STATISTICAL PHYSICS 215A

Final Exam – Spring 2013 – SOLUTIONS

1 Solution to Question 1

(a) The specific heats in question are defined by

$$C_M = \left(\frac{\partial E}{\partial T}\right)_M \qquad\qquad C_H = \left(\frac{\partial (E-MH)}{\partial T}\right)_H \tag{1.1}$$

(b) In the canonical ensemble, the independent variables are (T, M), and we have $E(T, M)$. To compute C_H, we use independent variables T, H, so that M is a function of T, H which we denote $M(T, H)$. Thus, we have

$$\begin{aligned} C_H &= \left.\frac{\partial(E-HM)}{\partial T}\right|_H = \left.\frac{\partial E}{\partial T}\right|_H - H\left.\frac{\partial M}{\partial T}\right|_H \\ \left.\frac{\partial E}{\partial T}\right|_H &= \left.\frac{\partial E(T, M(T,H))}{\partial T}\right|_H = \left.\frac{\partial E}{\partial T}\right|_M + \left.\frac{\partial E}{\partial M}\right|_T \left.\frac{\partial M}{\partial T}\right|_H \end{aligned} \tag{1.2}$$

from which the expression of (a) follows immediately.

(c) Curie's law allows us to express H in terms of (T, M), and we have,

$$dE = TdS + \frac{MT}{nD}dM \tag{1.3}$$

Since C_M is constant, and $\partial E/\partial M = 0$ at constant T, we have $dE = C_M dT$. During adiabatic transformations, we have $dS = 0$, so that we obtain the differential equation,

$$C_M\, dT = \frac{MT}{nD}dM \tag{1.4}$$

Dividing both sides by T separates the variables, and allows us to integrate by quadrature,

$$T(M) = T(M=0)\, \exp\left\{\frac{M^2}{2nDC_M}\right\} \tag{1.5}$$

(d) The transformations $2 \to 3$ and $4 \to 1$ are adiabatic; using (1.5), we have,

$$\begin{aligned} M_3^2 &= M_2^2 - 2nDC_M \ln\frac{T_h}{T_\ell} \\ M_4^2 &= M_1^2 - 2nDC_M \ln\frac{T_h}{T_\ell} \end{aligned} \tag{1.6}$$

Note that the signs work out as $T_h > T_\ell$, while we have $M_3^2 < M_2^2$ and $M_4^2 < M_1^2$.

The heat liberated along an isothermal may be computed from the fact that for fixed T the internal energy E remains constant for this specific material in view of the assumption that $\partial E/\partial M = 0$ at constant T,

$$0 = \delta Q + H dM = \delta Q + \frac{MT}{nD} dM \tag{1.7}$$

The heat liberated along the isothermal processes $1 \to 2$ and $3 \to 4$ is respectively given by,

$$\begin{aligned} Q_h &= -\frac{T_h}{2nD}(M_1^2 - M_2^2) \\ Q_\ell &= -\frac{T_\ell}{2nD}(M_4^2 - M_3^2) \end{aligned} \tag{1.8}$$

As internal energy is conserved along the isothermals, we have

$$W_{1\to 2} = Q_h \qquad\qquad W_{3\to 4} = Q_\ell \tag{1.9}$$

while along the adiabatics, the work equals the change in internal energy, and we have,

$$W_{2\to 3} = -C_M(T_h - T_\ell) \qquad\qquad W_{4\to 1} = -C_M(T_\ell - T_h) \tag{1.10}$$

Elimination T_h/T_ℓ between the two equations in (1.6), we find $M_1^2 - M_2^2 = M_4^2 - M_3^2$, so that we find $Q_h/Q_\ell = T_h/T_\ell$. The total work done by the system is given by conservation of total internal energy by $W = Q_h - Q_\ell$. By definition of η, we have,

$$\eta = \frac{W}{Q_h} = 1 - \frac{Q_\ell}{Q_h} = 1 - \frac{T_\ell}{T_h} \tag{1.11}$$

Note that by the second law of thermodynamics, W, Q_h, and Q_ℓ are all positive.

