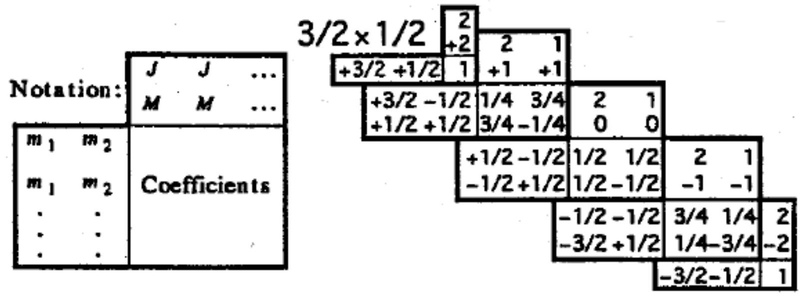
1. *Quantum Mechanics* (Spring 2004)

The table below shows some Clebsch-Gordan coefficients. If two particles have spin 1/2 and 3/2 respectively, write down all composite states $|sm\rangle$ in terms of the uncoupled states using Dirac notation. You may use the following table if you wish. (A square root is understood for all entries in the table below, with the $\pm$ sign outside the radical.)



2. *Quantum Mechanics* (Spring 2004)

A hydrogen atom is in the ground state ($n = 1, l = m = 0$) for $t < 0$. Suppose the atom is placed between the plates of a capacitor, and a weak, spatially uniform but time-dependent decaying field is applied at $t = 0$. The field (for $t > 0$) is

$$\mathbf{E} = \mathbf{E}_o e^{-\gamma t}$$

for some $\gamma > 0$. Take $\mathbf{E}_o$ along the z-axis. What is the probability (to first order in E_o) that the atom will be in each of the four $n = 2$ states as $t \to \infty$? Neglect spin.

You may need some of the functions $R_{nl}(r)$ and $Y_l^m(\theta, \phi)$ in the following table:

$a^{\frac{3}{2}} R_{10}(r) = 2e^{-r/a}$	$a^{\frac{3}{2}} R_{20}(r) = \frac{1}{\sqrt{2}} \left(1 - \frac{r}{2a}\right) e^{-r/2a}$	$a^{\frac{3}{2}} R_{21}(r) = \frac{1}{2\sqrt{6}} \frac{r}{a} e^{-r/2a}$
$Y_0^0(\theta, \phi) = \frac{1}{\sqrt{4\pi}}$	$Y_1^0(\theta, \phi) = \sqrt{\frac{3}{4\pi}} \cos\theta$	$Y_1^{\pm 1}(\theta, \phi) = \mp\sqrt{\frac{3}{8\pi}} \sin\theta \, e^{\pm i\phi}$

Table 1: Some hydrogen atom radial wave functions and spherical harmonics. a is the Bohr radius: $a = \hbar/mc\alpha$.

And an integral

$$\int_0^\infty x^n e^{-x/a} \, dx = a^{n+1} n!$$

3. *Quantum Mechanics* (Spring 2004)

The normalized wave function of a one-dimensional particle is

$$\psi(x) = Ne^{-\kappa x^2/2}$$

for some $\kappa > 0$. N is real and positive.

(a) What is N?

(b) What is the expectation value of x^2?

(c) What is the momentum space wave function $\langle p|\psi\rangle$?

(d) What is the expectation value of p^2?

(e) The Hamiltonian is

$$H = \frac{p^2}{2m} + V(x)$$

What is the potential $V(x)$?

4. *Quantum Mechanics* (Spring 2004)

The electron neutrino $|\nu_e\rangle$ and the muon neutrino $|\nu_\mu\rangle$ are the possible neutrino states produced and detected in experiments, but they are not necessarily eigenstates of the Hamiltonian. Rather, if the state is known to have momentum p, it is some linear combination of the energy eigenstates $|\nu_1\rangle$ and $|\nu_2\rangle$ of the form

$$|\nu_e\rangle = \cos(\theta)\,|\nu_1\rangle + \sin(\theta)\,|\nu_2\rangle$$

$$|\nu_\mu\rangle = -\sin(\theta)\,|\nu_1\rangle + \cos(\theta)\,|\nu_2\rangle$$

where

$$H\,|\nu_1\rangle = \sqrt{p^2c^2 + {m_1}^2c^4}\,|\nu_1\rangle$$

$$H\,|\nu_2\rangle = \sqrt{p^2c^2 + {m_2}^2c^4}\,|\nu_2\rangle$$

for two possibly different masses m_1 and m_2, and some "mixing angle" θ. If it is known that a neutrino was definitely a ν_μ when it was produced, what is the probability of detecting a ν_e after it has traveled a distance L? Assume that $m_1c \ll p$ and $m_2c \ll p$, so that the neutrinos are moving at almost (or even exactly) the speed of light, (so you can ignore corrections of the order $1 - v/c$ compared to terms of order 1) and state your result to first order in the difference $\Delta m^2 = {m_1}^2 - {m_2}^2$.

This is a simplified version of an actual neutrino oscillation experiment like the super-Kamiokande detector experiment a few years ago. In reality there is a third neutrino $|\nu_\tau\rangle$.

5. *Quantum Mechanics* (Spring 2004)

Calculate the transmission coefficient for a particle of energy $E > 0$ scattering off the 1D potential well $V(x) = V_0$ for $0 < x < a$, $V(x) = 0$ elsewhere, $V_0 < 0$. Are there resonance phenomena?

6. *Statistical Mechanics and Thermodynamics* (Spring 2004)

Consider a gas of relativistic, conserved bosons. The relation between energy and momentum is

$$E = |\mathbf{p}|\, c$$

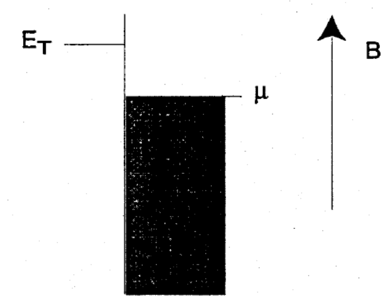
(a) Derive the condition for Bose-Einstein condensation in three dimensions.

(b) Does Bose-Einstein condensation occur in two dimensions? Justify your answer.

(c) What is the highest dimension for which Bose-Einstein condensation does not occur?

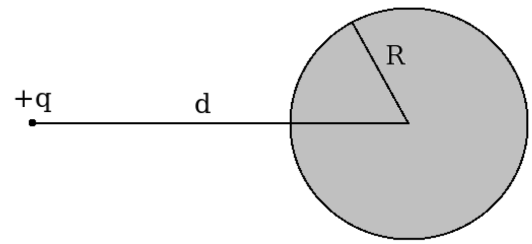
7. *Statistical Mechanics and Thermodynamics* (Spring 2004)

A quantum state at energy E_T is embedded in a system with a degenerate Fermi gas as, for instance, occurs with an impurity state with energy E_T in a degenerate semiconductor with a sea of conducting electrons at chemical potential μ. You may assume that $E_T > \mu$. The impurity, which has a spin of 1/2, can take an additional electron from the large bath of electrons (costs Coulomb energy U), to form a spin-singlet state. For a given temperature T and magnetic field B, calculate the ratio of the probability for the trap being empty to that for the trap being filled by an additional electron.



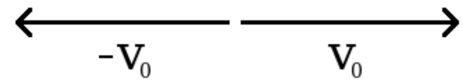
8. *Electricity and Magnetism* (Spring 2004)

A point charge q is located a distance d from the center of a conducting sphere of radius R. What must the total charge on the conducting sphere be for the force on the point charge to be zero?



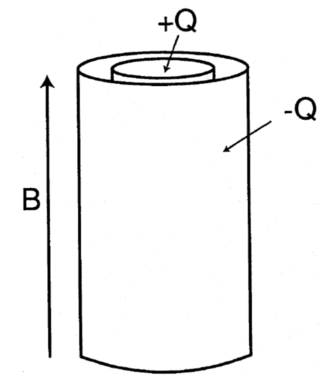
9. *Electricity and Magnetism* (Spring 2004)

Consider the infinite two-dimensional conducting plane depicted in the figure. The right half is maintained at electrostatic potential V_0 while the left half is maintained at potential $-V_0$. What is the potential above the plane?



10. *Electricity and Magnetism* (Spring 2004)

Consider a cylindrical capacitor of length L with charge $+Q$ on the inner cylinder of radius a and $-Q$ on the outer cylindrical shell of radius b. The capacitor is filled with a lossless dielectric with dielectric constant equal to 1. The capacitor is located in a region with a uniform magnetic field B, which points along the symmetry axis of the cylindrical capacitor. A flaw develops in the dielectric insulator, and a current flow develops between the two plates of the capacitor. Because of the magnetic field, this current flow results in a torque on the capacitor, which begins to rotate. After the capacitor is fully discharged (total charge on both plates is zero), what is the magnitude and direction of the angular velocity of the capacitor? The moment of inertia of the capacitor (about the axis of symmetry) is I, and you may ignore fringing fields in the calculation.



11. *Electricity and Magnetism* (Spring 2004)

Consider a plasma of free charges of mass m and charge e at constant density n. What is the index of refraction for electromagnetic waves of frequency ω which are incident upon this plasma?

12. *Electricity and Magnetism* (Spring 2004)

The fields due to a charge in motion are:

$$\mathbf{E}(\mathbf{x},t) = e\left[\frac{\mathbf{n}-\boldsymbol{\beta}}{\gamma^2(1-\boldsymbol{\beta}\cdot\mathbf{n})^3R^2}\right]_{\text{ret}} + \frac{e}{c}\left[\frac{\mathbf{n}\times\left[(\mathbf{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}\right]}{(1-\boldsymbol{\beta}\cdot\mathbf{n})^3R}\right]_{\text{ret}}$$

$$\mathbf{B}(\mathbf{x},t) = [\mathbf{n}\times\mathbf{E}]_{\text{ret}} \tag{1}$$

where $\boldsymbol{\beta} = \mathbf{v}/c$, $\mathbf{n}$ is a unit vector in the direction of the observation point $\mathbf{x}$, $\gamma = 1/\sqrt{1-\beta^2}$ and 'ret' means the quantities are evaluated at the retarded time (so e.g. $\mathbf{n}$ in (1) is the unit vector pointing from the retarded position of the charge to the observation point).

(a) Identify in the expression (1) 'static fields' and 'radiation fields'. Show how the static field part can be obtained from a Lorentz transformation of the fields of a static charge.

Hint: You may want to refer to Figure 1, where K' is the rest frame of the particle and P the observation point (which the particle passes at impact parameter b); suppose K and K' coincide at $t = t' = 0$. Write the fields in K', transform to the K coordinates, then transform the fields to K. Now you have the fields of the moving charge in terms of its present position. Show that the parallel and transverse components of E are the same as given in (1) in terms of the retarded position. Figure 2 may be useful, where R is the retarded distance and r the present distance. You have to express $R^2(1-\mathbf{b}\cdot\mathbf{n})^2$ in terms of r and b etc.

(b) Using the radiation field part of (1) in the non-relativistic limit ($\beta \ll 1$), calculate the average power radiated per unit solid angle by a charge q oscillating along the z-axis: $z(t) = A\cos(\omega t)$, where z is the position of the charge. The power is a function of the azimuthal angle θ, and 'average' means average in time (i.e. average over 1 oscillation).

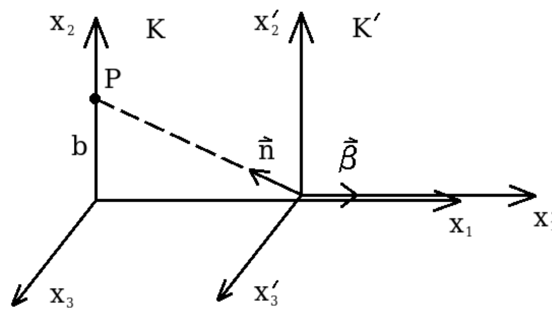


Figure 1: Rest frame K' versus observation frame K

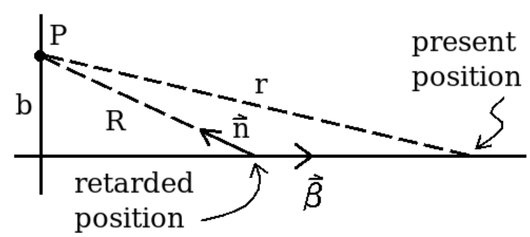


Figure 2: Retarded position versus present position

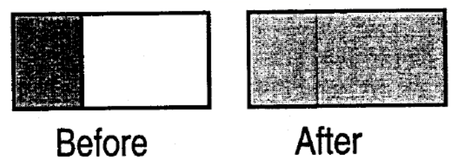
13. *Statistical Mechanics and Thermodynamics* (Spring 2004)

A van der Waals gas has the following equation of state:

$$P(T,V) = \frac{NkT}{(V - bN)} - a\left(\frac{N}{V}\right)^2$$

This gas is held in a container of negligible mass which is isolated from its surroundings. The gas is initially confined to 1/3 of the total volume of the container by a partition (a vacuum exists in the other 2/3 of the volume). The gas is initially in thermal equilibrium with temperature T_i. A hole is then opened in the partition, allowing the gas to irreversibly expand to fill the entire volume (V). What is the new temperature of the gas after thermal equilibrium has been re-established?

Hint: Note that the specific heat at constant volume for a van der Waals gas is the same as that for an ideal gas.



14. *Statistical Mechanics and Thermodynamics* (Spring 2004)

Imagine that the sites of a lattice are occupied with probability p and are unoccupied with probability $1-p$. If two neighboring sites are occupied, then we consider them to be part of the same cluster. As p is increased, larger clusters become more likely. When $p > p_c$ for some p_c (the 'percolation threshold') which depends on the dimension and the particular lattice, there will be a cluster which extends all the way across the system. For $p < p_c$, we will call the mean cluster size S.

(a) What is the percolation threshold, p_c, of a one-dimensional chain?

(b) In an infinite one-dimensional chain, what is the probability n_s that a given site is the left end of a cluster of length precisely s (in terms of p and s)?

(c) $n_s s$ is the probability that a given site is on a cluster (anywhere, not just the left end) of length s. p is the probability that a given site is on a cluster of any non-zero size. What is the mean cluster size, S, in terms of $n_s s$ ($s = 1, 2, ...$) and p?

(d) Using your results from parts (b) and (c), what is the mean cluster size, S, of a one-dimensional chain as a function of p alone?

1. *Quantum Mechanics* (Spring 2004)

The table below shows some Clebsch-Gordan coefficients. If two particles have spin 1/2 and 3/2 respectively, write down all composite states $|sm\rangle$ in terms of the uncoupled states using Dirac notation. You may use the following table if you wish. (A square root is understood for all entries in the table below, with the $\pm$ sign outside the radical.)

Notation:

		J	J	...
		M	M	...
m_1	m_2	Coefficients		
m_1	m_2			
⋮	⋮			

$3/2 \times 1/2$

		2
		+2
+3/2	+1/2	1

		2	1
		+1	+1
+3/2	−1/2	1/4	3/4
+1/2	+1/2	3/4	−1/4

		2	1
		0	0
+1/2	−1/2	1/2	1/2
−1/2	+1/2	1/2	−1/2

		2	1
		−1	−1
−1/2	−1/2	3/4	1/4
−3/2	+1/2	1/4	−3/4

		2
		−2
−3/2	−1/2	1

$S_1 = \frac{1}{2}$ $\quad S_2 = \frac{3}{2}$ $\quad \left|\frac{1}{2} - \frac{3}{2}\right| \leq S \leq \frac{1}{2} + \frac{3}{2} \Rightarrow 1 \leq S \leq 2$ and $|m| \leq S$

Uncoupled States $|S_1 S_2\, m_1 m_2\rangle$

$|\frac{1}{2}\ \frac{3}{2}\rangle$ $\quad |-\frac{1}{2}\ \frac{3}{2}\rangle$

$|\frac{1}{2}\ \frac{1}{2}\rangle$ $\quad |-\frac{1}{2}\ \frac{1}{2}\rangle$

$|\frac{1}{2}\ -\frac{1}{2}\rangle$ $\quad |-\frac{1}{2}\ -\frac{1}{2}\rangle$

$|\frac{1}{2}\ -\frac{3}{2}\rangle$ $\quad |-\frac{1}{2}\ -\frac{3}{2}\rangle$

Composite States $|S_1 S_2\, m S\rangle$

$|2\ 2\rangle$

$|1\ 1\rangle$ $\quad |2\ 1\rangle$

$|1\ 0\rangle$ $\quad |2\ 0\rangle$

$|1\ {-1}\rangle$ $\quad |2\ {-1}\rangle$

$|2\ {-2}\rangle$

The table is really a chain of tables stacked corner-to-corner.

$$|2\ 2\rangle = |\tfrac{3}{2}\ \tfrac{1}{2}\rangle$$

$$|2\ 1\rangle = \sqrt{\tfrac{1}{4}}\,|\tfrac{3}{2}\ -\tfrac{1}{2}\rangle + \sqrt{\tfrac{3}{4}}\,|\tfrac{1}{2}\ \tfrac{1}{2}\rangle$$

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$$|2\ {-2}\rangle = |-\tfrac{3}{2}\ -\tfrac{1}{2}\rangle$$

$$|1\ 1\rangle = \sqrt{\tfrac{3}{4}}\,|\tfrac{3}{2}\ -\tfrac{1}{2}\rangle - \sqrt{\tfrac{1}{4}}\,|\tfrac{1}{2}\ \tfrac{1}{2}\rangle$$

$$|1\ 0\rangle = \sqrt{\tfrac{1}{2}}\,|\tfrac{1}{2}\ -\tfrac{1}{2}\rangle - \sqrt{\tfrac{1}{2}}\,|-\tfrac{1}{2}\ \tfrac{1}{2}\rangle$$

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A hydrogen atom is in the ground state ($n = 1, l = m = 0$) for $t < 0$. Suppose the atom is placed between the plates of a capacitor, and a weak, spatially uniform but time-dependent decaying field is applied at $t = 0$. The field (for $t > 0$) is

$$\mathbf{E} = \mathbf{E}_o e^{-\gamma t}$$

for some $\gamma > 0$. Take $\mathbf{E}_o$ along the z-axis. What is the probability (to first order in E_o) that the atom will be in each of the four $n = 2$ states as $t \to \infty$? Neglect spin.

