STATISTICAL PHYSICS 215A

Final Exam – Spring 2013

Monday 10 June 2013 from 3pm to 6pm in room PAB-2-434

- Please write clearly;
- Present your arguments and calculations clearly;
- All four questions below are independent from one another.

- Print your name on every page used, including this one;
- Make clear which question you are answering on each page;
- No books, notes, computers, or calculators are allowed during the exam;
- Please turn off cell-phones, iPhones, iPods, iPads, Kindles, and other electronic devices.

Grades

Q1.

Q2.

Q3.

Q4.

Total /65

Some Useful formulas

- The Γ function obeys,

$$\Gamma(\nu+1) = \nu\Gamma(\nu) \qquad\qquad \Gamma(1/2) = \sqrt{\pi} \tag{0.1}$$

as well as Sterling's formula in the limit of large N,

$$\ln \Gamma(N+1) = N \ln N - N + \mathcal{O}(\ln N) \tag{0.2}$$

with $\Gamma(N+1) = N!$ for integer N.

- The following integral may come in useful for question 3,

$$\int_{-\infty}^{\infty} \frac{x^2 dx}{(e^x + e^{-x})^2} = \frac{\pi^2}{24} \tag{0.3}$$

- The Bose-Einstein functions in question 4 are defined by,

$$g_\nu(z) = \frac{1}{\Gamma(\nu)} \int_0^\infty \frac{x^{\nu-1} dx}{z^{-1} e^x - 1} \tag{0.4}$$

The integral is absolutely convergent for $|z| < 1$ and $\mathrm{Re}(\nu) > 0$, and admits the series expansion,

$$g_\nu(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^\nu} \tag{0.5}$$

and satisfies the differential recursion relation,

$$z g'_{\nu+1}(z) = g_\nu(z) \tag{0.6}$$

as well as the following asymptotic expansion near $z = e^{-\alpha}$ for $0 < \alpha \ll 1$,

$$g_{3/2}(e^{-\alpha}) = \zeta(3/2) - 2\sqrt{\pi}\alpha^{\frac{1}{2}} + \mathcal{O}(\alpha) \tag{0.7}$$

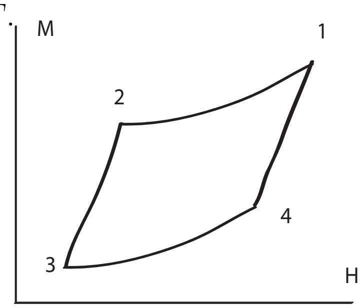
Finally, we have $g_\nu(1) = \zeta(\nu)$, namely the Riemann ζ-function.

QUESTION 1 [16 points]

For paramagnetic materials, the first law of thermodynamics states that $dE = \delta Q + HdM$, where H is the external magnetic field, and M is the magnetization.
(a) Write down the expressions for the specific heat C_M at constant M, and the specific heat C_H at constant H in terms of the internal energy E, temperature T, as well as H and M.
(b) Show that one has,

$$C_H = C_M - \left(\frac{\partial M}{\partial T}\right)_H \left[H - \left(\frac{\partial E}{\partial M}\right)_T\right]$$

(c) Consider now a paramagnetic material obeying Curie's law $M = nDH/T$, for some constant D, and $n = N/V$. This material is magnetized adiabatically from $M = 0$ to a non-zero value of M. Calculate the ratio of the temperatures $T(M)/T(M=0)$ as a function of M, under the assumption that C_M is constant and that $\partial E/\partial M = 0$ at constant T.
(d) This material is carried around a Carnot cycle with $T_h > T_\ell$.



$1 \to 2$ M is reduced (demagnetized) isothermally at $T = T_h$;
$2 \to 3$ M is reduced adiabatically;
$3 \to 4$ M is increased (magnetized) isothermally at $T = T_\ell$;
$4 \to 1$ M is increased adiabatically.

Express M_3, M_4 in terms of T_h, T_ℓ, D, C_M, n, and M_1, M_2.
Show that the efficiency of the cycle is given by $\eta = 1 - T_\ell/T_h$.

QUESTION 2 [16 points]

N *indistinguishable* quasi-classical particles move in one dimension of space which is a box of length L. The Hamiltonian is that of massless relativistic particles, and is given by,

$$H(\{p_i, q_i\}) = \sum_{i=1}^{N} c|p_i| \tag{0.8}$$

Use the micro-canonical ensemble, and fix the total energy to be E.
(a) Compute the number of states $\Omega(E, L, N)$ with energy less than E. Compute the number of states $\Omega'(E, L, N, \Delta)$ with energy between E and $E + \Delta$.
[Hint: Evaluate the non-trivial multiple integral recursively in N, using its scaling properties.]
(b) Compute the entropy $S(E, L, N)$. Show that in the thermodynamic limit, the entropy is an extensive quantity.
(c) Derive the relation between energy E and temperature T. Confirm your result using the canonical ensemble.
(d) Find the equation of state $P(N, L, T)$ and compute the specific heat C_L at constant L.

QUESTION 3 [16 points]

(a) Derive a formula for the specific heat C_V of a system of electrons at arbitrary temperature T and chemical potential μ in terms of the FD occupation number

$$f(\varepsilon) = \frac{1}{e^{\beta(\varepsilon-\mu)} + 1} \qquad\qquad \beta = \frac{1}{kT}$$

and the density of states $D(\varepsilon)$ for one electron per unit volume at energy ε.
(b) Derive a formula for C_V to leading non-zero order for strong degeneracy for general $D(\varepsilon)$.
(c) Derive a formula for C_V to leading non-zero order for weak degeneracy for general $D(\varepsilon)$.
(d) Apply these formulas to a system of free non-relativistic electrons. Do you recover the standard high T result for C_V ? Explain.

QUESTION 4 [17 points]

We consider an ideal non-relativistic Bose-Einstein gas whose constituents have mass m and no internal degrees of freedom. Total paticle number N, condensate particle number N_0, T, E, V, μ and fugacity $z = e^{\beta\mu}$, are related as follows,

$$N - N_0 = \frac{V}{\lambda^3}\, g_{3/2}(z) \qquad\qquad E = \frac{3kTV}{2\lambda^3}\, g_{5/2}(z) \tag{0.9}$$

As usual, we set $\lambda^2 = 2\pi\hbar^2/(mkT)$.

We now consider instead the BE gas confined to a vertical cylinder of height L in the presence of a uniform gravitational acceleration g.
(a) Calculate the critical temperature T_c at which BE condensation sets in under the assumption that the gravitational effects are weak, namely $mgL \ll kT_c$. Express your answer in terms m, g, L and the critical temperature T_c^0 for BE condensation in the absence of gravity.
(b) Show that the effect of gravity produces a discontinuity in the specific heat C_V at the BE transition T_c, and that the value of this discontinuity is given by,

$$\Delta C_V\Big|_{T_c} = -\frac{9}{8\sqrt{\pi}}\zeta\left(\frac{3}{2}\right) Nk \left(\frac{mgL}{kT_c^0}\right)^{\frac{1}{2}} \tag{0.10}$$