2 Solution to Question 2

(a) The total number of micro-states $\Omega(E, L, N)$ is given by the multiple integral,

$$\Omega(E, L, N) = \frac{1}{N!} \prod_{i=1}^{N} \int \frac{dq_i dp_i}{2\pi\hbar}\, \theta\left(E - c\sum_{i=1}^{N} |p_i|\right) \tag{2.1}$$

where θ denotes the Heaviside step function. The factor of $1/N!$ is included to account for the indistinguishability of the particles stated in the problem. The range of each integral in q_i is over the box of length L. By symmetry of the integrand under $p_i \to -p_i$, we restrict the range of integration in p_i to $[0, \infty]$, and include a factor of 2 for each integration. Thus, we end up with the simplified formula,

$$\begin{aligned}\Omega(E, L, N) &= \frac{1}{N!}\left(\frac{2L}{2\pi\hbar}\right)^N \mathcal{V}(N, E/c) \\ \mathcal{V}(N, \lambda) &= \prod_{i=1}^{N} \int_0^\infty dp_i \, \theta\left(\lambda - \sum_{i=1}^{N} |p_i|\right)\end{aligned} \tag{2.2}$$

By scaling all p_i, we see that we have,

$$\mathcal{V}(N, \lambda) = \lambda^N \mathcal{V}(N, 1) \tag{2.3}$$

On the other hand, the integral may be evaluated iteratively,

$$\begin{aligned}\mathcal{V}(N, \lambda) &= \int_0^\lambda dp_N \mathcal{V}(N-1, \lambda - p_N) \\ &= \int_0^\lambda dp_N (\lambda - p_N)^{N-1} \mathcal{V}(N-1, 1) = \frac{\lambda^N}{N} \mathcal{V}(N-1, 1)\end{aligned} \tag{2.4}$$

Putting all together, we have $\mathcal{V}(N, \lambda) = \lambda^N / N!$ so that,

$$\Omega(E, L, N) = \frac{1}{(N!)^2}\left(\frac{LE}{\pi\hbar c}\right)^N \tag{2.5}$$

It immediately follows that

$$\Omega'(E, L, N, \Delta) = \frac{1}{(N!)^2}\left(\frac{LE}{\pi\hbar c}\right)^N \frac{N\Delta}{E} \tag{2.6}$$

(b) The entropy is defined in terms of the number of micro-states at energy E, so it should be in terms of Ω'. We shall adopt the following definition,

$$S(E, L, N) = k \ln \frac{\Omega'(E, L, N, \Delta)}{\Delta} = k \ln \left(\frac{1}{(N!)^2}\left(\frac{LE}{\pi\hbar c}\right)^N \frac{N}{E}\right) \tag{2.7}$$

In the thermodynamic limit, the contribution from the factor N/E cancels out, and we omit it outright. The remaining expression in the limit may be rearranged as follows,

$$\frac{S}{N} = k \ln \left(\frac{LE}{\pi\hbar c N^2}\right) + 2k \tag{2.8}$$

We have used Sterling's formula to obtain the last term. The entropy is properly extensive.

(c) Using the micro-canonical ensemble, we have from the definition of temperature,

$$\frac{1}{T} = \left.\frac{\partial S}{\partial E}\right|_{N,L} = \frac{kN}{E} \tag{2.9}$$

The resulting relation $E = NkT$ violates the equipartition theorem by a factor of 2. In the canonical ensemble, we calculate the partition function,

$$Z = \frac{1}{N!}\prod_{i=1}^{N}\left(\int \frac{dp_i dq_i}{2\pi\hbar}\, e^{-\beta c|p_i|}\right) = \frac{1}{N!}\left(\frac{kTL}{\pi\hbar c}\right)^N \tag{2.10}$$

The internal energy is deduced directly from

$$E = -\frac{\partial \ln Z}{\partial \beta} = NkT \tag{2.11}$$

which agrees with the earlier calculation in the micro-canonical ensemble.