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$$a^{\frac{3}{2}} R_{10}(r) = 2e^{-r/a} \quad a^{\frac{3}{2}} R_{20}(r) = \frac{1}{\sqrt{2}}\left(1 - \frac{r}{2a}\right) e^{-r/2a} \quad a^{\frac{3}{2}} R_{21}(r) = \frac{1}{2\sqrt{6}}\frac{r}{a} e^{-r/2a}$$

$$Y_0^0(\theta,\phi) = \frac{1}{\sqrt{4\pi}} \quad Y_1^0(\theta,\phi) = \sqrt{\frac{3}{4\pi}}\cos(\theta) \quad Y_1^{\pm 1}(\theta,\phi) = \mp\sqrt{\frac{3}{8\pi}}\sin(\theta)e^{\pm i\phi}$$

Table 1: Some hydrogen atom radial wave functions and spherical harmonics. a is the Bohr radius: $a = \hbar/mc\alpha$.

This is Abers problem 9.1

And an integral

$$\int_0^\infty x^n e^{-x/a}\, dx = a^{n+1} n!$$

$P(f) = |\langle \Phi_f | U(t) | \Phi_i \rangle|^2 = |\langle \Psi_f | \Psi \rangle|^2$ where $|\Psi\rangle = U(t)|\Phi_i\rangle$ and $|\Psi_f\rangle = e^{-iH^0 t/\hbar}|\Phi_f\rangle$. H^0 is the unperturbed hydrogen atom Hamiltonian and $H = H^0 + H'$ where $H' = -eEz = -eE_0 e^{-\gamma t} z$

To derive the formula we need, start with the time-dependent S.E.

$$H(t)|\Psi\rangle = i\hbar|\dot{\Psi}\rangle \Rightarrow \langle \Psi_f | H(t) | \Psi \rangle = i\hbar \langle \Psi_f | \dot{\Psi} \rangle$$

$$\Rightarrow \langle \Psi_f | H(t) | \Psi \rangle = i\hbar \left[\frac{\partial}{\partial t}\langle \Psi_f | \Psi \rangle - \langle \dot{\Psi}_f | \Psi \rangle \right]$$

$$\Rightarrow \langle \Psi_f | H^0 | \Psi \rangle + \langle \Psi_f | H'(t) | \Psi \rangle = i\hbar \frac{\partial}{\partial t}\langle \Psi_f | \Psi \rangle - i\hbar\left(\frac{i}{\hbar}\langle \Psi_f | H^0\right)|\Psi\rangle$$

$$\Rightarrow \langle \Psi_f | H'(t) | \Psi \rangle = i\hbar \frac{\partial}{\partial t}\langle \Psi_f | \Psi \rangle \Rightarrow \langle \Psi_f | \Psi \rangle = \langle \Psi_f | \Psi \rangle|_{t=0} - \frac{i}{\hbar}\int_0^t \langle \Psi_f | H'(t') | \Psi \rangle dt'$$

Approximate $|\Psi\rangle$ in the integrand as $|\Psi_i\rangle$ like the Born approximation.

$$\Rightarrow \langle \Psi_f | \Psi \rangle \cong \delta_{fi} - \frac{i}{\hbar}\int_0^t \langle \Phi_f | H'(t') | \Phi_i \rangle e^{i\omega_{fi} t'} dt'$$

For our problem, the perturbation is the 0th spherical component of a rank 1 tensor so $|1-l| \le l' \le 1+l \Rightarrow l' = 1$ and $m' = m + 0 \Rightarrow m' = 0$ by the Wigner-Eckart Theorem selection rules, so only $|\Phi_f\rangle = |210\rangle$ is nonzero.

$$\langle 210 | z | 100 \rangle = \int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} R_{21}(r)\, Y_1^0(\theta,\phi)\, r\cos(\theta)\, R_{10}(r)\, Y_0^0(\theta,\phi)\, r^2 \sin(\theta)\, dr\, d\theta\, d\phi$$

$$= \int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} \frac{1}{\cancel{2}\sqrt{6}} a^{-5/2} r e^{-r/2a} \sqrt{\frac{3}{4\pi}} \cos(\theta) r\cos(\theta)\, \cancel{2}\, a^{-3/2} e^{-r/a} \frac{1}{\sqrt{4\pi}} r^2 \sin(\theta)\, dr\, d\theta\, d\phi$$

$$= \frac{2\pi}{4\pi}\frac{1}{\sqrt{2}} a^{-4} \int_0^{\pi} \cos^2(\theta)\sin(\theta)\, d\theta \int_0^{\infty} r^4 e^{-3r/2a} dr$$

$$= \frac{1}{2\sqrt{2}} a^{-4} \left(\frac{2}{3}\right)\left(\frac{2a}{3}\right)^5 4! = \frac{2^8 a}{3^5\sqrt{2}}$$

$$P(|210\rangle)|_{t\to\infty} = \left| \frac{-i}{\hbar}\int_0^t \langle 210 | -eE_0 e^{-\gamma t'} z | 100 \rangle e^{i\omega_{fi} t'} dt' \right|^2_{t\to\infty}$$

$$= \frac{e^2 E_0^2}{\hbar^2}\left(\frac{2^{15} a^2}{3^{10}}\right) \left| \int_0^{\infty} e^{(i\omega_{fi} - \gamma)t'} dt' \right|^2$$

$$= \frac{2^{15}}{3^{10}} \frac{e^2 a^2 E_0^2}{\hbar^2} \left| \frac{1}{i\omega_{fi} - \gamma}(-1) \right|^2 = \frac{2^{15}}{3^{10}} \frac{e^2 a^2 E_0^2}{\hbar^2} \frac{1}{\gamma^2 + \omega_{fi}^2} \quad \left(\omega_{fi} = \frac{E_2 - E_1}{\hbar} = -\frac{3E_1}{4\hbar}\right)$$

3. *Quantum Mechanics* (Spring 2004)

The normalized wave function of a one-dimensional particle is

$$\psi(x) = Ne^{-\kappa x^2/2}$$

for some $\kappa > 0$. N is real and positive.

(a) What is N?

(b) What is the expectation value of x^2?

(c) What is the momentum space wave function $\langle p|\psi\rangle$?

(d) What is the expectation value of p^2?

(e) The Hamiltonian is

$$H = \frac{p^2}{2m} + V(x)$$

What is the potential $V(x)$?

a. By normalization $1 = \langle\psi|\psi\rangle = \int_{-\infty}^{\infty} N^* N e^{-Kx^2}dx = 2|N|^2\int_0^\infty e^{-Kx^2}dx$

$= 2|N|^2\frac{1}{\sqrt{K}}\int_0^\infty e^{-u^2}du = 2|N|^2\frac{1}{\sqrt{K}}\frac{\sqrt{\pi}}{2} \Rightarrow |N|^2 = \sqrt{\frac{K}{\pi}}$

$\Rightarrow N = \left(\frac{K}{\pi}\right)^{1/4}$ since N is real and positive

b. $\langle\psi|x^2|\psi\rangle = \sqrt{\frac{K}{\pi}}\int_{-\infty}^{\infty} x^2 e^{-Kx^2}dx = \sqrt{\frac{K}{\pi}}K^{-3/2}\,2\int_0^\infty u^2 e^{-u^2}du$

$= \frac{1}{K\sqrt{\pi}}\cancel{2}\left(\frac{1}{\cancel{2}}\Gamma\left(\frac{2+1}{2}\right)\right) = \frac{1}{K\sqrt{\pi}}\left(\frac{1}{2}\Gamma\left(\frac{1}{2}\right)\right) = \frac{1}{K\sqrt{\pi}}\left(\frac{\sqrt{\pi}}{2}\right) = \frac{1}{2K}$

Using the formula $\int_0^\infty x^n e^{-x^2}dx = \frac{1}{2}\Gamma\left(\frac{n+1}{2}\right)$

c. Recall $\langle x|p\rangle = \frac{1}{\sqrt{2\pi\hbar}}\exp\left(\frac{ipx}{\hbar}\right)$

$\langle p|\psi\rangle = \int_{-\infty}^{\infty}\langle p|x\rangle\langle x|\psi\rangle dx = \int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi\hbar}}e^{ipx/\hbar}Ne^{-Kx^2/2}dx$

$= \frac{1}{\sqrt{2\pi\hbar}}\left(\frac{K}{\pi}\right)^{1/4}\int_{-\infty}^{\infty}e^{-Kx^2/2 + ipx/\hbar}dx$

$= \frac{1}{\sqrt{2\pi\hbar}}\left(\frac{K}{\pi}\right)^{1/4}\int_{-\infty}^{\infty}e^{-\frac{K}{2}\left[\left(x-\frac{ip}{K\hbar}\right)^2+\frac{p^2}{K^2\hbar^2}\right]}dx$ (completing the square)

$= \frac{1}{\sqrt{2\pi\hbar}}\left(\frac{K}{\pi}\right)^{1/4}e^{-p^2/2K\hbar^2}\int_{-\infty}^{\infty}e^{-\frac{K}{2}u^2}du$

$= \frac{1}{\sqrt{2\pi\hbar}}\left(\frac{K}{\pi}\right)^{1/4}e^{-p^2/2K\hbar^2}\sqrt{\frac{2}{K}}\sqrt{\pi} = \frac{1}{\sqrt{\hbar}(K\pi)^{1/4}}e^{-p^2/2K\hbar^2}$

d. $\langle\psi|p^2|\psi\rangle = \int_{-\infty}^{\infty}p^2|\langle p|\psi\rangle|^2dp = \frac{1}{\hbar\sqrt{K\pi}}\int_{-\infty}^{\infty}p^2e^{-p^2/K\hbar^2}dp$

$= \frac{1}{\hbar\sqrt{K\pi}}(K\hbar^2)^{3/2}\frac{\sqrt{\pi}}{2} = \frac{1}{2}\hbar^2K$

e. $-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + V\psi = E\psi \Rightarrow \frac{\partial^2\psi}{\partial x^2} = \frac{2m(V-E)}{\hbar^2}\psi$

$\frac{\partial\psi}{\partial x} = -KxNe^{-Kx^2/2}$ $\quad\frac{\partial^2\psi}{\partial x^2} = K^2x^2Ne^{-Kx^2/2} - KNe^{-Kx^2/2} = (K^2x^2-K)\psi$

$\Rightarrow \frac{2m(V-E)}{\hbar^2} = K^2x^2 - K \Rightarrow V - E = \frac{\hbar^2K^2}{2m}x^2 - \frac{\hbar^2K}{2m}$

$\Rightarrow V(x) = \frac{\hbar^2K^2}{2m}x^2 + C$ where C is a constant equal to $E - \frac{\hbar^2K}{2m}$

which makes sense because the ground state of the SHO is gaussian like $\psi(x)$ and the SHO potential is quadratic.

4. *Quantum Mechanics* (Spring 2004)

The electron neutrino $|\nu_e\rangle$ and the muon neutrino $|\nu_\mu\rangle$ are the possible neutrino states produced and detected in experiments, but they are not necessarily eigenstates of the Hamiltonian. Rather, if the state is known to have momentum p, it is some linear combination of the energy eigenstates $|\nu_1\rangle$ and $|\nu_2\rangle$ of the form

$$|\nu_e\rangle = \cos(\theta)\,|\nu_1\rangle + \sin(\theta)\,|\nu_2\rangle$$
$$|\nu_\mu\rangle = -\sin(\theta)\,|\nu_1\rangle + \cos(\theta)\,|\nu_2\rangle$$

where

$$H\,|\nu_1\rangle = \sqrt{p^2c^2 + m_1^2c^4}\,|\nu_1\rangle$$
$$H\,|\nu_2\rangle = \sqrt{p^2c^2 + m_2^2c^4}\,|\nu_2\rangle$$

for two possibly different masses m_1 and m_2, and some "mixing angle" θ. If it is known that a neutrino was definitely a ν_μ when it was produced, what is the probability of detecting a ν_e after it has traveled a distance L? Assume that $m_1c \ll p$ and $m_2c \ll p$, so that the neutrinos are moving at almost (or even exactly) the speed of light, (so you can ignore corrections of the order $1 - v/c$ compared to terms of order 1) and state your result to first order in the difference $\Delta m^2 = m_1^2 - m_2^2$.

This is a simplified version of an actual neutrino oscillation experiment like the super-Kamiokande detector experiment a few years ago. In reality there is a third neutrino $|\nu_\tau\rangle$.

$$|\Psi(t)\rangle = U(t)\,|\Psi(0)\rangle = e^{-iHt/\hbar}\,|\Psi(0)\rangle = e^{-iHt/\hbar}\,|\nu_\mu\rangle$$
$$= -\sin(\theta)\,e^{-iHt/\hbar}\,|\nu_1\rangle + \cos(\theta)\,e^{-iHt/\hbar}\,|\nu_2\rangle$$
$$= -\sin(\theta)\exp\left[-i\sqrt{p^2c^2+m_1^2c^4}\,t/\hbar\right]|\nu_1\rangle + \cos(\theta)\exp\left[-i\sqrt{p^2c^2+m_2^2c^4}\,t/\hbar\right]|\nu_2\rangle$$

$$\langle \nu_e|\Psi(t)\rangle = -\sin(\theta)\cos(\theta)\exp\left[-i\sqrt{p^2c^2+m_1^2c^4}\,t/\hbar\right] + \sin(\theta)\cos(\theta)\exp\left[-i\sqrt{p^2c^2+m_2^2c^4}\,t/\hbar\right]$$

Now $\sqrt{1+x} \cong 1 + \frac{x}{2}$ for small x, so $\sqrt{p^2c^2+m^2c^4} = pc\sqrt{1+\frac{m^2c^2}{p^2}} \cong pc\left(1+\frac{m^2c^2}{2p}\right)$

since we are assuming $m_1c \ll p$ and $m_2 c \ll p$

$$\langle \nu_e|\Psi(t)\rangle \cong \tfrac{1}{2}\sin(2\theta)\left\{\exp\left[-ipc\left(1+\frac{m_2^2c^2}{2p^2}\right)t/\hbar\right] - \exp\left[-ipc\left(1+\frac{m_1^2c^2}{2p^2}\right)t/\hbar\right]\right\}$$

$$P(\nu_e) = |\langle \nu_e|\Psi(t)\rangle|^2 \cong \tfrac{1}{4}\sin^2(2\theta)\left\{1 - \exp\left[i\,\frac{m_2^2-m_1^2}{2p}c^3t/\hbar\right] - \exp\left[i\,\frac{m_1^2-m_2^2}{2p}c^3t/\hbar\right] + 1\right\}$$
$$= \tfrac{1}{4}\sin^2(2\theta)\left\{2 - 2\cos\left(\tfrac{1}{2}\,\frac{\Delta m^2}{p}c^3t/\hbar\right)\right\}$$
$$= \tfrac{1}{2}\sin^2(2\theta)\left\{1 - \cos\left(\tfrac{1}{2}\,\frac{\Delta m^2c^2}{p}L/\hbar\right)\right\} \quad \text{since } t = \frac{L}{c}$$

Now $\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta) = 1 - 2\sin^2(\theta)$

$$\Rightarrow 1 - \cos(2\theta) = 2\sin^2(\theta) \Rightarrow 1 - \cos(\theta) = 2\sin^2(\theta/2)$$

$$P(\nu_e) = \sin^2(2\theta)\,\sin^2\left(\frac{\Delta m^2c^2}{4p}L/\hbar\right)$$

5. *Quantum Mechanics* (Spring 2005)

Calculate the transmission coefficient for a particle of energy $E > 0$ scattering off the $1D$ potential well $V(x) = V_0$ for $0 < x < a$, $V(x) = 0$ elsewhere, $V_0 < 0$. Are there resonance phenomena?

See Griffiths Section 2.6

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + V\psi = E\psi$$

$$\Rightarrow \frac{\partial^2\psi}{\partial x^2} = -\frac{2m(E-V)}{\hbar^2}\psi$$

Let $k = \sqrt{\frac{2mE}{\hbar^2}}$ and $\ell = \sqrt{\frac{2m(E-V_0)}{\hbar^2}}$ which is real because $V_0 < 0 < E$

$\Rightarrow \psi_1(x) = Ae^{ikx} + Be^{-ikx}$, $\psi_2(x) = Ce^{i\ell x} + De^{-i\ell x}$, $\psi_3(x) = Fe^{ikx} + \cancel{Ge^{-ikx}}$

$G = 0$ since there is no wave coming from the right.

