Solution**

(a) We obtain the frequency in the rest frame by Lorentz transformation as follows:

$$\omega' = \gamma(\omega + \beta kc) \cong 2\gamma\omega, \text{ as } \beta \cong 1 - \frac{1}{2\gamma^2} \cong 1, \text{ and } \omega = kc.$$

For the parameters given $\gamma \sim 200$, and $\hbar\omega' \approx 2 \cdot 200 \cdot 1.55 \approx 500 \text{ eV}$, which is much less $m_e c^2$, and the Thomson approximation is valid.

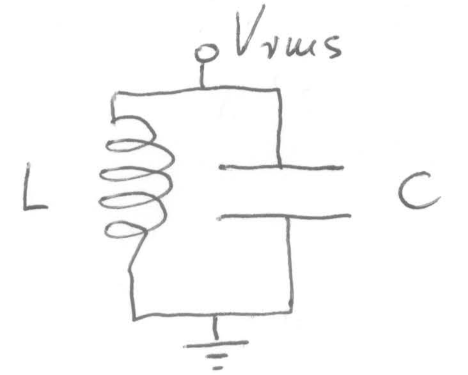
(b) The final scattered frequency is similarly obtained through

$$\omega_{sc} = \gamma\left(\omega' + \beta \frac{k'}{c}\right) \cong 2\gamma\omega' \cong 4\gamma^2\omega$$

and the scattered photon energy is approximate well by

$$\hbar\omega_{sc} \cong 4\gamma^2\hbar\omega \cong 250 \text{ keV}.$$

GZ SM:




The voltage contributions are $L\frac{dI}{dt}$ and $\frac{1}{C}Q$ where $I = \dot{Q}$, $Q$ is the charge on the capacitor. Therefore,

$$L\ddot{Q} + \frac{1}{C}Q = 0 \quad \text{or} \quad \ddot{Q} + \frac{1}{LC}Q = 0$$

a harmonic oscillator of freq. $\omega^2 = \frac{1}{LC}$

Therefore the energy levels are $E_n = (n+\frac{1}{2})\hbar\omega$

and the partition sum:

$$Z = \sum_{n=0}^{\infty} e^{-(n+\frac{1}{2})\hbar\omega/T} = e^{-\frac{\hbar\omega}{2T}} \frac{1}{1-e^{-\hbar\omega/T}}$$

If you want, you can write the Hamiltonian $H = \frac{1}{2}L\dot{Q}^2 + \frac{1}{2C}Q^2$ which is evidently a harmonic oscillator.

The energy is $\langle H \rangle = \langle \frac{1}{2}L\dot{Q}^2 \rangle + \langle \frac{1}{2C}Q^2 \rangle$

$$= \frac{1}{2}L\omega^2\langle Q^2 \rangle + \frac{1}{2C}\langle Q^2 \rangle = \frac{1}{C}\langle Q^2 \rangle$$

and $Q = CV \Rightarrow \langle H \rangle = C\langle V^2 \rangle$

From the partition sum, the energy is

$$\langle H \rangle = T^2 \frac{\partial \ln Z}{\partial T} = \frac{1}{2}\hbar\omega \frac{e^{\frac{\hbar\omega}{2T}} + e^{-\frac{\hbar\omega}{2T}}}{e^{\frac{\hbar\omega}{2T}} - e^{-\frac{\hbar\omega}{2T}}}$$

therefore $$\langle V^2 \rangle = \frac{\hbar\omega}{2C} \frac{1}{\tanh\left(\frac{\hbar\omega}{2T}\right)} , \quad \omega = \frac{1}{\sqrt{LC}}$$

and $V_{rms} = \langle V^2 \rangle^{1/2}$ (since $\langle V \rangle = 0$).

High $T$ : $\frac{\hbar\omega}{T} \ll 1$ then $\tanh\left(\frac{\hbar\omega}{2T}\right) \simeq \frac{\hbar\omega}{2T}$

and $\langle V^2 \rangle^{1/2} = \sqrt{\frac{T}{C}}$

Low $T$ : $\frac{\hbar\omega}{T} \gg 1$ then $\tanh(\,) \to 1$

and $\langle V^2 \rangle^{1/2} \simeq \sqrt{\frac{\hbar\omega}{2C}}$ indep. of $T$.

SM 2 SOLUTION

(a)

The entire polymer can be treated as $N$ independent segments with free rotation.

When there is no force or bias, we simply have the mean displacement $\langle \mathrm{r}_{1N} \rangle = 0$.

(b)

Define the angle of rotation relative to the force as $\theta$, then the energy of each segment is $-fd\cos\theta$.

The total partition function is $Z = \left[2\pi \int_{-1}^{1} d(\cos\theta) e^{\beta f d \cos\theta}\right]^N = \left[\frac{4\pi}{\beta f d} \sinh \beta f d\right]^N$.

The free energy is $F = -\frac{N}{\beta} \ln[\frac{4\pi}{\beta f d} \sinh \beta f d]$.

Thus, $\langle r_{1N} \rangle = -\frac{\partial F}{\partial f} = N(-\frac{1}{\beta f} + d \coth \beta f d)$.

(c)

The situation is similar to (b), except that the i-th segment (counted from the suspending end) experiences a force $f_i = mg(N-i)$.

Thus, the partition function becomes $Z = \Pi_{i=1}^{N} Z_i$, where $Z_i = \frac{4\pi}{\beta f_i d} \sinh \beta f_i d$.

The free energy is $F = \sum_i F_i$, where $F_i = -\frac{1}{\beta} \ln Z_i$.

Defining $a = mg\beta d$ and converting the sum to an integral,

we have $\langle r_{1N} \rangle = -\sum_i \frac{\partial F_i}{\partial f_i} = d \int_0^N dx \frac{\partial}{\partial(ax)} \ln \frac{\sinh ax}{ax} = \frac{d}{a} \ln \frac{\sinh aN}{aN} = \frac{k_B T}{mg} \ln[\frac{\sinh(Nmgd/k_B T)}{Nmgd/k_B T}]$.