(d) Returning to the micro-canonical ensemble, the pressure is given by,

$$\frac{P}{T} = \left.\frac{\partial S}{\partial L}\right|_{E,N} = \frac{Nk}{L} \tag{2.12}$$

so that $P = NkT/L$, while we have,

$$C_L = \left.\frac{\partial E}{\partial T}\right|_{L,N} = kN \tag{2.13}$$

3 Solution to Question 3

(a) In terms of the density of one-particle states $D(\varepsilon)$ and the Fermi occupation number f, the total number of particles N and the internal energy E are given by,

$$\begin{aligned} N &= V\int_0^\infty d\varepsilon\, D(\varepsilon) f(\varepsilon) \\ E &= V\int_0^\infty d\varepsilon\, \varepsilon D(\varepsilon) f(\varepsilon) \end{aligned} \tag{3.1}$$

We assume that $D(\varepsilon) = 0$ for $\varepsilon < 0$. Since the function $D(\varepsilon)$ does not involve temperature (but may involve V which is held constant in computing C_V), the specific heat is given by,

$$C_V = \frac{\partial E}{\partial T} = V\int_0^\infty d\varepsilon\, \varepsilon D(\varepsilon)\frac{\partial f(\varepsilon)}{\partial T} \tag{3.2}$$

Note that since we are using the grand-canonical ensemble here, this specific heat corresponds to holding μ fixed (instead of in the canonical ensemble where we would hold N fixed instead). Working this out, and after some minor simplifications, we get,

$$C_V = \frac{V}{kT^2} \int_0^\infty d\varepsilon \, \frac{\varepsilon(\varepsilon - \mu) D(\varepsilon)}{\left(e^{\beta(\varepsilon-\mu)/2} + e^{-\beta(\varepsilon-\mu)/2}\right)^2} \tag{3.3}$$

(b) Strong degeneracy corresponds to low temperatures. The denominator is then responsible for concentrating the support of the integral over ε near μ, so we may extend the integration region all the way to $-\infty$. Also, to leading order, we may evaluate $D(\varepsilon)$ at the central value μ. The parity of the remaining integral allows us to replace the factor ε in the numerator by $\varepsilon - \mu$, so that we end up with the following expression,

$$C_V = \frac{V D(\mu)}{kT^2} \int_{-\infty}^\infty d\varepsilon \, \frac{(\varepsilon - \mu)^2}{\left(e^{\beta(\varepsilon-\mu)/2} + e^{-\beta(\varepsilon-\mu)/2}\right)^2} \tag{3.4}$$

Changing variables from ε to x with $\varepsilon = \mu + 2kTx$ gives,

$$C_V = 8k^2 T V D(\mu) \int_{-\infty}^\infty dx \, \frac{x^2}{(e^x + e^{-x})^2} \tag{3.5}$$

Using the value of the integral stated in the problem, and the fact that for small temperatures we have $\mu = \mu_0 + \mathcal{O}(T^2)$, with μ_0 defined by,

$$N = V \int_0^{\mu_0} d\varepsilon \, D(\varepsilon) \tag{3.6}$$

we approximate this result by setting $D(\mu) = D(\mu_0)$, so that the final result is given by,

$$C_V = \frac{1}{3} \pi^2 k^2 V D(\mu_0) \, T \tag{3.7}$$

Observe from equation (3.6) that holding μ fixed is equivalent to holding N fixed in the approximation of $T = 0$; thus C_V computed at fixed μ as was done here in the grand-canonical ensemble coincides with C_V computed at fixed N.

(c) Weak degeneracy corresponds to high temperature T, in which case the FD occupation number becomes the Boltzmann number, and we have,

$$C_V = \frac{V}{kT^2} \int_0^\infty d\varepsilon \, \varepsilon(\varepsilon - \mu) D(\varepsilon) \, e^{-\beta(\varepsilon-\mu)} \tag{3.8}$$

with μ obtained from,

$$N = V \int_0^\infty d\varepsilon \, D(\varepsilon) e^{-\beta(\varepsilon-\mu)} \tag{3.9}$$

In both the formulas for C_V and N, the fugacity $e^{\beta\mu}$ factors out from under the integrations, and we may eliminate it entirely, to obtain the formula,

$$C_V = \frac{N}{kT^2} \frac{\int_0^\infty d\varepsilon\, \varepsilon(\varepsilon - \mu) D(\varepsilon)\, e^{-\beta\varepsilon}}{\int_0^\infty d\varepsilon\, D(\varepsilon) e^{-\beta\varepsilon}} \tag{3.10}$$

Note that,as earlier, this is still the specific heat at constant μ, not constant N.