Since $V(x) < \infty$ $\forall x$, we impose continuity on $\psi(x)$ and $\psi'(x)$:

$$\psi_1(0) = \psi_2(0) \Rightarrow A + B = C + D$$
$$\psi_1'(0) = \psi_2'(0) \Rightarrow k(A-B) = \ell(C-D)$$
$$\psi_2(a) = \psi_3(a) \Rightarrow Ce^{i\ell a} + De^{-i\ell a} = Fe^{ika}$$
$$\psi_2'(a) = \psi_3'(a) \Rightarrow \ell(Ce^{i\ell a} - De^{-i\ell a}) = kFe^{ika}$$

Now use the second two equations to solve for C and D:

$$2Ce^{i\ell a} = \left(1+\frac{k}{\ell}\right)Fe^{ika} \Rightarrow C = \frac{1}{2}\left(1+\frac{k}{\ell}\right)Fe^{ika}e^{-i\ell a}$$
$$2De^{-i\ell a} = \left(1-\frac{k}{\ell}\right)Fe^{ika} \Rightarrow D = \frac{1}{2}\left(1-\frac{k}{\ell}\right)Fe^{ika}e^{i\ell a}$$

Next use the first two equations to eliminate B and insert C, D:

$$2A = C + D + \frac{\ell}{k}(C-D) \Rightarrow A = \frac{1}{2}\left(1+\frac{\ell}{k}\right)C + \frac{1}{2}\left(1-\frac{\ell}{k}\right)D$$

$$\Rightarrow A = \frac{1}{4}\left(2+\frac{k}{\ell}+\frac{\ell}{k}\right)Fe^{ika}e^{-i\ell a} + \frac{1}{4}\left(2-\frac{k}{\ell}-\frac{\ell}{k}\right)Fe^{ika}e^{i\ell a}$$

$$= Fe^{ika}\cos(\ell a) - \frac{i}{2}\frac{k^2+\ell^2}{k\ell}Fe^{ika}\sin(\ell a)$$

$$\Rightarrow F = Ae^{-ika}\left[\cos(\ell a) - \frac{i}{2}\frac{k^2+\ell^2}{k\ell}\sin(\ell a)\right]^{-1}$$

Therefore $T \equiv \frac{|F|^2}{|A|^2} = \left[\cos^2(\ell a) + \frac{1}{4}\left(\frac{k^2+\ell^2}{k\ell}\right)^2\sin^2(\ell a)\right]^{-1}$

$$= \left[1 + \left(\frac{1}{4}\frac{k^4+2k^2\ell^2+\ell^4}{k^2\ell^2} - \frac{4k^2\ell^2}{4k^2\ell^2}\right)\sin^2(\ell a)\right]^{-1} \quad \left(\cos^2(\ell a) = 1-\sin^2(\ell a)\right)$$

$$= \left[1 + \frac{1}{4}\left(\frac{k^2-\ell^2}{k\ell}\right)^2\sin^2(\ell a)\right]^{-1}$$

$$= \left[1 + \frac{1}{4}\left(\frac{V_0}{\sqrt{E(E-V_0)}}\right)^2\sin^2\left(\frac{a}{\hbar}\sqrt{2m(E-V_0)}\right)\right]^{-1}$$

$$= \left[1 + \frac{1}{4}\frac{V_0^2}{E(E-V_0)}\sin^2\left(\frac{a}{\hbar}\sqrt{2m(E-V_0)}\right)\right]^{-1}$$

Resonance phenomena can occur if the energy is just right. The Ramsauer-Townsend effect gives perfect transmission

so $T = 1 \Leftrightarrow \frac{a}{\hbar}\sqrt{2m(E-V_0)} = n\pi \Leftrightarrow 2m(E-V_0) = \left(\frac{n\pi\hbar}{a}\right)^2 \Leftrightarrow E = \frac{n^2\pi^2\hbar^2}{2ma^2} + V_0$

6. *Statistical Mechanics and Thermodynamics* (Spring 2006)

Consider a gas of relativistic, conserved bosons. The relation between energy and momentum is

$$E = |\mathbf{p}|c$$

(a) Derive the condition for Bose-Einstein condensation in three dimensions.

(b) Does Bose-Einstein condensation occur in two dimensions? Justify your answer.

(c) What is the highest dimension for which Bose-Einstein condensation does not occur?

The simplest definition of T_c is the minimum temperature for which all particles in the system are expected to be in excited states. Our strategy:

1. Find the density of states
2. Integrate occupancy times density of states to get the total number of particles in excited states N_e (since $\epsilon=0$ for ground state, they aren't counted in this integral because $\rho(0)=0$).
3. Maximize N_e by setting $\mu=0$ so the minimum temperature comes out.
4. Set $N_e=N$ and solve for $T=T_c$.

a. $\epsilon = pc = \hbar k c = \left(\frac{\hbar\pi c}{L}\right)n \Rightarrow n = \left(\frac{L}{\hbar\pi c}\right)\epsilon \Rightarrow dn = \left(\frac{L}{\hbar\pi c}\right)d\epsilon$

$$\rho(\epsilon) = \frac{1}{8}4\pi n^2 dn = \frac{\pi}{2}\left(\frac{L}{\hbar\pi c}\right)^3\epsilon^2 d\epsilon = \frac{V}{2\pi^2}\frac{\epsilon^2}{(\hbar c)^3}d\epsilon$$

$$\begin{aligned}
N_e &= \int_0^\infty f(\epsilon)\rho(\epsilon)\,d\epsilon \\
&= \frac{V}{2\pi^2}\frac{1}{(\hbar c)^3}\int_0^\infty \frac{\epsilon^2}{e^{\beta(\epsilon-\mu)}-1}\,d\epsilon \\
&= \frac{V}{2\pi^2}\frac{1}{(\hbar c)^3}\int_0^\infty \frac{\epsilon^2}{e^{\beta(\epsilon-\mu)}}\frac{1}{1-e^{-\beta(\epsilon-\mu)}}\,d\epsilon \\
&= \frac{V}{2\pi^2}\frac{1}{(\hbar c)^3}\int_0^\infty \frac{\epsilon^2}{e^{\beta(\epsilon-\mu)}}\sum_{\ell=0}^\infty e^{-\beta\ell(\epsilon-\mu)}\,d\epsilon \quad \text{(must have } \epsilon > \mu \text{ for } \bar{n}>0\text{)} \\
&= \frac{V}{2\pi^2}\frac{1}{(\hbar c)^3}\int_0^\infty \epsilon^2\sum_{\ell=1}^\infty e^{-\beta\ell(\epsilon-\mu)}\,d\epsilon \\
&= \frac{V}{2\pi^2}\frac{1}{(\hbar c)^3}\sum_{\ell=1}^\infty\left[e^{\beta\ell\mu}\int_0^\infty \epsilon^2 e^{-\beta\ell\epsilon}\,d\epsilon\right] \\
&= \frac{V}{2\pi^2}\frac{1}{(\hbar c)^3}\sum_{\ell=1}^\infty\left[e^{\beta\ell\mu}\left(\frac{1}{\beta\ell}\right)^3\int_0^\infty x^2 e^{-x}\,dx\right] \\
&= \frac{V}{2\pi^2}\frac{1}{(\hbar c)^3}\frac{2!}{\beta^3}\sum_{\ell=1}^\infty\frac{e^{\beta\ell\mu}}{\ell^3} \\
&\xrightarrow{\mu\to 0} \frac{V}{\pi^2}\frac{1}{(\hbar c\beta)^3}\sum_{\ell=1}^\infty\frac{1}{\ell^3} \quad \text{and } \sum_{\ell=1}^\infty \frac{1}{\ell^3} \equiv \zeta(3)
\end{aligned}$$

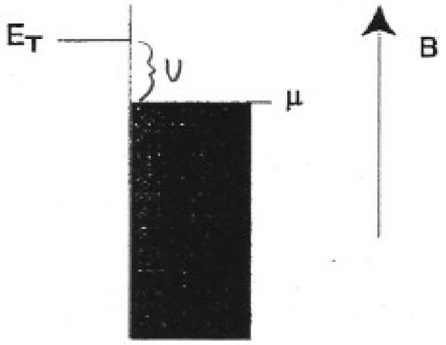
$$\begin{aligned}
N_e = N \Rightarrow \frac{1}{\beta^3} &= \frac{N}{V}\pi^2\frac{(\hbar c)^3}{\zeta(3)} \\
\Rightarrow \frac{1}{\beta} &= \hbar c\left(\pi^2\frac{N}{V}/\zeta(3)\right)^{1/3} \\
\Rightarrow T_c &= \frac{\hbar c}{k}\left(\pi^2\frac{N}{V}/\zeta(3)\right)^{1/3}
\end{aligned}$$

b. In 2D, $\rho(\epsilon)d\epsilon = \frac{1}{4}2\pi n\,dn = \frac{\pi}{2}\left(\frac{L}{\hbar\pi c}\right)^2\epsilon\,d\epsilon$ gives $\zeta(2)$ with a similar procedure and $\zeta(2)$ converges so everything is fine and condensation does occur.

c. In 1D, $\rho(\epsilon)d\epsilon = dn = \frac{L}{\hbar\pi c}d\epsilon$ gives $\zeta(1)$ with a similar procedure, but $\zeta(1)$ diverges, so the resulting T_c is $T_c=0$, so BEC does not occur in 1D, making 1 the highest dimension for which BEC does not occur.

7. *Statistical Mechanics and Thermodynamics* (Spring 2004)

A quantum state at energy E_T is embedded in a system with a degenerate Fermi gas as, for instance, occurs with an impurity state with energy E_T in a degenerate semiconductor with a sea of conducting electrons at chemical potential μ. You may assume that $E_T > \mu$. The impurity, which has a spin of 1/2, can take an additional electron from the large bath of electrons (costs Coulomb energy U), to form a spin-singlet state. For a given temperature T and magnetic field B, calculate the ratio of the probability for the trap being empty to that for the trap being filled by an additional electron.



When the trap is empty it has energy associated with the spin of $\frac{\hbar}{2}$ interacting with the magnetic field. When the trap is filled, the total spin is zero, so there is no interaction with the magnetic field, but it has energy $U = E_T - \mu$. $\vec{\mu} = g\mu_B\vec{s}$ where $\hbar\vec{s}$ is the spin

$$E_B = -\vec{\mu}\cdot\vec{B} = -\frac{1}{2}g\mu_B B$$

Therefore $E_{empty} = -\frac{1}{2}g\mu_B B$ and $E_{filled} = U$.

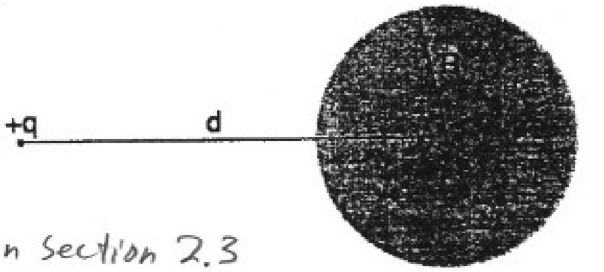
$$P_{empty} = \frac{e^{-\beta E_{empty}}}{e^{-\beta E_{empty}} + e^{-\beta E_{filled}}} \qquad P_{filled} = \frac{e^{-\beta E_{filled}}}{e^{-\beta E_{empty}} + e^{-\beta E_{filled}}}$$

$$\frac{P_{empty}}{P_{filled}} = \frac{e^{-\beta E_{empty}}}{e^{-\beta E_{filled}}} = \frac{e^{\beta(\frac{1}{2}g\mu_B B)}}{e^{-\beta U}} = e^{(U + \frac{1}{2}g\mu_B B)/kT}$$

$$= e^{(E_T - \mu + \frac{1}{2}g\mu_B B)/kT}$$

8. *Electricity and Magnetism* (Spring 2004)

A point charge q is located a distance d from the center of a conducting sphere of radius R. What must the total charge on the conducting sphere be for the force on the point charge to be zero?



See Jackson Section 2.3

First assume the sphere is grounded. Then we know by the method of images a charge $q' = -q\left(\frac{R}{d}\right)$ flows onto the sphere and distributes itself so that the field outside is like a point charge of charge q' at $x' = R\left(\frac{R}{d}\right)$.

Now remove the ground connection and add charge q'' to the sphere (which already has charge q' on it). Since the surface is an equipotential, the new charge will distribute itself uniformly over the surface, which is equivalent to an image charge of charge q'' at $x''=0$.

The field at $x=d$ is zero when

$$E(x=d) = \frac{1}{4\pi\epsilon_0}\left(\frac{q'}{(x'-d)^2} + \frac{q''}{(x''-d)^2}\right) = \frac{1}{4\pi\epsilon_0}\left(\frac{-qR/d}{\left(\frac{R^2}{d}-d\right)^2} + \frac{q''}{d^2}\right) = 0$$

$$\Leftrightarrow \quad q'' = \frac{qRd}{\left(\frac{R^2}{d}-d\right)^2} = \frac{qd^3R}{(R^2-d^2)^2}$$

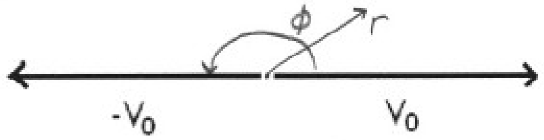
The total charge on the sphere is then

$$Q = q' + q'' = \frac{qd^3R}{(R^2-d^2)^2} - \frac{qR}{d} = \frac{qd^4R - qR(R^2-d^2)^2}{d(R^2-d^2)^2}$$

$$= qR\,\frac{2d^2R^2 - R^4}{d(R^2-d^2)^2}$$

9. *Electricity and Magnetism* (Spring 2004)

Consider the infinite two-dimensional conducting plane depicted in the figure. The right half is maintained at electrostatic potential V_0 while the left half is maintained at potential $-V_0$. What is the potential above the plane?



See Jackson Section 2.11

We solve Laplace's equation in cylindrical coordinates

$$\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \phi^2} + \frac{\partial^2}{\partial z^2}$$

There is no z dependence by symmetry so we use separation of variables and seek solutions of the form $\Phi(r,\phi) = R(r)Q(\phi)$.

(Or you could recall that the solution is $\Phi(r,\phi) = (A + B\ln(r))(C + D\phi)$ when r ranges from 0 to ∞).

$$\nabla^2 \Phi = 0 \Rightarrow \frac{Q}{r}\frac{\partial}{\partial r}\left(r\frac{\partial R}{\partial r}\right) + \frac{R}{r^2}\frac{\partial^2 Q}{\partial \phi^2} = 0$$

$$\Rightarrow \frac{r}{R}\frac{\partial}{\partial r}\left(r\frac{\partial R}{\partial r}\right) + \frac{1}{Q}\frac{\partial^2 Q}{\partial \phi^2} = 0$$

$$\Rightarrow \frac{r}{R}\frac{\partial}{\partial r}\left(r\frac{\partial R}{\partial r}\right) = \lambda \text{ and } \frac{1}{Q}\frac{\partial^2 Q}{\partial \phi^2} = -\lambda$$

by independence of variables

$$\Rightarrow r\frac{\partial}{\partial r}\left(r\frac{\partial R}{\partial r}\right) = \lambda R \text{ and } \frac{\partial^2 Q}{\partial \phi^2} = -\lambda Q$$

$$\Rightarrow \begin{cases} R(r) = Ar^{\sqrt{\lambda}} + Br^{-\sqrt{\lambda}} & \text{and } Q(\phi) = C\sin(\sqrt{\lambda}\phi) + D\cos(\sqrt{\lambda}\phi) \quad (\lambda \neq 0) \\ R(r) = A' + B'\ln(r) & \text{and } Q(\phi) = C' + D'\phi \quad (\lambda = 0) \end{cases}$$

The conditions that $|\Phi(r=\infty)| < \infty$ and $|\Phi(r=0)| < \infty$ imply $A = B = B' = 0$, so the $\lambda \neq 0$ case is excluded.

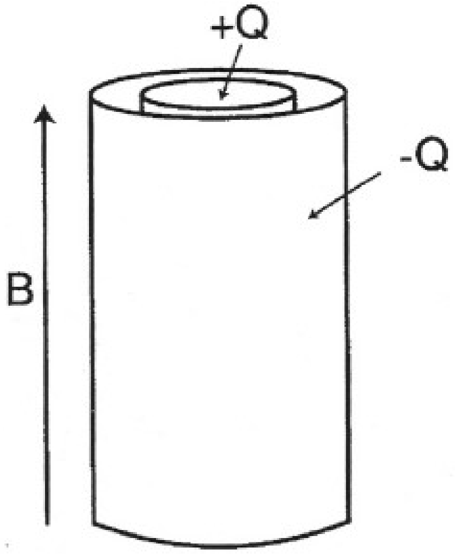
$$\Rightarrow \Phi(r,\phi) = C' + D'\phi$$

$\Phi(\phi=0) = V_0 \Rightarrow C' = V_0$ and $\Phi(\phi=\pi) = -V_0 \Rightarrow D' = -\frac{2V_0}{\pi}$

Therefore $\Phi(r,\phi) = V_0\left(1 - \frac{2}{\pi}\phi\right)$

10. *Electricity and Magnetism* (Spring 2004)

Consider a cylindrical capacitor of length L with charge $+Q$ on the inner cylinder of radius a and $-Q$ on the outer cylindrical shell of radius b. The capacitor is filled with a lossless dielectric with dielectric constant equal to 1. The capacitor is located in a region with a uniform magnetic field B, which points along the symmetry axis of the cylindrical capacitor. A flaw develops in the dielectric insulator, and a current flow develops between the two plates of the capacitor. Because of the magnetic field, this current flow results in a torque on the capacitor, which begins to rotate. After the capacitor is fully discharged (total charge on both plates is zero), what is the magnitude and direction of the angular velocity of the capacitor? The moment of inertia of the capacitor (about the axis of symmetry) is I, and you may ignore fringing fields in the calculation.



Let $d\vec{L}$ be the change in angular momentum due to the flow of an infinitesimal amount of charge dq. Then $\vec{L} = \int_0^Q d\vec{L}$.

$$d\vec{L} = \int_0^t \vec{\tau}\, dt = \int_a^b \vec{\tau}(r) \frac{dt}{dr} dr = \int_a^b \vec{r} \times \vec{F}(\vec{r}) \frac{1}{v} dr$$

$$= dq \int_a^b \vec{r} \times (\vec{v} \times \vec{B}) \frac{1}{v} dr = dq \int_a^b r B\, \hat{r} \times (\hat{r} \times \hat{z})\, dr$$

$$= -\frac{1}{2}(b^2 - a^2) B\, dq\, \hat{z}$$

$$\vec{L} = \int_0^Q d\vec{L} = -\frac{1}{2}(b^2 - a^2) B Q\, \hat{z}$$

$$\vec{L} = I\vec{\omega} \Rightarrow \vec{\omega} = -\frac{1}{2} \frac{QB}{I} (b^2 - a^2)\, \hat{z}$$

11. *Electricity and Magnetism* (Spring 2004)

Consider a plasma of free charges of mass m and charge e at constant density n. What is the index of refraction for electromagnetic waves of frequency ω which are incident upon this plasma?

$$V = \frac{c}{n_i} \Rightarrow \frac{1}{\sqrt{\mu\epsilon}} = \frac{1}{n_i\sqrt{\mu_0\epsilon_0}} \Rightarrow n_i = \sqrt{\frac{\mu\epsilon}{\mu_0\epsilon_0}}$$

Unless if a substance is ferromagnetic, its magnetic susceptibility μ will be approximately μ_0, so $n_i \cong \sqrt{\frac{\epsilon}{\epsilon_0}}$

Recall $\frac{\epsilon(\omega)}{\epsilon_0} \cong 1 - \frac{\omega_p^2}{\omega^2}$ and $\omega_p^2 = \frac{ne^2}{\epsilon_0 m}$

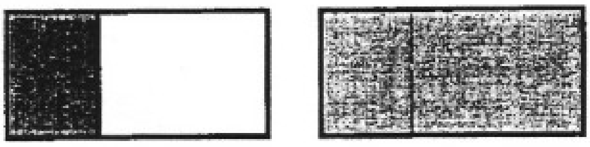
$$\text{So } n_i(\omega) \cong \sqrt{\frac{\epsilon(\omega)}{\epsilon_0}} \cong \sqrt{1 - \frac{\omega_p^2}{\omega^2}} = \sqrt{1 - \frac{ne^2}{\epsilon_0 m \omega}}$$

13. *Statistical Mechanics and Thermodynamics* (Spring 2004)

A van der Waals gas has the following equation of state:

$$P(T,V) = \frac{NkT}{(V-bN)} - a\left(\frac{N}{V}\right)^2$$

This gas is held in a container of negligible mass which is isolated from its surroundings. The gas is initially confined to 1/3 of the total volume of the container by a partition (a vacuum exists in the other 2/3 of the volume). The gas is initially in thermal equilibrium with temperature T_i. A hole is then opened in the partition, allowing the gas to irreversibly expand to fill the entire volume (V). What is the new temperature of the gas after thermal equilibrium has been re-established? (Hint: Note that the specific heat at constant volume for a van der Waals gas is the same as that for an ideal gas.)



Before **After**

See Reif Page 177. The concept here is that the gas will do work against its own van der Waals attraction forces when it expands, which lowers the temperature of the gas.

Start with $C_v = \left(\frac{đQ}{dT}\right)_v = T\left(\frac{\partial S}{\partial T}\right)_v$ and $\left(\frac{\partial S}{\partial V}\right)_T = \left(\frac{\partial p}{\partial T}\right)_V$.

The first is only true for quasistatic situations and this expansion is not quasistatic, but we will use $đQ$ as the change in internal energy of the gas rather than heat exchanged with the environment, and the change in energy of the gas can be quasistatic. By the chain rule,

$$dS = \left(\frac{\partial S}{\partial T}\right)_V dT + \left(\frac{\partial S}{\partial V}\right)_T dV$$

$$= \frac{C_V}{T}dT + \left(\frac{\partial p}{\partial T}\right)_V dV$$

$$\Rightarrow TdS = C_V dT + T\left(\frac{\partial p}{\partial T}\right)_V dV$$

$$dE = TdS - pdV \Rightarrow dE = C_V dT + T\left(\frac{\partial p}{\partial T}\right)_V dV - p\,dV$$

$$= C_V dT + \frac{NKT}{(V-bN)}dV - \left(\frac{NKT}{(V-bN)} - a\left(\frac{N}{V}\right)^2\right)dV$$

$$= C_V dT + a\left(\frac{N}{V}\right)^2 dV$$

No work done on environment and no heat exchanged $\Rightarrow dE = 0$

$$\Rightarrow C_V dT = -a\left(\frac{N}{V}\right)^2 dV$$

Assume $C_V = \frac{3}{2}NK$ like ideal gas even though it isn't true,

$$\Rightarrow \Delta T = \int dT = -\frac{2}{3}\frac{a}{K}N\int_{V/3}^{V}\frac{1}{V^2}dV = \frac{2}{3}\frac{a}{K}N\left(\frac{1}{V} - \frac{1}{V/3}\right) = -\frac{4}{3}\frac{a}{K}\frac{N}{V}$$

$$\Rightarrow T_f = T_i - \frac{4}{3}\frac{a}{K}\frac{N}{V}$$

QM S'04 #1

$S_1 = 3/2\ ;\ S_2 = 1/2 \quad ; \quad S = S_1 + S_2 = 2 \quad (\text{max. total } S)$

$\qquad\qquad\qquad\qquad S_1 - S_2 = 1 \quad (\text{min. total } S)$

$$|2,2\rangle = |3/2, 3/2\rangle |1/2, 1/2\rangle$$
$$|2,1\rangle = \frac{1}{\sqrt{4}} |3/2, 3/2\rangle |1/2, -1/2\rangle + \sqrt{\frac{3}{4}} |3/2, 1/2\rangle |1/2, 1/2\rangle$$
$$|2,0\rangle = \frac{1}{\sqrt{2}} |3/2, 1/2\rangle |1/2, -1/2\rangle + \frac{1}{\sqrt{2}} |3/2, -1/2\rangle |1/2, 1/2\rangle$$
$$|2,-1\rangle = \sqrt{\frac{3}{4}} |3/2, -1/2\rangle |1/2, -1/2\rangle + \frac{1}{\sqrt{4}} |3/2, -3/2\rangle |1/2, 1/2\rangle$$
$$|2,-2\rangle = |3/2, -3/2\rangle |1/2, -1/2\rangle$$

$$|1,1\rangle = \sqrt{\frac{3}{4}} |3/2, 3/2\rangle |1/2, -1/2\rangle - \frac{1}{\sqrt{4}} |3/2, 1/2\rangle |1/2, 1/2\rangle$$
$$|1,0\rangle = \frac{1}{\sqrt{2}} |3/2, 1/2\rangle |1/2, -1/2\rangle - \frac{1}{\sqrt{2}} |3/2, -1/2\rangle |1/2, 1/2\rangle$$
$$|1,-1\rangle = \frac{1}{\sqrt{4}} |3/2, -1/2\rangle |1/2, -1/2\rangle - \sqrt{\frac{3}{4}} |3/2, -3/2\rangle |1/2, 1/2\rangle$$

Spring 2004 #1

$|s\ m\rangle \quad |s_1\ m_1\rangle|s_2\ m_2\rangle \quad s_1 = 1/2 \quad s_2 = 3/2$

$|2\ 2\rangle = |3/2\ 3/2\rangle|1/2\ 1/2\rangle$

$|2\ 1\rangle = \frac{1}{\sqrt{4}}|3/2, 3/2\rangle|1/2\ -1/2\rangle + \sqrt{3/4}\,|3/2\ 1/2\rangle|1/2\ 1/2\rangle$

$|2\ 0\rangle = \frac{1}{\sqrt{2}}|3/2\ 1/2\rangle|1/2, -1/2\rangle + 1/\sqrt{2}\,|3/2\ -1/2\rangle|1/2\ 1/2\rangle$

$|2\ -1\rangle = \sqrt{3/4}\,|3/2, -1/2\rangle|1/2\ -1/2\rangle + \frac{1}{\sqrt{4}}|3/2\ -3/2\rangle|1/2\ 1/2\rangle$

$|2\ -2\rangle = |3/2\ -3/2\rangle|1/2\ -1/2\rangle$

$|1\ 1\rangle = \sqrt{\frac{3}{4}}\,|3/2\ 3/2\rangle|1/2\ -1/2\rangle\ -\sqrt{1/4}\,|3/2\ 1/2\rangle|1/2\ 1/2\rangle$

$|1\ 0\rangle = \sqrt{\frac{1}{2}}\,|3/2\ 1/2\rangle|1/2\ -1/2\rangle - \sqrt{\frac{1}{2}}\,|3/2\ -1/2\rangle|1/2\ 1/2\rangle$

$|1\ -1\rangle = \sqrt{\frac{1}{4}}\,|3/2\ -1/2\rangle|1/2\ -1/2\rangle - \sqrt{\frac{3}{4}}\,|3/2\ -3/2\rangle|1/2\ 1/2\rangle$

Spring 2004 #1 (p 1 of 2)

The table below shows some Clebsch-Gordan coefficients. If two particles have spin 1/2 and 3/2 respectively, write down all composite states $|s\,m\rangle$ in terms of the uncoupled states using Dirac notation.

The possible values of s is given by

$$|s_1 - s_2| \le s \le |s_1 + s_3|$$

$$\Rightarrow \left|\tfrac{1}{2} - \tfrac{3}{2}\right| \le s \le \left|\tfrac{1}{2} + \tfrac{3}{2}\right| \quad \Rightarrow \quad 1 \le s \le 2$$

Thus, s can be either 1 or 2.

if you want to read a column on the table, it has the form

$$|s\,m\rangle = \sum_{m_1+m_2=m} C^{s_1 s_2 s}_{m_1 m_2 m}\, |s_1 m_1\rangle |s_2\, m_2\rangle$$

where

$$\begin{array}{l|c}
\multicolumn{2}{l}{s_1 \times s_2} \\
 & s \\
 & m \\
\hline
m_1\, m_2 & \left(C^{s_1 s_2 s}_{m_1 m_2 m}\right)^2
\end{array}$$

So, for $s=2$ $(s_1 = 3/2,\ s_2 = 1/2)$

$$|2\,2\rangle = |3/2, 3/2\rangle\, |1/2, 1/2\rangle$$

$$|2\,1\rangle = \frac{1}{\sqrt{4}} |3/2, 3/2\rangle\, |1/2, -1/2\rangle + \sqrt{\frac{3}{4}}\, |3/2, 1/2\rangle\, |1/2, 1/2\rangle$$

$$|2\,0\rangle = \frac{1}{\sqrt{2}} |3/2, \tfrac{1}{2}\rangle\, |1/2, -1/2\rangle + \frac{1}{\sqrt{2}} |3/2, -1/2\rangle\, |1/2, 1/2\rangle$$

$$|2,-1\rangle = \sqrt{\frac{3}{4}}\, |3/2, -1/2\rangle\, |1/2, -1/2\rangle + \frac{1}{\sqrt{4}} |3/2, -3/2\rangle\, |1/2, 1/2\rangle$$

$$|2,-2\rangle = |3/2, -3/2\rangle\, |1/2, -1/2\rangle$$

) For $s=1$

$$|11\rangle = \sqrt{\tfrac{3}{4}}\,|3/2, 3/2\rangle|\tfrac{1}{2}, -1/2\rangle - \frac{1}{\sqrt{4}}\,|3/2, 1/2\rangle|1/2, 1/2\rangle$$

$$|10\rangle = \frac{1}{\sqrt{2}}\,|3/2, 1/2\rangle|1/2, -1/2\rangle - \frac{1}{\sqrt{2}}\,|3/2, -1/2\rangle|3/2, \tfrac{1}{2}\rangle$$

$$|1,-1\rangle = \frac{1}{\sqrt{4}}\,|3/2, -1/2\rangle|\tfrac{1}{2}, -1/2\rangle - \sqrt{\tfrac{3}{4}}\,|3/2, -3/2\rangle|1/2, 1/2\rangle$$

QM S'04 #2

H-atom is in the ground state ($n=1, \ell=m=0$) at $t<0$

A time dependent E-field is applied:

$$\vec{E} = \vec{E}_0 e^{-\gamma t} \quad \gamma > 0 \; ; \; \vec{E}_0 = E_0 \hat{z}$$

What is the probability that for $t \to \infty$ the atom is in each of the four $n=2$ states?

Start: $n=1, \ell=0=m$ $\quad |1,0,0\rangle$

Finish $n=2$; $\underset{m=0}{\ell=0}$ $\quad |2,0,0\rangle$; $\underset{m=1,0,-1}{\ell=1}$ $\quad |2,1,1\rangle, |2,1,0\rangle, |2,1,-1\rangle$

$$c_{1\to 2}(t) = \frac{-i}{\hbar}\int_0^\infty \langle H'\rangle e^{i\omega_0 t}\,dt \; ; \; \omega_0 = \frac{E_2 - E_1}{\hbar} \; ; \; \begin{array}{l} E_1 = -13.6\,eV \\ E_2 = -\frac{13.6\,eV}{4} \end{array}$$

$$= -\frac{13.6\,eV}{\hbar}\left(\frac{1}{4} - 1\right) = \frac{13.6\,eV}{\hbar}\cdot\frac{3}{4}$$

$\langle H'\rangle = \quad \langle 100|H'|200\rangle ; \langle 100|H'|211\rangle ; \langle 100|H'|210\rangle ; \langle 100|H'|21-1\rangle$

$$\bullet \; \langle 100|H'|200\rangle = \int_0^{2\pi}\int_0^{\pi}\int_0^{\infty}\left(\frac{2e^{-r/a}}{\sqrt{4\pi}\,a^{3/2}}\right) E_0 \overset{r\cos\theta}{\downarrow}{z}\, e^{-\gamma t}\left(\frac{1}{\sqrt{2}\,a^{3/2}\sqrt{4\pi}}\left(1 - \frac{r}{2a}\right)e^{-r/2a}\right) r^2 \sin\theta\, d\theta\, d\phi\, dr$$

$$= \frac{2}{4\pi a^3\sqrt{2}} E_0 e^{-\gamma t} \underbrace{\int_0^{2\pi} d\phi}_{2\pi} \underbrace{\int_0^{\pi} \cos\theta \sin\theta\, d\theta}_{} \int_0^{\infty} r^3\left(1 - \frac{r}{2a}\right) e^{-\frac{3r}{2a}}\, dr = 0$$

$u = \sin\theta$
$du = \cos\theta\, d\theta$

$$\Rightarrow \int_0^0 u\,du = 0$$

So $\underline{\langle 100|H'|200\rangle = 0}$

$$\cdot \langle 100|H'|210\rangle = \int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} \frac{\cancel{2}e^{-r/a}}{a^{3/2}\sqrt{4\pi}}\left(E_0 r\cos\theta\, e^{-\gamma t}\right)\left(\frac{1}{\cancel{2}\sqrt{6}}\frac{r}{a\,a^{3/2}}e^{-r/2a}\sqrt{\frac{3}{4\pi}}\cos\theta\right) r^2 dr\,\sin\theta\, d\theta\, d\phi$$

$$= \frac{E_0 e^{-\gamma t}}{4\pi a^3}\frac{1}{\sqrt{2}}\frac{1}{a}\underbrace{\int_0^{2\pi} d\phi}_{2\pi}\underbrace{\int_0^{\pi}\cos^2\theta\sin\theta\, d\theta}\underbrace{\int_0^{\infty} r^4 e^{-\frac{3r}{2a}}dr} = \frac{E_0 e^{-\gamma t}}{\cancel{4\pi} a^4}\frac{1}{\sqrt{2}}\cdot\cancel{2\pi}\cdot\frac{\cancel{2}}{3}\cdot\left(\frac{2a}{3}\right)^5\cdot 4!$$

$u = \cos\theta$

$du = -\sin\theta\, d\theta$

$$\int_0^{\infty} x^n e^{-x/a'}dx = a'^{n+1} n! \quad n = 4;\ a' = \frac{2a}{3}$$

$$= \left(\frac{2a}{3}\right)^5 4!$$

$$-\int_1^{-1} u^2 du = \int_{-1}^{1} u^2 du = \frac{1}{3}u^3\Big|_{-1}^{1} = \frac{1}{3}(1-(-1)) = \frac{2}{3}$$

So $\langle 100|H'|210\rangle = \dfrac{E_0 e^{-\gamma t}}{3\sqrt{2}a^4}\left(\dfrac{2a}{3}\right)^5 4! = \boxed{\dfrac{E_0 e^{-\gamma t} a}{\sqrt{2}}\,8\left(\dfrac{2}{3}\right)^5}$

$$\cdot\langle 100|H'|211\rangle = \int_0^{2\pi}\int_0^{\pi}\int_0^{\infty}\frac{\cancel{2}e^{-r/a}}{a^{3/2}\sqrt{4\pi}}\left(E_0 r\cos\theta\, e^{-\gamma t}\right)\left(\frac{1}{\cancel{2}\sqrt{6}}\frac{r}{a\,a^{3/2}}e^{-r/2a}\cdot -\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{i\phi}\right) r^2 dr\sin\theta\, d\theta\, d\phi$$

$$= \frac{-E_0 e^{-\gamma t}}{a^4 4\pi}\frac{1}{\sqrt{4}}\underbrace{\int_0^{2\pi} e^{i\phi}d\phi}\int_0^{\pi}\cos\theta\sin^2\theta\, d\theta\int_0^{\infty} r^4 e^{-\frac{3r}{2a}}dr = 0$$

$$\frac{1}{i}e^{i\phi}\Big|_0^{2\pi} = \frac{1}{i}[1-1] = 0$$

So $\langle 100|H'|211\rangle = 0$ and similarly for $\langle 100|H'|21-1\rangle$.