(d) The density of states for a non-relativistic free particle is given by

$$g \int \frac{d^3p d^3q}{(2\pi\hbar)^3} f(\varepsilon) = p^2/2m) = \frac{2\pi g V (2m)^{3/2}}{(2\pi\hbar)^3} \int_0^\infty d\varepsilon\, \sqrt{\varepsilon} f(\varepsilon) \tag{3.11}$$

where $g = 2$ for the electron. Thus, we conclude that

$$D(\varepsilon) = \frac{2\pi g V (2m)^{3/2}}{(2\pi\hbar)^3} \sqrt{\varepsilon} \tag{3.12}$$

For low T, formula (3.6) gives μ_0 as a function of the number density,

$$\frac{N}{V} = \frac{4\pi g}{3} \left(\frac{2m\mu_0}{(2\pi\hbar)^2} \right)^{3/2} \tag{3.13}$$

so that the specific heat takes the form,

$$\frac{C_V}{Nk} = \frac{mkT}{4\hbar^2} \left(\frac{4\pi g V}{3N} \right)^{2/3} \tag{3.14}$$

For high T, the specific heat at constant μ is given by

$$\begin{aligned} \frac{C_V}{Nk} &= \beta^2 \frac{\int_0^\infty d\varepsilon\, \varepsilon^{3/2}(\varepsilon - \mu)\, e^{-\beta\varepsilon}}{\int_0^\infty d\varepsilon\, \varepsilon^{1/2} e^{-\beta\varepsilon}} \\ &= \frac{\int_0^\infty dx\, x^{3/2}(x - \beta\mu)\, e^{-x}}{\int_0^\infty dx\, x^{1/2} e^{-x}} \\ &= \frac{\Gamma(7/2)}{\Gamma(3/2)} - \frac{\Gamma(5/2)}{\Gamma(3/2)} \beta\mu = \frac{15}{4} - \frac{3}{2}\beta\mu \end{aligned} \tag{3.15}$$

At high T the last term may be neglected. We do not find the standard 3/2 because this is not the specific heat at constant N. To obtain the latter in the large T limit, one should first eliminate μ in favor of N, so that one first obtains,

$$\frac{E}{N} = \frac{\int_0^\infty d\varepsilon\, \varepsilon\, D(\varepsilon)\, e^{-\beta\varepsilon}}{\int_0^\infty d\varepsilon D(\varepsilon)\, e^{-\beta\varepsilon}} = \frac{3}{2} kT \tag{3.16}$$

The standard result $C_V = 3Nk/2$ then follows.

4 Solution to Question 4

(a) The Hamiltonian for this system depends on the height ℓ of the matter in the cylinder of total height L, with $0 < \ell < L$, and is given by,

$$H = \frac{\mathbf{p}^2}{2m} + mg\ell \tag{4.1}$$

The corresponding number density is given by,

$$N - N_0 = \frac{V}{L} \int_0^L d\ell \int \frac{d^3 p}{(2\pi\hbar)^3} \frac{1}{e^{\beta(\mathbf{p}^2/2m + mg\ell - \mu)} - 1} \tag{4.2}$$

Th expression may be recast in terms of the function $g_{3/2}$. Setting $\ell = Ly$, we obtain,

$$N - N_0 = \frac{V}{\lambda^3} \int_0^1 dy \, g_{3/2} \left(z \, e^{-\beta mgLy} \right) \tag{4.3}$$

Criticality in the absence of gravity is at temperature T_c^0, with corresponding value λ_0, and is determined by setting $z = 1$, so that we have,

$$N = \frac{V}{\lambda_0^3} g_{3/2}(1) = \frac{V}{\lambda_0^3} \zeta(3/2) \tag{4.4}$$

Criticality in the presence of gravity at temperature T_c with corresponding value λ is determined by setting $z = 1$, so that we have the relation,

$$N = \frac{V}{\lambda^3} \int_0^1 dy \, g_{3/2} \left(e^{-\beta mgLy} \right) \tag{4.5}$$

Since we have $mgL \ll kT_c$, we may uniformly expand the integrand as follows,

$$g_{3/2} \left(e^{-\beta mgLy} \right) = \zeta(3/2) - 2\sqrt{\pi} (\beta mgLy)^{\frac{1}{2}} + \mathcal{O}(\beta mgL) \tag{4.6}$$