So $$c_{1\to 2}(t) = \frac{-i}{\hbar}\int_0^{\infty} 8\frac{E_0 e^{-\gamma t}a}{\sqrt{2}}\left(\frac{2}{3}\right)^5 e^{i\omega_0 t}dt = \frac{-i E_0 a 8}{\hbar\sqrt{2}}\left(\frac{2}{3}\right)^5\underbrace{\int_0^{\infty} e^{-(\gamma - i\omega_0)t}dt}$$

$$\frac{1}{-(\gamma - i\omega_0)}e^{-(\gamma-i\omega_0)t}\Big|_0^{\infty} = \frac{+1}{(\gamma - i\omega_0)}$$

$$c_{1\to 2}(t) = \frac{-i E_0 a 8}{\hbar\sqrt{2}}\left(\frac{2}{3}\right)^5\frac{1}{(\gamma - i\omega_0)}$$

So the probability is: $|c_{1\to 2}(t)|^2 = \dfrac{E_0^2 a^2}{2\hbar^2}\left(\dfrac{2}{3}\right)^{10}\dfrac{8^2}{(\gamma - i\omega_0)^2} = \boxed{\dfrac{E_0^2 a^2}{2\hbar^2}\left(\dfrac{2}{3}\right)^{10}\dfrac{64}{\gamma^2+\omega_0^2}}$

$$\uparrow$$

$$\frac{8}{\gamma^2+\omega_0^2}$$

Spring 2004 #

$$H' = e E_0 z e^{-\gamma t}$$ (reference problem 9.1 abers

$$|\Psi(t)\rangle = C(t) e^{-iE_n t} |\Psi_n\rangle$$

$$\langle n\ell m | \Psi(t)\rangle = C_{n\ell m}(t) e^{-iE_n t}$$

$$C_{n\ell m} = -i\langle n\ell m | z | 100\rangle e E_0 \int_0^t e^{i\omega_{n\ell m,100} t'} e^{-\gamma t'} dt'$$

– to first order

$\omega = E_n - E_1$

abers 9.3

$\langle 200 | z | 100\rangle$ goes to zero from parity conservation

$\langle 21\pm1 | z | 100\rangle$ go away due to z being the 0th component of a rank one spherical tensor

$\langle 210 | z | 100\rangle$ only excited state

$$\int_0^t e^{iE_{21}t'} e^{-\gamma t} dt' = \frac{1}{iE_{21} - \gamma}\left(e^{iE_{21}t - \gamma t} - 1\right)$$

$E_{21} = E_2 - E_1$

$$E_{21} = \frac{3\alpha^2 m}{8}$$

$$\langle 210 | z | 100\rangle = \sqrt{\frac{2^{15} a^2}{3^{10}}}$$

(used a lot)

$$|C_{210}(t)|^2 = e^2 E_0^2 \left(\frac{2^{15} a^2}{3^{10}}\right) \frac{1}{E_{21}^2 + \gamma^2}\left(1 + e^{-2\gamma t} - 2e^{-\gamma t}\cos(E_{21} t)\right)$$

$$|C_{210}|^2_{t\to\infty} = e^2 E_0^2 \left(\frac{2^{15} a^2}{3^{10}}\right) \frac{1}{E_{21}^2 + \gamma^2}$$

Spring 2004 #2 (p 1 of 2)

A hydrogen atom is in the ground state ($n=1$, $\ell=m=0$) for $t<0$. Suppose the atom is placed between the plates of a capacitor, and a weak, spatially uniform but time-dependent decaying field is applied at $t=0$. The field (for $t>0$) is

$$\vec{E} = \vec{E}_0 e^{-\gamma t}$$

for some $\gamma>0$. Take E_0 along the z-axis. What is the probability (to first order in E_0) that the atom will be in each of the four $n=2$ states as $t\to\infty$? Neglect spin.

this is a time dependent perturbation problem. (see also Fall 2003 #5 and Spring 2003 #1)

the transition probability for $t\to\infty$ is given by Zettili eq 10.11

$$P_{fi}(t) = \left| \int_0^\infty \langle \psi_f | V'(t') | \psi_i \rangle e^{-i\omega_{fi} t'} dt' \right|^2$$

where

$$V'(t') = e E_0 e^{-\gamma t} z$$ ← time dependent Stark effect

and

$$\omega_{fi} = E_f - E_i = -\frac{\alpha^2 m}{2}\left(\frac{1}{n_f^2} - \frac{1}{n_i^2}\right)\Bigg|_{\substack{n_f=2\\ n_i=1}} = \frac{3\alpha^2 m}{8}$$

For

$$\langle \psi_f | V'(t') | \psi_i \rangle = e E_0 e^{-\gamma t} \langle 2\ell' m' | z | 100 \rangle$$

we know the following selection rules since z is odd and a first rank tensor. for the matrix element to be non zero, we need $|\Delta \ell| = 1$ and $|\Delta m| = 0$

Thus, $\ell' = 1$ and $m' = 0$. The other elements vanish.

From Fall 2003 #5, we know that

$$\langle 210 | z | 100 \rangle = \frac{a\, 2^8}{\sqrt{2}\, 3^5}$$

so,

$$P(t) = \frac{e^2 E_0^2 a^2 2^{15}}{3^{10}} \left| \int_0^\infty e^{-(i\omega_{21} + \gamma)t'} dt' \right|^2$$

$$\Rightarrow \boxed{P(t) = \frac{e^2 a^2 E_0^2 \, 2^{15}}{3^{10} (\omega_{21}^2 + \gamma^2)}} \quad , \omega_{21} = \frac{3\alpha^2 m}{8}$$

QM S'04 #3

$$\psi(x) = N e^{-Kx^2/2} \quad ; \quad K > 0$$

a) $N = ?$

$$1 = \int_{-\infty}^{\infty} N e^{-Kx^2/2} N e^{-Kx^2/2} dx = N^2 \int_{-\infty}^{\infty} e^{-Kx^2} dx = N^2 \sqrt{\frac{\pi}{K}}$$

$$\uparrow$$

$$2\int_0^{\infty} e^{-ax^2} dx = \sqrt{\frac{\pi}{a}}$$

$$\Rightarrow \quad N = \left(\frac{K}{\pi}\right)^{1/4}$$

b) $$\langle x^2 \rangle = N^2 \int_{-\infty}^{\infty} x^2 e^{-Kx^2} dx = \frac{N^2 \sqrt{\pi}}{2K^{3/2}} = \frac{K^{1/2}}{\sqrt{\pi}} \frac{\sqrt{\pi}}{2K^{3/2}} = \frac{K^{-1}}{2} = \frac{1}{2K}$$

$$2\int_0^{\infty} x^m e^{-ax^2} dx = \frac{\Gamma[(m+1)/2]}{a^{(m+1)/2}} \quad ; \; m = 2; \quad a = K$$

$$= \frac{\sqrt{\pi}}{2K^{3/2}}$$

$$\langle x^2 \rangle = \frac{1}{2K}$$

c) $$\langle p | \psi \rangle = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} e^{-\frac{ipx}{\hbar}} \psi(x) dx = \frac{N}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} e^{-\frac{ipx}{\hbar}} e^{-Kx^2/2} dx$$

$$= \frac{N}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} e^{-\left(\frac{K}{2}x^2 + \frac{ip}{\hbar}x\right)} dx$$

Let $y = \sqrt{a}\,x + \frac{b}{2\sqrt{a}} \Rightarrow y^2 = ax^2 + bx + \frac{b^2}{4a}$; $dy = \sqrt{a}\,dx$
$dx = \frac{dy}{\sqrt{a}}$

$a = \frac{K}{2}$; $b = \frac{ip}{\hbar}$ This is completing the square

$$\langle p|\psi\rangle = \frac{N}{\sqrt{2\pi\hbar}}\frac{e^{b^2/4a}}{\sqrt{a}}\underbrace{\int_{-\infty}^{\infty} e^{-y^2}dy}_{\sqrt{\pi}} = \frac{N}{\sqrt{2\pi a\hbar}}\sqrt{\pi}\, e^{b^2/4a}$$

$$= \frac{(K/\pi)^{1/4}}{\sqrt{2a\hbar}} e^{b^2/4a} \quad ; \quad a = \frac{K}{2} \; ; \; b = \frac{ip}{\hbar}$$

so $\dfrac{b^2}{4a} = \dfrac{(ip/\hbar)^2}{2\cancel{4}\frac{K}{\cancel{2}}} = \boxed{\dfrac{-p^2}{2\hbar^2 K}}$ and $\dfrac{(K/\pi)^{1/4}}{\sqrt{\hbar}\,K^{1/2}} = \dfrac{1}{\sqrt{\hbar}(\pi K)^{1/4}}$

$$\therefore \quad \langle p|\psi\rangle = \frac{e^{-\frac{p^2}{2\hbar^2 K}}}{\sqrt{\hbar}(\pi K)^{1/4}}$$

d) $\langle p^2\rangle = \dfrac{1}{\hbar\sqrt{\pi K}}\displaystyle\int_{-\infty}^{\infty} p^2 e^{-p^2/\hbar^2 K}dp = \dfrac{1}{\hbar\sqrt{\pi K}}\dfrac{\sqrt{\pi}(\hbar^2 K)^{3/2}}{2} = \dfrac{\hbar^2 K}{2}$

$$2\int_0^{\infty} x^m e^{-ax^2}dx = \frac{\Gamma[(m+1)/2]}{a^{(m+1)/2}} = \frac{\frac{\sqrt{\pi}}{2}}{\left(\frac{1}{\hbar^2 K}\right)^{3/2}} = \frac{\sqrt{\pi}}{2}(\hbar^2 K)^{3/2}$$

e) $H\psi = E\psi$ need to assume this:

$$p = -i\hbar\frac{d}{dx} \Rightarrow p^2 = -\hbar^2\frac{d^2}{dx^2}$$

$$\frac{p^2}{2m}\psi + V(x)\psi = E\psi \Rightarrow \frac{-\hbar^2}{2m}\frac{d^2}{dx^2}\psi + V(x)\psi = E\psi$$

$$\frac{d^2}{dx^2}\psi = \frac{d}{dx}\frac{d}{dx}\psi = \frac{d}{dx}\left(\frac{d}{dx}Ne^{-Kx^2/2}\right) = N\frac{d}{dx}\left(-Kxe^{-Kx^2/2}\right) = -KN\left(-Kx^2e^{-Kx^2/2} + e^{-Kx^2/2}\right)$$

$$= -K(-Kx^2+1)Ne^{-Kx^2/2} = -K(-Kx^2+1)\psi$$

$$\Rightarrow \frac{-\hbar^2}{2m}\left[-K(-Kx^2+1)\right]\cancel{\psi} + V(x)\cancel{\psi} = E\cancel{\psi} \Rightarrow -\frac{\hbar^2K^2x^2}{2m} + \frac{\hbar^2K}{2m} + V(x)\cancel{\psi} = E$$

so $V(x) - \dfrac{\hbar^2K^2x^2}{2m} = E - \dfrac{\hbar^2}{2m}K$ for all x $\therefore$

or $V(x) - \dfrac{\hbar^2K^2}{2m}x^2 = \text{const.} \Rightarrow V(x) = \dfrac{\hbar^2K^2}{2m}x^2 + \text{const.}$

same

$E(x) - \dfrac{\hbar^2}{2m}K = \text{const.} \Rightarrow E(x) = \dfrac{\hbar^2}{2m}K + \text{const.}$

Spring 2004 #3

$\psi(x) = N e^{-kx^2/2}$ $\qquad k > 0$

a) find N

$1 = N^2 \int_{-\infty}^{\infty} e^{-kx^2} dx = N^2 \sqrt{\pi/k} \qquad N^2 = \sqrt{k/\pi} \qquad N = (k/\pi)^{1/4}$

b) $\langle \psi | x^2 | \psi \rangle = N^2 \int_{-\infty}^{\infty} e^{-kx^2} x^2 dx = -N^2 \int_{-\infty}^{\infty} \frac{\partial}{\partial k} e^{-kx^2} dx$

$= -N^2 \frac{\partial}{\partial k} \sqrt{\pi/k} = -N^2 \sqrt{\pi} \cdot (-1/2)(k)^{-3/2}$

$= \frac{N^2 \sqrt{\pi}}{2(k)^{3/2}}$

c) $\langle p | \psi \rangle \overset{\hbar = 1}{=} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \psi(x) e^{-ipx} dx = \frac{N}{\sqrt{2\pi}} \int e^{-\frac{kx^2}{2}} e^{-ipx} dx$

$= \frac{N}{\sqrt{2\pi}} \int e^{-\frac{kx^2}{2} - ipx} dx = \frac{e^{-p^2/\cancel{4}k} \sqrt{\pi\, 2}}{\sqrt{k}} \frac{N}{\sqrt{2\pi}} = \frac{N e^{-p^2/2k}}{\sqrt{k}}$

$\int_{-\infty}^{\infty} e^{-(at^2 + bt + c)} dt = \sqrt{\pi/a}\, e^{\left(\frac{b^2 - 4ac}{4a}\right)} \qquad a = \frac{k}{2} \quad b = ip$

d) $\langle \psi | p^2 | \psi \rangle = \frac{N^2}{k} \int_{-\infty}^{\infty} e^{-p^2/k} p^2 dp = \frac{N^2}{k} \frac{\sqrt{\pi}}{2(\alpha)^{3/2}} = \frac{N^2}{k} \frac{\sqrt{\pi}}{2} (k)^{3/2}$

$\frac{1}{k} = \alpha \qquad \frac{1}{\alpha} = k$

$e^{-\alpha p^2}$

$= \frac{N^2 \sqrt{\pi}}{2k} (k)^{3/2}$

e) $H = \frac{p^2}{2m} + V(X)$

$\psi(x)$ has form of ψ_0 of harmonic ossilator.

$\Rightarrow m\omega = k \qquad \omega^2 = \frac{k^2}{m^2} \qquad \omega = \frac{k}{m}$

$V(x) = \frac{1}{2} k \omega x^2 = \frac{1}{2} \frac{k^2 x^2}{m} = \frac{k^2 x^2}{2m}$

The normalized wave function of a one-dimensional particle is

$$\psi(x) = N e^{-kx^2/2}$$

for some $k>0$, N is real and positive.

(a) what is N?

use normalization condition to find N.

$$1 = |N|^2 \int_{-\infty}^{\infty} e^{-kx^2}\,dx = |N|^2 \sqrt{\frac{\pi}{k}} \Rightarrow \boxed{N = \left(\frac{k}{\pi}\right)^{1/4}}$$

(b) what is expectation value of x^2?

$$\langle x^2 \rangle = |N|^2 \int_{-\infty}^{\infty} x^2 e^{-kx^2}\,dx = |N|^2 \frac{\sqrt{\pi}}{2k^{3/2}} = \sqrt{\frac{k}{\pi}}\,\frac{\sqrt{\pi}}{2k^{3/2}} = \boxed{\frac{1}{2k}}$$

(c) what is the momentum space wave function $\langle p|\psi\rangle$?

(see Abers eq 2.192) ($\hbar = 1$)

$$\langle p|\psi\rangle = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ipx}\,\psi(x)\,dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ipx} N e^{-kx^2/2}\,dx$$

$$= \frac{N}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left(\frac{k}{2}x^2 + ipx\right)}\,dx$$

note: $\displaystyle\int_{-\infty}^{\infty} e^{-(at^2 - bt + c)}\,dt = \sqrt{\frac{\pi}{a}}\,\exp\left(\frac{b^2 - 4ac}{4a}\right)$

So,

$$\langle p|\psi\rangle = \frac{N}{\sqrt{2\pi}} \sqrt{\frac{2\pi}{k}}\, e^{-p^2/2k} = \left(\frac{k}{\pi}\right)^{1/4} \frac{1}{k^{1/2}}\, e^{-p^2/2k}$$

$$\therefore \boxed{\langle p|\psi\rangle = \frac{1}{(\pi k)^{1/4}}\, e^{-p^2/2k}}$$

(d) what is the expectation value of p^2? -- I assume they mean in x-space

$$\langle p^2 \rangle = -|N|^2 \int_{-\infty}^{\infty} e^{-Kx^2/2} \frac{\partial^2}{\partial x^2} e^{-Kx^2/2} \, dx = -|N|^2 \int_{-\infty}^{\infty} e^{-Kx^2/2} \frac{d}{dx} (-Kx) e^{-Kx^2/2} \, dx$$

$$= |N|^2 K \int_{-\infty}^{\infty} e^{-Kx^2/2} \left(e^{-Kx^2/2} - Kx^2 e^{-Kx^2/2} \right) dx$$

$$= |N|^2 K \int_{-\infty}^{\infty} e^{-Kx^2} \left(1 - Kx^2 \right) dx = |N|^2 K \left[\sqrt{\frac{\pi}{K}} - K \frac{\sqrt{\pi}}{2K^{3/2}} \right]$$

$$= |N|^2 K \left(\frac{1}{2} \sqrt{\frac{\pi}{K}} \right) = \frac{1}{2} \sqrt{\frac{K}{\pi}} \, K \sqrt{\frac{\pi}{K}} = \frac{K}{2}$$

$$\therefore \boxed{\langle p^2 \rangle = \frac{K}{2}}$$

(e) The hamiltonian is

$$H = \frac{p^2}{2m} + V(x)$$

what is the potential $V(x)$?

the wave function given is the ground state wave function for a harmonic oscillator with $K \Rightarrow m\omega$ in this case. So,

$$\boxed{V(x) = \frac{1}{2} K x^2 = \frac{m\omega}{2} x^2}$$

Spring 2004. #4

$|\nu_e\rangle = \cos\theta|\nu_1\rangle + \sin\theta|\nu_2\rangle \qquad \hbar = 1$

$|\nu_\mu\rangle = -\sin\theta|\nu_1\rangle + \cos\theta|\nu_2\rangle$

$H|\nu_1\rangle = \sqrt{p^2c^2 + m_1^2c^4}\,|\nu_1\rangle$

$H|\nu_2\rangle = \sqrt{p^2c^2 + m_2^2c^4}\,|\nu_2\rangle$

$|\psi(0)\rangle = |\nu_\mu\rangle$

$|\psi(t)\rangle = -e^{-i\sqrt{p^2c^2+m_1^2c^4}\,t}\sin\theta|\nu_1\rangle + e^{-i\sqrt{p^2c^2+m_2^2c^4}\,t}\cos\theta|\nu_2\rangle$

in limit where $ct = L$

$\langle\nu_e|\psi(t)\rangle = -\sin\theta\cos\theta\, e^{-i\sqrt{p^2c^2+m_1^2c^4}\,t} + \sin\theta\cos\theta\, e^{-i\sqrt{p^2c^2+m_2^2c^4}\,t}$

$= \sin\theta\cos\theta\left(-e^{-iL\sqrt{p^2+m_1^2c^2}} + e^{-iL\sqrt{p^2+m_2^2c^2}}\right)$

But $iL\sqrt{p^2+m^2c^2} = iLp\sqrt{1+\frac{m^2c^2}{p^2}} \approx 1 + \frac{m^2c^2}{2p^2}$

$= \sin\theta\cos\theta\left(e^{-iLp}\left(-e^{-iLp\frac{m_1^2c^2}{2p^2}} + e^{-iLp\frac{m_2^2c^2}{2p^2}}\right)\right)$

$P(t\to L) = |\langle\nu_e|\psi(t)\rangle|^2 = \sin^2\cos^2\theta\left(-e^{iLp\frac{m_1^2c^2}{2p^2}} + e^{iLp\frac{m_2^2c^2}{2p^2}}\right)\left(-e^{-iLp\frac{m_1^2c^2}{2p^2}} + e^{-iLp\frac{m_2^2c^2}{2p^2}}\right)$

$= \sin^2\theta\cos^2\theta\left(+1 + 1 - e^{-\frac{iLpc^2}{2p^2}(m_2^2-m_1^2)} - e^{-\frac{iLpc^2}{2p^2}(m_1^2-m_2^2)}\right)$

$= \sin^2\theta\cos^2\theta\left(-e^{+\frac{iLc^2}{2p}\Delta m^2} - e^{-\frac{iLc^2}{2p}\Delta m^2} + 2\right)$

$\Delta m^2 = m_1^2 - m_2^2$

$= \sin^2\theta\cos^2\theta\left(2 - 2\cos\left(\frac{Lc^2\Delta m^2}{2p}\right)\right]$

$= \frac{\sin^2(2\theta)}{4}\,\cancel{4}\,\sin\left(\frac{Lc^2\Delta m^2}{4p}\right)$ $\Downarrow$

$$P(t) = \sin^2(2\theta)\, \sin^2\left(\frac{L\, c^2 \Delta m^2}{4p}\right)$$

Spring 2004 # 4 (p 1 of 3)

(→ see also Fall 2001 # 3 and Fall 1999 # 10)

4. *Quantum Mechanics*

The electron neutrino $|\nu_e\rangle$ and the muon neutrino $|\nu_\mu\rangle$ are the possible neutrino states produced and detected in experiments, but they are not necessarily eigenstates of the Hamiltonian. Rather, if the state is known to have momentum p, it is some linear combination of the energy eigenstates $|\nu_1\rangle$ and $|\nu_2\rangle$ of the form

$$|\nu_e\rangle = \cos\theta|\nu_1\rangle + \sin\theta|\nu_2\rangle$$
$$|\nu_\mu\rangle = -\sin\theta|\nu_1\rangle + \cos\theta|\nu_2\rangle$$

where

$$H|\nu_1\rangle = \sqrt{p^2c^2 + m_1^2c^4}\,|\nu_1\rangle$$
$$H|\nu_2\rangle = \sqrt{p^2c^2 + m_2^2c^4}\,|\nu_2\rangle$$

for two possibly different masses m_1 and m_2, and some "mixing angle" θ. If it is known that a neutrino was definitely a ν_μ when it was produced, what is the probability of detecting a ν_e after it has traveled a distance L? Assume that $m_1c \ll p$ and $m_2c \ll p$, so that the neutrinos are moving at almost (or even exactly) the speed of light, (so you can ignore corrections of the order $1 - v/c$ compared to terms of order 1) and state your result to first order in the difference $\Delta m^2 = m_1^2 - m_2^2$.

> This is a simplified version of an actual neutrino oscillation experiment like the super-Kamiokande detector experiment a few years ago. In reality there is a third neutrino $|\nu_\tau\rangle$.

let $\hbar = 1$.

we are given that the state at $t=0$ is

$$|\psi(t=0)\rangle = |\nu_\mu\rangle = -\sin\theta\,|\nu_1\rangle + \cos\theta\,|\nu_2\rangle$$

applying the time evolution operator yields

$$|\psi(t)\rangle = e^{-iHt}\,|\psi(t=0)\rangle$$ ↳ the energies are given in the problem

$$\Rightarrow \boxed{|\psi(t)\rangle = -e^{-iE_1t}\sin\theta\,|\nu_1\rangle + e^{-iE_2t}\cos\theta\,|\nu_2\rangle}$$

where $E_1 = \sqrt{p^2c^2 + m_1^2c^4} = \sqrt{p^2 + m_1^2}$ ↳ natural units

$$E_2 = \sqrt{p^2 + m_2^2}$$

Spring 2004 #4 (p 2 of 3)

) the probability of detecting a ν_e at a later time t is given by the magnitude square of the projection of $|\nu_e\rangle$ onto $|\Psi(t)\rangle$ A-1ii so

$$P(t) = |\langle \nu_e | \Psi(t) \rangle|^2$$

$$\Rightarrow P(t) = \left| \left(\cos\theta \langle \nu_1 | + \sin\theta \langle \nu_2 | \right) \left(-e^{-iE_1 t} \sin\theta |\nu_1\rangle + e^{-iE_2 t} \cos\theta |\nu_2\rangle \right) \right|^2$$

since $\langle \nu_i | \nu_j \rangle = \delta_{ij}$, we have

$$P(t) = \left| -e^{-iE_1 t} \cos\theta \sin\theta + e^{-iE_2 t} \sin\theta \cos\theta \right|^2$$

$$= \left| \cos\theta \sin\theta \left(e^{-iE_2 t} - e^{-iE_1 t} \right) \right|^2$$

) note: $\cos\theta \sin\theta = \frac{1}{2} \sin 2\theta$

and $$\left| e^{-iE_2 t} - e^{-iE_1 t} \right|^2 = \left(e^{-iE_2 t} - e^{-iE_1 t} \right) \left(e^{iE_2 t} - e^{+iE_1 t} \right)$$

$$= 1 - e^{-i(E_2 - E_1)t} - e^{+i(E_2 - E_1)t} + 1$$

$$= 2 - 2\cos\left[(E_2 - E_1) t \right]$$

So,

$$P(t) = \frac{1}{4} \sin^2(2\theta) \left[2 - 2\cos\left[(E_2 - E_1)t \right] \right] = \boxed{\frac{\sin^2(2\theta)}{2} \left(1 - \cos\left[(E_2 - E_1)t \right] \right)}$$

where

$$E_2 - E_1 = \sqrt{p^2 + m_2^2} - \sqrt{p^2 + m_1^2} = p\left[\sqrt{1 + \left(\frac{m_2}{p}\right)^2} - \sqrt{1 + \left(\frac{m_1}{p}\right)^2} \right]$$

) Now, we are told that $m_1 \ll p$ and $m_2 \ll p$. So, using binomial expansion, we have

$$E_2 - E_1 \simeq p\left[1 + \frac{1}{2}\left(\frac{m_2}{p}\right)^2 - 1 - \frac{1}{2}\left(\frac{m_1}{p}\right)^2 \right]$$

$$\Rightarrow E_2 - E_1 \simeq \frac{1}{2p}\left(m_2^2 - m_1^2\right)$$

Substituting this result into our expression for $P(t)$ yields

$$\boxed{P(t) = \frac{\sin^2(2\theta)}{2}\left[1 - \cos\left[\frac{t}{2p}(m_2^2 - m_1^2)\right]\right]}$$

the time it takes to trave some distance L is given by

$$t = \frac{L}{c} = L \text{ (in natural units)}$$

so,

$$P(L) = \frac{\sin^2(2\theta)}{2}\left[1 - \cos\left[\frac{L}{2p}(m_2^2 - m_1^2)\right]\right]$$

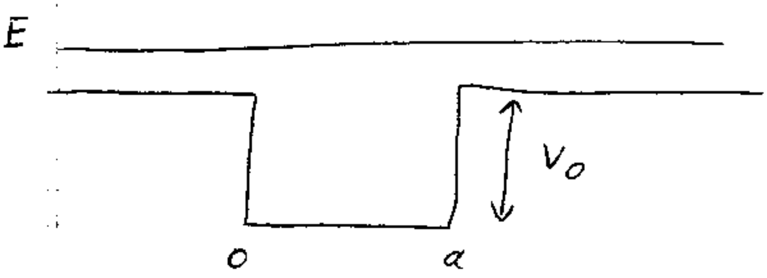
note $\sin^2\theta = \frac{1}{2} - \frac{1}{2}\cos 2\theta \Rightarrow 2\sin^2(\theta) = 1 - \cos(2\theta)$

Thus

$$\boxed{P(L) = \sin^2(2\theta)\sin^2\left[\frac{L}{4p}(m_2^2 - m_1^2)\right]}$$

→ this is the probability that a neutrino starting off as a muon neutrino will change flavors to an electron neutrino after a distance L is traveled ..

QM S'04 #5



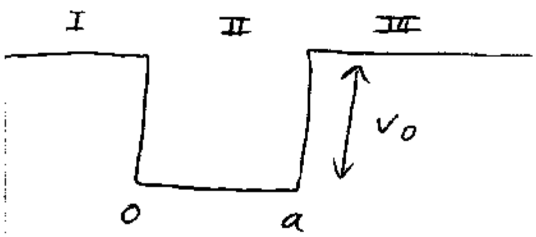
$$T^{-1} = 1 + \frac{V_0^2}{4E(E+V_0)} \sin^2\left(\frac{a}{\hbar}\sqrt{2m(E+V_0)}\right)$$

resonance is when $T = 1$ which happens when

$$\frac{a}{\hbar}\sqrt{2m(E+V_0)} = n\pi \quad \text{or} \quad 2m(E+V_0) = \frac{n^2\pi^2\hbar^2}{a^2}$$

$$\boxed{E + V_0 = \frac{n^2\pi^2\hbar^2}{2ma^2}}$$

Or the derivation:



$E > 0$

I) $V(x) = 0 \Rightarrow -\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi = E\psi$

$$\frac{d^2}{dx^2}\psi = \frac{-2mE}{\hbar^2}\psi \qquad k_1 \equiv \frac{\sqrt{2mE}}{\hbar}$$

$$= -k_1^2\psi$$

$$\Rightarrow \psi(x) = Ae^{ik_1x} + Be^{-ik_1x} \quad \text{for} \quad x < 0$$

by a similar argument for region III we end up with

$$\psi(x) = F e^{ik_1x} + G e^{-ik_1x} \qquad x > a$$

For region II $\quad V(x) = -V_0$

$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi - V_0\psi = E\psi \Rightarrow \frac{d^2}{dx^2}\psi = -\frac{2m(E+V_0)}{\hbar^2}\psi \; ; \; k_2 \equiv \frac{\sqrt{2m(E+V_0)}}{\hbar}$$

$$\frac{d^2}{dx^2}\psi = -k_2^2\psi \Rightarrow \quad \psi(x) = Ce^{ik_2x} + De^{-ik_2a} \qquad \text{for } \; 0<x<a$$

In summary

$$\psi(x) = \begin{cases} Ae^{ik_1x} + Be^{-ik_1x} & x<0 \\ Ce^{ik_2x} + De^{-ik_2x} & 0<x<a \\ Fe^{ik_1x} + Ge^{-ik_1x} & x>a \end{cases}$$

Now we need to match up the wavefunctions on the boundary conditions:

$x=0$

$$A + B = C + D \qquad (1)$$

$$\left.\frac{d}{dx}\right|_{x=0} \qquad \cancel{i}k_1A - \cancel{i}k_1B = \cancel{i}k_2C - \cancel{i}k_2D \qquad (2)$$

$$(2)/k_1 \qquad A - B = \frac{k_2}{k_1}(C - D) \qquad (2')$$

$$(1)+(2') \qquad 2A = \left(1+\frac{k_2}{k_1}\right)C + \left(1-\frac{k_2}{k_1}\right)D$$

$$A = \frac{1}{2}\left\{\left(1+\frac{k_2}{k_1}\right)C + \left(1-\frac{k_2}{k_1}\right)D\right\}$$

$$(1)-(2') \qquad 2B = \left(1-\frac{k_2}{k_1}\right)C + \left(1+\frac{k_2}{k_1}\right)D$$

$$B = \frac{1}{2}\left\{\left(1-\frac{k_2}{k_1}\right)C + \left(1+\frac{k_2}{k_1}\right)D\right\}$$

QM S'04 #5

So we have

$$\begin{pmatrix} A \\ B \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \left(1+\frac{k_2}{k_1}\right) & \left(1-\frac{k_2}{k_1}\right) \\ \left(1-\frac{k_2}{k_1}\right) & \left(1+\frac{k_2}{k_1}\right) \end{pmatrix}\begin{pmatrix} C \\ D \end{pmatrix}$$

Now for the other boundary:

$x=a$

$$Ce^{ik_2a} + De^{-ik_2a} = Fe^{ik_1a} + Ge^{-ik_1a} \qquad (3)$$

$$\left.\frac{d}{dx}\right|_{x=a} \quad ik_2Ce^{ik_2a} - ik_2De^{-ik_2a} = ik_1Fe^{ik_1a} - ik_1Ge^{-ik_1a} \qquad (4)$$

$$(4)/k_2 \qquad Ce^{ik_2a} - De^{-ik_2a} = \frac{k_1}{k_2}\left(Fe^{ik_1a} - Ge^{-ik_1a}\right) \qquad (4')$$

$$(3)+(4) \qquad 2Ce^{ik_2a} = \left(1+\frac{k_1}{k_2}\right)Fe^{ik_1a} + \left(1-\frac{k_1}{k_2}\right)Ge^{-ik_1a}$$

$$C = \frac{1}{2}\left\{\left(1+\frac{k_1}{k_2}\right)Fe^{i(k_1-k_2)a} + \left(1-\frac{k_1}{k_2}\right)Ge^{-i(k_1+k_2)a}\right\}$$

$$(3)-(4) \qquad 2De^{-ik_2a} = \left(1-\frac{k_1}{k_2}\right)Fe^{ik_1a} + \left(1+\frac{k_1}{k_2}\right)Ge^{-ik_1a}$$

$$D = \frac{1}{2}\left\{\left(1-\frac{k_1}{k_2}\right)Fe^{i(k_1+k_2)a} + \left(1+\frac{k_1}{k_2}\right)Ge^{i(k_2-k_1)a}\right\}$$

So

$$\begin{pmatrix} C \\ D \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \left(1+\frac{k_1}{k_2}\right)e^{i(k_1-k_2)a} & \left(1-\frac{k_1}{k_2}\right)e^{-i(k_1+k_2)a} \\ \left(1-\frac{k_1}{k_2}\right)e^{i(k_1+k_2)a} & \left(1+\frac{k_1}{k_2}\right)e^{i(k_2-k_1)a} \end{pmatrix}\begin{pmatrix} F \\ G \end{pmatrix}$$

Combining the two matrices in order to get A, B in terms of F, G:

$$\begin{pmatrix} A \\ B \end{pmatrix} = \frac{1}{2}\begin{pmatrix} (1+\frac{k_2}{k_1}) & (1-\frac{k_2}{k_1}) \\ (1-\frac{k_2}{k_1}) & (1+\frac{k_2}{k_1}) \end{pmatrix} \frac{1}{2}\begin{pmatrix} (1+\frac{k_1}{k_2})e^{i(k_1-k_2)a} & (1-\frac{k_1}{k_2})e^{-i(k_1+k_2)a} \\ (1-\frac{k_1}{k_2})e^{i(k_1+k_2)a} & (1+\frac{k_1}{k_2})e^{-i(k_2-k_1)a} \end{pmatrix}\begin{pmatrix} F \\ G \end{pmatrix}$$

Now what we care for is the transmission coefficient which is:

$$T = \frac{|F|^2}{|A|^2} \quad \text{or} \quad T^{-1} = \frac{|A|^2}{|F|^2}$$

now the latter is more useful as the above matrix has A in terms of F. So we need to multiply the first row by the first column to get what we want:

$$A = \frac{1}{4}\left[(1+\frac{k_2}{k_1})(1+\frac{k_1}{k_2})e^{i(k_1-k_2)a} + (1-\frac{k_2}{k_1})(1-\frac{k_1}{k_2})e^{i(k_1+k_2)a}\right]F$$

$$= \frac{e^{ik_1}}{4}\left[(1+\frac{k_1}{k_2}+\frac{k_2}{k_1}+1)e^{-ik_2a} + (1-\frac{k_1}{k_2}-\frac{k_2}{k_1}+1)e^{ik_2a}\right]F$$

$$= \frac{e^{ik_1}}{4}\left[\underbrace{2e^{-ik_2a}} + \left(\frac{k_1}{k_2}+\frac{k_2}{k_1}\right)e^{-ik_2a} + \underbrace{2e^{ik_2a}} - \left(\frac{k_1}{k_2}+\frac{k_2}{k_1}\right)e^{ik_2a}\right]F$$

$$= \frac{e^{ik_1}}{4}\left[4\cos(k_2a) \mp \left(\frac{k_1}{k_2}+\frac{k_2}{k_1}\right)\underbrace{\left[e^{-ik_2a} - e^{-ik_2a}\right]}_{2i\sin(k_2a)}\right]F$$

$$\text{Now } |A|^2 = \frac{\overbrace{e^{ik_1}e^{-ik_1}}^{=1}}{16}\left[16\cos^2(k_2a) - \left(\frac{k_1}{k_2}+\frac{k_2}{k_1}\right)\left[\cancel{2i\sin(k_2a)}\cos(k_2a) - 2i\cancel{\sin(k_2a)}\cos(k_2a)\right] + \left(\frac{k_1}{k_2}+\frac{k_2}{k_1}\right)^2 4\sin^2(k_2a)\right]|F|^2$$

QM S'04 #5

So $|A|^2 = \frac{1}{16}\left[16\cos^2(k_2 a) + 4\frac{(k_1^2+k_2^2)^2}{k_1^2 k_2^2}\sin^2(k_2 a)\right]|F|^2$

$$= \left[\cos^2(k_2 a) + \frac{(k_1^2+k_2^2)^2}{4k_1^2k_2^2}\sin^2(k_2 a)\right]|F|^2$$

$$= \left[\frac{4k_1^2k_2^2\cos^2(k_2 a) + (k_1^2+k_2^2)^2\sin^2(k_2 a)}{4k_1^2k_2^2}\right]|F|^2$$

$$= \left[\frac{4k_1^2k_2^2 - 4k_1^2k_2^2\sin^2(k_2 a) + k_1^4\sin^2(k_2 a) + k_2^4\sin^2(k_2 a) + 2k_1^2k_2^2\sin^2(k_2 a)}{4k_1^2k_2^2}\right]|F|^2$$

$$= \left[1 + \frac{(k_1^4 + k_2^4 - 2k_1^2k_2^2)\sin^2(k_2 a)}{4k_1^2k_2^2}\right]|F|^2$$

$$= \left[1 + \frac{(k_1^2-k_2^2)^2}{4k_1^2k_2^2}\sin^2(k_2 a)\right]|F|^2$$

So $T^{-1} = \frac{|A|^2}{|F|^2} = 1 + \frac{(k_1^2-k_2^2)^2}{4k_1^2k_2^2}\sin^2(k_2 a)$

now $k_1 = \frac{\sqrt{2mE}}{\hbar}$; $k_2 = \frac{\sqrt{2m(E-V_0)}}{\hbar}$

so $\frac{(k_1^2-k_2^2)^2}{4k_1^2k_2^2} = \frac{\left(\frac{\cancel{2mE} - \cancel{2mE} - 2mV_0}{\hbar^2}\right)^2}{4\left(\frac{4m^2E(E-V_0)}{\hbar^4}\right)} = \frac{\cancel{4m^2}V_0^2}{4(\cancel{4m^2})(E(E-V_0))} = \frac{V_0^2}{4E(E-V_0)}$

$$\sin^2(k_2 a) = \sin^2\left(\frac{a}{\hbar}\sqrt{2m(E-V_0)}\right)$$

hence

$$T^{-1} = 1 + \frac{V_0^2}{4E(E-V_0)}\sin^2\left(\frac{a}{\hbar}\sqrt{2m(E-V_0)}\right)$$

and for $T=1$ $\sin^2(\ldots) = 0$ or $\frac{a}{\hbar}\sqrt{2m(E-V_0)} = n\pi$

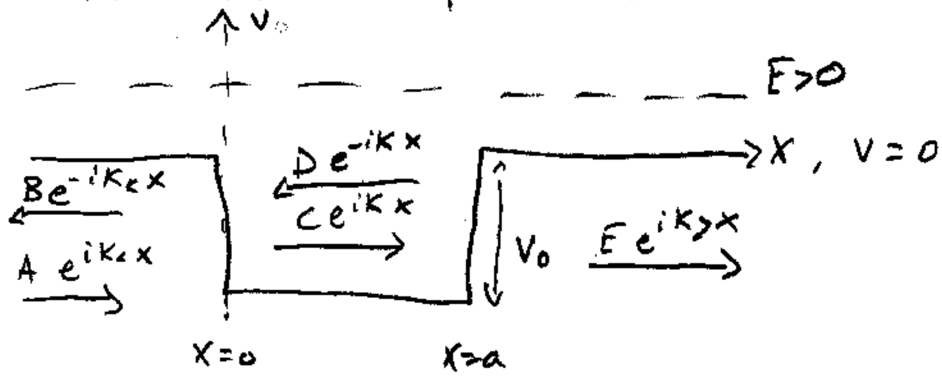
$\Rightarrow$ $E + V_0 = \frac{n^2\pi^2\hbar^2}{2ma^2}$

Spring 2004 #5 (p 1 of 2)

Calculate the transmission coefficient for a particle of energy $E>0$ scattering off the 1 D potential well

$$V(x) = \begin{cases} V_0 & 0 < x < a \\ 0 & \text{elsewhere} \end{cases}$$

where $V_0 < 0$. Are there resonance phenomena?