Combining equations (4.4) with (4.5) in the above approximation, we find,

$$\frac{1}{\lambda_0^3} = \frac{1}{\lambda^3} \int_0^1 dy \left(1 - \frac{2\sqrt{\pi}}{\zeta(3/2)} (\beta mgLy)^{\frac{1}{2}} + \mathcal{O}(\beta mgLy) \right) \tag{4.7}$$

Within the approximation of small gravitational effects, we have,

$$T_c = T_c^0 \left(1 + \frac{8\sqrt{\pi}}{9\zeta(3/2)} \left(\frac{mgL}{kT_c^0} \right)^{\frac{1}{2}} + \mathcal{O}(g) \right) \tag{4.8}$$

(b) In the condensed phase, we set $z = 1$, so that the internal energy is given by,

$$\frac{E}{V} = \frac{3kT}{2\lambda^3} \int_0^1 dy \, g_{5/2} \left(e^{-\beta mgLy} \right) \tag{4.9}$$

From the asymptotics of $g_{3/2}$, we deduce the asymptotics of $g_{5/2}$, and we find,

$$g_{5/2}(e^{-\alpha}) = \zeta(5/2) - \zeta(3/2)\alpha + \frac{4}{3}\sqrt{\pi}\alpha^{3/2} + \mathcal{O}(\alpha^2) \tag{4.10}$$

From this expansion, it is clear that the next order correction arising from the integral is suppressed by a factor of mgL/kT_c, so this is subleading compared to the square root behavior. As a result, we find,

$$\frac{E}{V} = \frac{3kT}{2\lambda^3} \zeta(5/2) \tag{4.11}$$

so that the specific heat is given by,

$$\frac{C_V}{Nk} = \frac{15}{4} \frac{\zeta(5/2)}{\zeta(3/2)} \frac{\lambda_0^3}{\lambda^3} \tag{4.12}$$

where λ corresponds to T_c, while λ_0 corresponds to T_c^0.

In the gas phase, we have instead,

$$\frac{E}{V} = \frac{3kT}{2\lambda^3} \int_0^1 dy \, g_{5/2} \left(z e^{-\beta mgLy} \right) \tag{4.13}$$

Since $g_{5/2}(z)$ has a regular expansion near $z = 1$ to leading non-trivial order, the exponential correction in the argument may be neglected, as it is of higher order. Thus we are left with,

$$\frac{E}{V} = \frac{3kT}{2\lambda^3} g_{5/2}(z) \tag{4.14}$$

The specific heat is obtained by differentiating with respect to T, including the dependence on z,

$$\frac{C_V}{Nk} = \frac{15}{4} \frac{g_{5/2}(z)}{g_{3/2}(z)} \frac{\lambda_0^3}{\lambda^3} + \frac{3}{2} \frac{\lambda_0^3}{\lambda^3} \frac{\partial \ln z}{\partial \ln T} \bigg|_{N,V} \tag{4.15}$$

The remaining derivative is obtained from differentiating the relation

$$\frac{N}{V}\lambda^3 = \int_0^1 dy \, g_{3/2} \left(z e^{-\beta mgLy} \right) \tag{4.16}$$

with respect to T, while keeping N and V fixed. Setting $z = 1$ to match with the condensed phase, and again neglecting higher orders in mgL/kT_c, we find the discontinuity to be given by,

$$\frac{\Delta C_V}{Nk} = -\frac{9}{4}\frac{\zeta(3/2)}{\int_0^1 dy\, g_{1/2}\left(e^{-\beta mgLy}\right)} \tag{4.17}$$

Using the expansion of $g_{1/2}$ near 1, derived from that of $g_{3/2}$, we find,

$$g_{1/2}(e^{-\alpha}) = \sqrt{\frac{\pi}{\alpha}} + \mathcal{O}(1) \tag{4.18}$$

so that

$$\frac{\Delta C_V}{Nk} = -\frac{9\zeta(3/2)}{8\sqrt{\pi}}\left(\frac{mgL}{kT_c^0}\right)^{\frac{1}{2}} \tag{4.19}$$