Solutions are shown on the figure above where

$$K_<^2 = K_>^2 = 2mE$$
$$K^2 = 2m(E + |V_0|)$$

Boundary conditions ($\psi(x)$ and $\frac{d\psi}{dx}$ are continuous at $x=0$ and $x=a$) yields

at $x=0$: $\quad A + B = C + D \quad$ and $\quad K_<(A - B) = K(C - D)$

at $x=a$: $\quad Ce^{iKa} + De^{-iKa} = Ee^{iK_<a} \quad$ and $\quad K(Ce^{iKa} - De^{-iKa}) = K_< E e^{iK_<a}$

solving this system of equations for E yields (see Zettili p 214-215)

$$E = 4K_<K\,A\,e^{-iK_<a}\left[4K_<K\cos(Ka) - 2i\left(K_<^2 + K^2\right)\sin(Ka)\right]^{-1}$$

Since the transmission coefficient is defined as

$$T = \frac{K_<|E|^2}{K_<|A|^2} = \left[1 + \frac{1}{4}\left(\frac{K_<^2 - K^2}{K_<K}\right)^2 \sin^2(Ka)\right]^{-1}$$

Now let's substitute in for $K_<$ and K.

Spring 2004 #5 (p2 of 2)

$$T = \left[1 + \frac{1}{4} \left(\frac{2mE - 2mE - 2m|V_0|}{\sqrt{2mE(2mE + 2m|V_0|)}} \right)^2 \sin^2\left[\sqrt{2m(E+|V_0|)}\, a\right] \right]^{-1}$$

$$= \left[1 + \frac{1}{4} \left(\frac{4m^2 V_0^2}{2mE(2mE+2m|V_0|)} \right) \sin^2\left[\sqrt{2m(E+|V_0|)}\, a\right] \right]^{-1}$$

$$\boxed{T = \left[1 + \frac{|V_0| \sin^2\left[\sqrt{2m(E+|V_0|)}\, a\right]}{4E\left[1 + \left(\frac{E}{|V_0|}\right)\right]} \right]^{-1}}$$

resonance phenomenon occur when the maxima of the transmission coefficient coincides with the energy eigenvalues. This does not occur classically ... it results from a constructive interference between the incident and reflected waves. This phenomenon is observed experimentally when low-energy ($E \sim 0.1\,eV$) electrons scatter off noble atoms (Ramsauer-Townsend effect) and neutrons off nuclei.

So,

when $T=1$, we have resonance. This occurs when $\sin^2[\ ] = 0$

Thus, when

$$\boxed{a\sqrt{2m(E+|V_0|)} = n\pi} \quad , n = 0, 1, 2, 3, \ldots$$

Selected Answers

Spring 2004

6) (a) $\frac{V}{N} \frac{4\pi}{h^3} \int_0^\infty \frac{p^2\,dp}{e^{\beta pc}-1}$ (b) $\frac{A}{N} \frac{2\pi}{h^2} \int_0^\infty \frac{p\,dp}{e^{\beta pc}-1}$ (c) $\frac{L}{N} \frac{2}{h} \int_0^\infty \frac{dp}{e^{\beta(pc-\mu)}-1}$

Stat. Mech. S'04 #6, S'01 # 13

For relativistic bosons

$$E = |\vec{p}|c$$

a) First we need the density of states $D(E)$ for 3-D:

$$\frac{L}{\lambda/2} = n \Rightarrow \frac{2L}{n} = \lambda; \quad p = \frac{h}{\lambda} = \frac{h}{2L} n$$

$$E = c|\vec{p}| = c\left(p_x^2 + p_y^2 + p_z^2\right)^{1/2} = \frac{hc}{2L}\left(n_x^2 + n_y^2 + n_z^2\right)^{1/2}$$

$$= \frac{hc}{2L} n \Rightarrow n = \frac{2L}{hc} E \Rightarrow dn = \frac{2L}{hc} dE$$

$$\frac{(2s+1)}{8}\int_0^\infty 4\pi n^2 dn = \frac{(2s+1)}{8}\int 4\pi\left(\frac{2L}{hc}\right)^3 E^2 dE = \int_0^\infty \underbrace{\frac{(2s+1)4\pi \overset{L^3}{V}}{(hc)^3}} E^2 dE$$

$$D(E) = \frac{(2s+1)4\pi V}{(hc)^3} E^2$$

The condition for BEC is determined by the boson temperature T_B, which can be derived followingly:

$$\int_0^\infty \frac{1}{e^{E/kT}-1} D(E)\,dE = N \quad \text{for } T = T_B$$

$$\Rightarrow \frac{(2s+1)4\pi V}{(hc)^3}\int_0^\infty \frac{E^2}{e^{E/kT}-1} dE = \frac{(2s+1)4\pi V (kT)^3}{(hc)^3} \underbrace{\int_0^\infty \frac{x^2}{e^x - 1} dx}_{=2.404}$$

$$x = E/kT \Rightarrow E = kTx \Rightarrow dE = kT dx$$

hence

$$\frac{(2s+1)\,4\pi V}{(hc)^3}(kT_B)^3\ 2.404 = N$$

$$\Rightarrow\ (kT_B)^3 = \frac{N}{V}\ \frac{(hc)^3}{(2s+1)4\pi\cdot 2.404}$$

$$\Rightarrow\ \boxed{T_B = \left(\frac{N}{k^3 V}\ \frac{(hc)^3}{(2s+1)4\pi\cdot 2.404}\right)^{1/3}}$$

b) yes it does occur - just derive $D(E)$ for 2-D case and repeat above steps:

$$E = \frac{hc}{2L}\left(n_x^2 + n_y^2\right) = \frac{hc}{2L}\,n\ \Rightarrow\ n = \frac{2L}{hc}E\ \Rightarrow\ dn = \frac{2L}{hc}\,dE$$

$$\frac{(2s+1)}{4}\int_0^\infty 2\pi n\,dn = \frac{(2s+1)}{4}\int_0^\infty 2\pi\left(\frac{2L}{hc}\right)E\,dE = \int_0^\infty \underbrace{\frac{(2s+1)2\pi A}{(hc)^2}}\,E\,dE \qquad A = L^2$$

$$D(E) = \frac{(2s+1)2\pi A}{(hc)^2}\,E$$

$$\int_0^\infty \frac{(2s+1)2\pi A}{(hc)^2}\ \frac{E}{e^{E/kT}-1}\,dE = \frac{(2s+1)2\pi A}{(hc)^2}(kT)^2\underbrace{\int_0^\infty \frac{x}{e^x-1}\,dx}_{\frac{\pi^2}{6}} = N$$

$$x \equiv E/kT \Rightarrow E = kTx \Rightarrow dE = kT\,dx$$

so

$$(kT_B)^2 = \frac{N}{A}\ \frac{3(2s+1)}{\pi^3 (hc)^2}\ \Rightarrow\ \boxed{T_B = \left(\frac{N}{k^2 A}\ \frac{3(2s+1)}{\pi^3 (hc)^2}\right)^{1/2}}$$

Stat. Mech. S'04 #6

c) BEC does not occur in 1-D case.

$$E = c|\vec{p}| = \frac{hc}{2L} n \Rightarrow n = \frac{2L}{hc} E \Rightarrow dn = \frac{2L}{hc} dE$$

$$\frac{(2s+1)}{2} \int_0^\infty dn = \int_0^\infty \underbrace{\frac{(2s+1)}{\cancel{2}} \frac{\cancel{2}L}{hc}}_{D(E) = \frac{(2s+1)L}{hc}} dE$$

$$\int_0^\infty \frac{(2s+1)L}{hc} \frac{dE}{e^{-E/kT}-1} = \frac{(2s+1)L}{hc} kT \underbrace{\int_0^\infty \frac{dx}{e^x - 1}}_{\text{undefined}} = \text{undefined.}$$

$x \equiv E/kT \Rightarrow dE = kT dx$

Consider a gas of relativistic, conserved bosons. The relation between energy and momentum is

$$E = |\vec{p}|c$$

(a) Derive the condition for Bose-Einstein condensation in three dimensions.

(See spring 2001 #13 and Erik's wonderful explanation of general BEC problems for massive/massless bosons in d-dimensions)

Since the energy is given by $E = |\vec{p}|c$, we assume that we are talking about massless particles. Let's further assume that they are spin 0, so, the degeneracy is one.

The procedure is to find the transition temperature at which a BEC forms. We can get an expression for the transition temperature from the expression for the total number of bosons.

$$N = \int_0^\infty \bar{n}(\epsilon)\,dN \quad , \quad dN = \underbrace{D(\epsilon)}_{\text{density of states}}\,d\epsilon \qquad (1)$$

The convention used to find the density of states is to take a very large cube (if 3-D) each of side L and force the wave functions representing the bosons to vanish at the walls. This leads to the condition for the quantized wave vector to be

$$k_i = \frac{n_i \pi}{L} \quad , \quad i = x, y, z$$

So, in 3D with $\epsilon = |\vec{p}|c$, we have

$$\epsilon = |\vec{p}|c = \hbar|\vec{k}|c = \frac{\hbar c \pi}{L} n_i$$

Solving for n_i yields

$$n_i = \frac{\epsilon L}{\hbar c \pi}$$

Then the density of states (in n space) is given by (in 3-D)

$$D(\epsilon) = \frac{dN}{d\epsilon} = \frac{dN}{dn}\frac{dn}{d\epsilon} = 4\pi n^2 \frac{dn}{d\epsilon}$$

So,

$$D(\epsilon) = 4\pi\left(\frac{\epsilon L}{\hbar c\pi}\right)^2 \frac{L}{\hbar c\pi} = 4\pi\left(\frac{L}{\hbar c\pi}\right)^3 \epsilon^2$$

Let $\hbar = c = 1$ $\Rightarrow \boxed{D(\epsilon) = \frac{4L^3}{\pi^2}\epsilon^2}$

Substituting this result into eq (1) for the total N yields

$$N_{3D} = \frac{4L^3}{8\pi^2}\int_0^\infty \frac{\epsilon^2}{e^{\beta(\epsilon-\mu)}-1}\,d\epsilon$$

where we used $\bar{n}(\epsilon) = \frac{1}{e^{\beta(\epsilon-\mu)}-1}$ for bosons and a factor of "$\frac{1}{8}$" because we only care about the positive values of the sphere in "n-space" which is ⅛ of the total sphere. So, we have

$$N_{3D} = \frac{L^3}{2\pi^2}\int_0^\infty \frac{\epsilon^2}{e^{\beta(\epsilon-\mu)}-1}\,d\epsilon = \frac{V}{\pi^2\beta^3}\sum_{\ell=1}^{\infty}\frac{e^{\beta\ell\mu}}{\ell^3}$$

Now, N is at a maximum when $\mu = 0$. The maximum is when condensation occurs.

So, $\mu \to 0$

$$\Rightarrow N_{3D} = \frac{V}{\pi^2\beta^3}\zeta(3) \approx \frac{V}{\pi^2\beta^3}1.1202$$

↑ gamma function

Solving for the temperture, T_c, required for a BEC to form, we get

$$\boxed{T_c \approx \frac{1}{K}\left[\frac{\pi^2 N_{3D}}{V(1.1202)}\right]^{1/3}}$$

(b) Does Bose-Einstein condensation occur in two-dimensions? justify your answer.

for massless particles, a BEC does occur in 2D. for massive particles, it does not. Since we are dealing with massless particles, the answer is yes.

in 2D, the density of states is

$$D(\epsilon) = 2\pi n \frac{dn}{d\epsilon} = 2\pi\left(\frac{L}{c\hbar\pi}\right)^2 \epsilon$$

so, the total number of particles is

$$(+\ c = 1 = \hbar) \quad N_{2D} = \frac{2\pi}{4}\left(\frac{L}{\pi}\right)^2 \int_0^\infty \frac{\epsilon}{e^{\beta(\epsilon-\mu)}-1}\,d\epsilon = \frac{A}{2\pi\beta^2}\sum_{l=1}^{\infty}\frac{e^{\beta l \mu}}{l^2}$$

L^2

this factor comes in for the same reason the 1/8 did in the sphere 3-D part. Now the pos. values of n are 1/4 of the area of a circle

when $\mu \to 0$

$$N_{2D} = \frac{A}{2\pi\beta_c^2}\zeta(2) = \frac{A}{2\pi\beta_c^2}\frac{\pi^2}{6}$$

so,

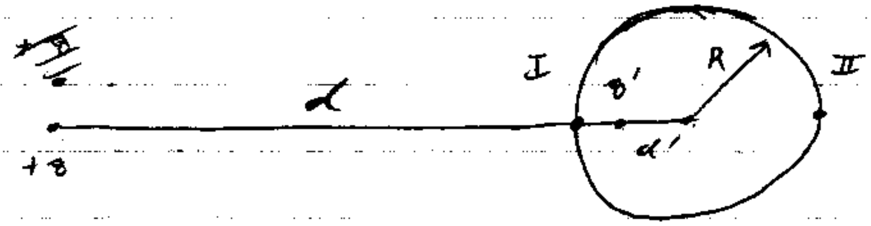
$$\boxed{T_c = \frac{1}{K}\left(\frac{12\ N_{2D}}{A\pi}\right)^{1/2}}$$

(c) what is the highest dimension for which Bose-Einstein condensation does not occur?

for massive particles, 2-D

for massless particles, 1-D

EM S'04 # 8



Step 1 is to determine q' & d':

At I & II $V=0$ for grounded conductor:

	q	q'
I	$d-R$	$R-d'$
II	$d+R$	$R+d'$

$$\text{I} \Rightarrow \frac{q}{d-R} + \frac{q'}{R-d'} = 0 \qquad (1)$$

$$\text{II} \Rightarrow \frac{q}{d+R} + \frac{q'}{R+d'} = 0 \qquad (2)$$

$$(1): \quad \frac{q}{d-R} = \frac{-q'}{R-d'} \Rightarrow q = -q'\frac{(d-R)}{R-d'}$$

plug into (2):

$$\frac{-\cancel{q'}(d-R)}{(R-d')(d+R)} = \frac{-\cancel{q'}}{R+d'} \Rightarrow (d-R)(R+d') = (R-d')(d+R)$$

$$\cancel{dR} + dd' - R^2 - \cancel{Rd'} = \cancel{dR} + R^2 - dd' - \cancel{d'R}$$

$$\Rightarrow \cancel{2}dd' = \cancel{2}R^2 \Rightarrow \boxed{d' = \frac{R^2}{d}}$$

plug this back into (1)

$$q' = -\frac{q(R-d')}{(d-R)} = \frac{-q\left(R - \frac{R^2}{d}\right)}{(d-R)} \times \frac{d}{d} = -q\frac{(Rd - R^2)}{d(d-R)} = \frac{-qR}{d}\frac{\cancel{(d-R)}}{\cancel{(d-R)}}$$

$$\boxed{q' = -\frac{qR}{d}}$$

Now for the force between q & q'

$$F = \frac{1}{4\pi\epsilon_0}\frac{qq'}{(d-d')^2} = \frac{-q^2}{4\pi\epsilon_0}\frac{R}{d\left(d - \frac{R^2}{d}\right)^2} = \boxed{\frac{-q^2}{4\pi\epsilon_0}\frac{Rd}{(d^2-R^2)^2}}$$

Now we place a charge $Q - q'$ on the conductor (after removing the ground). Then the force on $+q$ will be the old force plus the new force due to the charged sphere:

$$F = F_{old} + F_{new} = \frac{-q^2 R d}{4\pi\epsilon_0 (d^2 - R^2)^2} + \frac{q}{4\pi\epsilon_0} \frac{(Q - q')}{d^2}$$

$$= \frac{-q^2 R d}{4\pi\epsilon_0 (d^2 - R^2)^2} + \frac{q}{4\pi\epsilon_0} \frac{(Q + \frac{qR}{d})}{d^2}$$

Now we want the force on $+q$ to be 0:

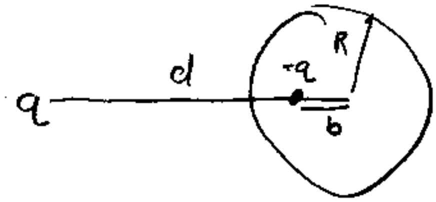
$$\frac{\cancel{q}^2 R d}{\cancel{4\pi\epsilon_0}(d^2 - R^2)^2} = \frac{\cancel{q}}{\cancel{4\pi\epsilon_0}} \frac{(Q + \frac{qR}{d})}{d^2} \Rightarrow \frac{qRd}{(d^2 - R^2)^2} = \frac{Q}{d^2} + \frac{qR}{d^3}$$

$$\Rightarrow \quad Q = d^2 \left[\frac{qRd}{(d^2 - R^2)^2} - \frac{qR}{d^3} \right] = \cancel{d^2} \left[\frac{qRd^4 - qR(d^2 - R^2)^2}{d^{\cancel{3}}(d^2 - R^2)^2} \right]$$

$$= \frac{qRd^4 - qR(d^4 - 2d^2R^2 + R^4)}{d(d^2 - R^2)^2} = \frac{\cancel{qRd^4} - \cancel{qRd^4} + 2qd^2R^3 - qR^5}{d(d^2 - R^2)^2}$$

$$\boxed{Q = q \left[\frac{2d^2R^3 - R^5}{d(d^2 - R^2)^2} \right]}$$

Spring 2004 # . 8



$$q' = \frac{-Rq}{d} \qquad b = \frac{R^2}{d}$$

what charge must sphere have for force on point charge to be zero

This is the same as if you had a charge $(Q - q')$ at the origin since the charge is uniformly spread.
so that total enclosed charge is Q

$$F = \frac{q q'}{(d-b)^2} + \frac{q(Q-q')}{d^2} = \frac{-q^2}{(d - \frac{R^2}{d})^2} + \frac{q(Q + \frac{Rq}{d})}{d^2}$$

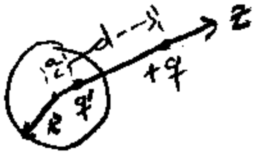
$$= \frac{-q^2 R d}{(d^2 - R^2)^2} + \frac{q}{d^2}\left(Q + \frac{Rq}{d}\right) = 0$$

$$\frac{qQ}{d^2} = \frac{q^2 R d}{(d^2-R^2)^2} - \frac{q^2 R}{d^3}$$

$$Q = \frac{qRd^3}{(d^2-R^2)^2} - \frac{qR}{d} = \frac{qRd^4 - qR(d^2-R^2)^2}{d(d^2-R^2)^2}$$

$$\boxed{Q = \frac{qR^3(2d^2 - R^2)}{d(d^2 - R^2)}}$$

A point charge q is located a distance d from the center of a conducting sphere of radius R. What must the total charge on the conducting sphere be for the force on the point charge to be zero?



we know that with conducting image charge problems with spheres that the location, a, of the image charge and charge, q', is

$$a = \frac{R^2}{d} \quad \text{and} \quad q' = -q\frac{R}{d}$$

→ this applies when the sphere is grounded to have $V=0$ on surface of sphere.

Now the force corresponding to the sphere if grounded is (see Fall 2002 #10(c))

$$F = \frac{qq'}{|d-a|^2} = \frac{-q^2(R/d)}{|d-\frac{R^2}{d}|^2} = \frac{-q^2Rd}{|d^2-R^2|^2}$$

Now, if sphere is not grounded, there is some charge on the surface, $Q-q'$. (see Griffiths' problem 3.8 for a similar problem). So, now the total charge on the surface of the sphere is $(Q-q')+q' = Q$. Thus the force on the charge q is

$$F = \frac{-q^2Rd}{|d^2-R^2|^2} + \frac{q(Q-q')}{d^2} \quad , \quad q' = -q\frac{R}{d}$$

Setting this force equal to zero and solving for Q yields

$$\frac{q^2Rd}{|d^2-R^2|^2} = \frac{qQ}{d^2} + \frac{q^2R}{d^3}$$

Spring 2004 #8 (p 2 of 2)

$$\Rightarrow \quad Q = \frac{d^2}{q}\left[\frac{q^2 R d}{|d^2-R^2|^2} - \frac{q^2 R}{d^3}\right]$$

$$= q\left[\frac{R d^3 \cdot d^3}{d^3|d^2-R^2|^2} - \frac{d^2 R (d^2-R^2)^2}{d^3|d^2-R^2|^2}\right]$$

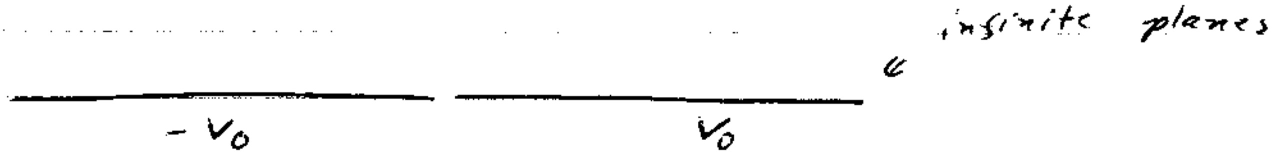
$$= qR\left[\frac{d^6 - d^6 - R^4 d^2 + 2d^4 R^2}{d^3 |d^2-R^2|^2}\right]$$

Thus, the charge must be

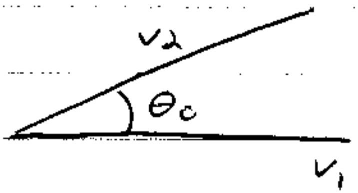
$$\boxed{Q = qR\left[\frac{2d^2R^2 - R^4}{d(d^2-R^2)^2}\right]}$$

EM S'04 #9; S'03 #9

Find the potential above the plane:



This is just a wedge potential problem with the opening angle being $\theta_0 = \pi$:



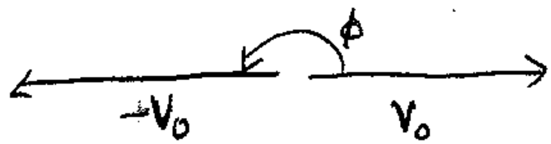
The general solution according to C. Wong is:

$$V(\theta) = A + B\theta \quad ; \quad A = V_1 \quad ; \quad B = \frac{V_2 - V_1}{\theta_0}$$

So we have $V_1 = V_0$; $V_2 = -V_0$ and $\theta_0 = \pi$

$$\boxed{V(\theta) = V_0 + \frac{(-V_0 - V_0)}{\pi}\theta = V_0\left(1 - \frac{2\theta}{\pi}\right)}$$

Consider the infinite two-dimensional conducting plane depicted in the figure. The right half is maintained at electrostatic potential V_0 while the left half is maintained at potential $-V_0$. What is the potential above the plane?



(see Fall 2003 #10, Spring 2003 #9, Spring 2005 #8)

since ϕ is restricted (does not range to 2π), the general solution to the potential is given by

$$\Phi(r,\phi) = (a_0 + b_0 \ln r)(c_0 + d_0 \phi)$$

Now, apply the boundary conditions

- $\Phi(r, \phi = 0) = V_0 = (a_0 + b_0 \ln r)\, c_0$

  since $V_0 \neq V_0(r)$, $b_0 = 0$

  Thus, $$V_0 = a_0 c_0$$

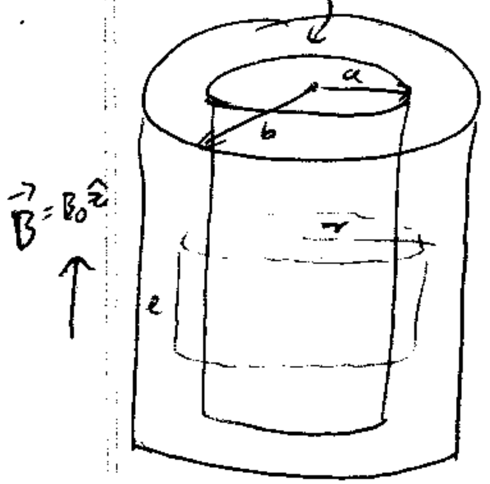
- $\Phi(r, \phi = \pi) = -V_0 = a_0 c_0 + a_0 d_0 \pi = V_0 + a_0 d_0 \pi$

$$\Rightarrow \quad a_0 d_0 = \frac{-2V_0}{\pi}$$

Thus, the potential is

$$\boxed{\Phi(r,\phi) = V_0 - \frac{2V_0}{\pi}\phi = V_0\left(1 - \frac{2}{\pi}\phi\right)}$$

+Q E-M. S'04 #10



$\omega_f = ?$

$\vec{\ell}_{em} = \mu_0 \epsilon_0 (\vec{r} \times \vec{S})$

$\vec{S} = \frac{1}{\mu_0} (\vec{E} \times \vec{B})$

$\frac{Q}{\cancel{2\pi a l}} \cdot \cancel{2\pi a l}$

$\vec{B} = B_0 \hat{z}$; $E = ?$ $\int \vec{E} \cdot d\vec{a} = \frac{Q_{enc}}{\epsilon_0}$

$2\pi r l$

So $\vec{E} = \frac{+Q}{2\pi L \epsilon_0 r} \hat{r}$

R.H.R.

hence $\vec{S} = \frac{1}{\mu_0 \epsilon_0} \frac{Q B_0}{2\pi L r} (-\hat{\phi})$

So $\vec{\ell}_{em} = -\frac{Q B_0}{2\pi L} \hat{z}$ R.H.R ($\frac{1}{r}$ vanished because $\vec{r} = r\hat{r}$

To get the total angular momentum:

$$L = \int_0^L \int_0^{2\pi} \int_a^b \ell_{em} \, r\,dr\,d\phi\,dz = \frac{-Q B_0}{2\pi L} [L][2\pi] \frac{1}{2} r^2 \Big|_a^b$$

$$= -Q B_0 \frac{(b^2 - a^2)}{2}$$

now for a rigid rotator:

$$L = I\omega \rightarrow \vec{\omega} = \frac{L}{I} = \frac{-Q B_0 (b^2 - a^2)}{2 I} \hat{z}$$

moment of inertia

E.M. S'04 # 11

The index of refraction is given by:

$$n = \sqrt{\frac{\mu}{\mu_0}\frac{\epsilon}{\epsilon_0}}$$

but $\mu \approx \mu_0$, so $n \approx \sqrt{\epsilon_r}$

Now for a plasma:

$$\epsilon_r = \frac{\epsilon(\omega)}{\epsilon_0} \approx 1 - \frac{\omega_p^2}{\omega^2} \; ; \quad \omega_p^2 = \frac{\overbrace{NZ}^{n} e^2}{\epsilon_0 m}$$

So

$$\boxed{n(\omega) = \sqrt{1 - \frac{\omega_p^2}{\omega^2}} = \sqrt{1 - \frac{ne^2}{\epsilon_0 m \omega^2}}}$$

Spring 2004 #11

plasma frequency

$$\varepsilon_r = \frac{\varepsilon(\omega)}{\varepsilon_0} = 1 - \frac{\omega_p^2}{\omega^2} \qquad \omega_p = \sqrt{\frac{n e^2}{\varepsilon_0 m_e}}$$

$$n = \sqrt{\frac{\varepsilon}{\varepsilon_0} \frac{\mu}{\mu_0}} \qquad \mu = \mu_0$$

$$\Rightarrow n = \sqrt{\frac{\varepsilon}{\varepsilon_0}} = \sqrt{1 - \frac{n e^2}{\varepsilon_0 m_e \omega^2}} \quad \checkmark$$

Consider a plasma of free charges of mass m and charge e at constant density n. What is the index of refraction for electromagnetic waves of frequency ω which are incident upon this plasma? (see Spring 2003 #10)

the index of refraction of a plasma is given by

$$n = \sqrt{1+\chi_e} \qquad (1)$$

where χ_e can be found from the induced polarization. where

$$P = \chi_e E = np \qquad (2)$$

, p is the dipole moment and n is the density

where p is

$$p = ex \qquad (3)$$

So, what is x? x can be found from the equation of motion. That is, we have

$$m\ddot{x} = eE_0 e^{-i\omega t} = eE \qquad (4)$$

$$\Rightarrow x = x_0 e^{-i\omega t} \qquad \Rightarrow \ddot{x} = -\omega^2 x$$

So, substituting this result back into eq (4) yields

$$m\omega^2 x = -eE \qquad \Rightarrow \qquad x = \frac{-eE}{m\omega^2}$$

substituting this result into eq 3, then p into eq (2) yields

$$P = \chi_e E = \frac{-ne^2 E}{m\omega^2}$$

thus,

$$\chi_e = \frac{-ne^2}{m\omega^2} = -\frac{\omega_p^2}{\omega^2} \qquad , \text{where } \omega_p^2 = \frac{ne^2}{m}$$

Finally

$$\boxed{n = \sqrt{1 - \frac{\omega_p^2}{\omega^2}}}$$

Stat. Mech. S'04 #13

Unlike the ideal gas case where E is only a function of T (i.e. $E = \frac{3}{2} NkT$), for a van der Waals gas E is also a function of V.

(Reif p. 173)

$$dE = C_V \, dT + \left[T \left(\frac{\partial p}{\partial T} \right)_V - p \right] dV$$

now $\left(\frac{\partial p}{\partial T} \right)_V = \frac{Nk}{(V - bN)}$

then

$$T \left(\frac{\partial p}{\partial T} \right)_V - p = \frac{NkT}{(V-bN)} - \frac{NkT}{(V-bN)} + a\left(\frac{N}{V}\right)^2 = a\left(\frac{N}{V}\right)^2$$

hence

$$E(T,V) = \int C_V \, dT + aN^2 \int \frac{dV}{V^2}$$

we are told $C_V = \frac{3}{2} Nk$ (as for an ideal gas)

hence

$$E(T,V) = \frac{3}{2} Nk \, \Delta T + aN^2 \int \frac{dV}{V^2}$$

As this is a free adiabatic expansion

$$\Delta E = 0 \quad (\text{as } \underset{\text{adiabatic}}{Q = 0} \text{ and } \underset{\text{free expansion}}{pdV = 0})$$

$$0 = \frac{3}{2} \cancel{N} k \underbrace{\Delta T}_{T_F - T_i} - a N^{\cancel{2}} \left. \frac{1}{V'} \right|_{\frac{V}{3}}^{V}$$

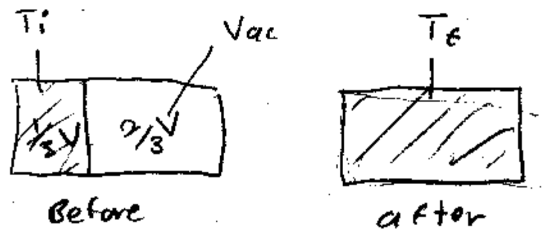
$$\frac{3}{2} k \Delta T = a N^2 \left(\frac{1}{V} - \frac{1}{V/3} \right) = a N^2 \left(\frac{1}{V} - \frac{3}{V} \right) = -\frac{2aN}{V}$$

hence

$$\Delta T = -\frac{4 N a}{3 k V} \Rightarrow T_f = T_i - \frac{4}{3} \frac{N}{k} \frac{a}{V}$$

Spring 2004. #13

$$p(T,V) = \frac{NkT}{(V-bN)} - a\left(\frac{N}{V}\right)^2$$



volume $\frac{V}{N}$

$C_V = \frac{3}{2}Nk$ since the same as an ideal gas.

$$dE = TdS - pdV = d(TS) - SdT - pdV$$

work do by system

$$\left(\frac{\partial p}{\partial T}\right)_V = \frac{Nk}{(V-bN)} \qquad \left(\frac{\partial E}{\partial V}\right)_T = T\left(\frac{\partial p}{\partial T}\right)_V - p$$

$$= \frac{TNk}{(V-bN)} - p = a\left(\frac{N}{V}\right)^2$$

$C_V(T) \rightarrow$ since it is the same as an ideal gas

$$dE = C_V dT + a\left(\frac{N}{V}\right)^2 dV \qquad \text{eq 5.8.10 Reif}$$

Since

$$dS = \left(\frac{\partial S}{\partial T}\right)_V dT + \left(\frac{\partial S}{\partial V}\right)_T dV \qquad \left(\frac{\partial S}{\partial T}\right)_V = \frac{1}{T}C_V$$

$$dS = \frac{C_V}{T}dT + \left(\frac{\partial p}{\partial T}\right)_V dV \qquad \left(\frac{\partial S}{\partial V}\right)_T = \left(\frac{\partial p}{\partial T}\right)_V$$

plug into

$dE = TdS - pdV$

Side note

$$E(T,V) = \int_{T_0}^{T} C_V(T')dT' - a\left(\frac{N^2}{V}\right) + \text{constant}$$

$$= C_V T - \frac{aN^2}{V} + \text{constant}$$

In a free expansion $\Rightarrow \Delta Q = 0$

$\Delta W = 0$

$\boxed{\Delta E = 0}$

$$E(T_2, V_2) = E(T_1, V_1)$$

$$\int_{T_0}^{T_2} C_V(T')\,dT' - \frac{aN^2}{V_2} = \int_{T_0}^{T_1} C_V(T')\,dT' - \frac{aN^2}{V_1}$$

$$\Rightarrow \int_{T_0}^{T_2} C_V(T')\,dT' - \int_{T_0}^{T_1} C_V(T')\,dT' = a\left(\frac{N^2}{V_2} - \frac{N^2}{V_1}\right)$$

$$\int_{T_1}^{T_2} C_V(T')\,dT' = aN^2\left(\frac{1}{V_2} - \frac{1}{V_1}\right)$$

Van der Wall gas $T_1 < T' < T_2$ has a constant specific heat at fixed volume

$$\Rightarrow C_V(T_2 - T_1) = aN^2\left(\frac{1}{V_2} - \frac{1}{V_1}\right)$$

$$T_2 - T_1 = -\frac{aN^2}{C_V}\left(\frac{1}{V_1} - \frac{1}{V_2}\right) \qquad C_V = \tfrac{3}{2}NK$$

$$T_2 = T_1 - \frac{2aN^2}{3NK}\left(\frac{1}{V_1} - \frac{1}{V_2}\right)$$

$$\boxed{T_f = T_i - \frac{2aN}{3K}\left(\frac{1}{V_1} - \frac{1}{V_2}\right)}$$

when $a=0$ we get the free expansion of an ideal gas.

$$= T_i - \frac{2aN}{3K}\left(\frac{3}{V} - \frac{1}{V}\right)$$

$$T_f = T_i - \frac{4aN}{3KV} \qquad V_1 = \tfrac{1}{3}V, \quad V_2 = V$$