1. *Quantum Mechanics* (Spring 2007)

Consider a hydrogen atom in a weak uniform magnetic field $\mathbf{B} = B\hat{\mathbf{e}}_z$.

(a) Write down the total Hamiltonian of this system, including the spin degree of freedom of the electron, but neglecting spin-orbit interactions.

(b) Show that the vector potential of the applied magnetic field in the Coulomb gauge, $\boldsymbol{\nabla} \cdot \mathbf{A} = 0$, can be taken to be $\mathbf{A} = \frac{B}{2}(-y, x, 0)$.

(c) Now expand the Hamiltonian of part a) to first order in $\mathbf{A}$ ($\mathbf{B}$), i.e., neglect terms quadratic in $\mathbf{A}$. Also make the (unphysical[1]) assumption that the electron spin can be neglected. Show that then $H = H_0 + \omega_L l_Z$ with $\omega_L = eB/(2m_e c)$ and with H_0 the Hamiltonian of this system without magnetic field.

(d) Find the eigenfunctions and eigenvalues of H of part c) and discuss the resulting splitting of the degenerate hydrogen energy levels of H_0. Make the same simplifying assumptions as in part c). Also neglect relativistic effects.

[1] A charged but spinless hypothetical particle, the charged Higgs bosons $H^{\pm}$, is suggested by Supersymmetry. Finding one in a hydrogen-like bound state with the proton would, of course, be a sensation.

2. *Quantum Mechanics* (Spring 2007)

A particle moves in a three-dimensional potential

$$V(\mathbf{r}) = -V_0 a \, \delta(r - a)$$

V_0 and a are positive numbers. Is there always an $l = 0$ bound state? Is there ever an $l = 0$ bound state? If your answers are no and yes, respectively (that is, there is an $l = 0$ bound state for some, but not all, values of V_0 and a), what is the condition on V_0 and a for the bound state to exist?

3. *Quantum Mechanics* (Spring 2007)

Let H be the Hamiltonian for the hydrogen atom, including spin. $\hbar\mathbf{L} = \mathbf{r}\times\mathbf{p}$ and $\hbar\mathbf{s}$ are the orbital and spin angular momentum, respectively, and $\mathbf{J} = \mathbf{L} + \mathbf{s}$. Conventionally, the states are labelled $\left|n, l, j, m_j\right\rangle$ and they are eigenstates of H, $\mathbf{L}^2$, $\mathbf{J}^2$, and J_z.

In parts (a) and (d) you may ignore spin-orbit and relativistic effects.

(a) If the electron is in the state $\left|n, l, j, m_j\right\rangle$, what values will be measured for these four observables in terms of $\hbar$, c, the fine-structure constant α, and the electron mass m?

(b) What are the restrictions on the possible values of n, l, j, and m_j?

(c) Let $J_\pm = J_x \pm iJ_y$. What are

(i) $\left\langle 3, 1, \frac{3}{2}, \frac{3}{2}\right|J_+\left|3, 1, \frac{3}{2}, -\frac{1}{2}\right\rangle = ?$

(ii) $\left\langle 3, 1, \frac{3}{2}, \frac{3}{2}\right|J_+\left|3, 1, \frac{3}{2}, \frac{1}{2}\right\rangle = ?$

(iii) $\left\langle 2, 1, \frac{1}{2}, -\frac{1}{2}\right|\mathbf{L}^2\left|2, 1, \frac{1}{2}, -\frac{1}{2}\right\rangle = ?$

(iv) $\left\langle 3, 2, \frac{3}{2}, -\frac{1}{2}\right|\mathbf{J}^2\left|3, 2, \frac{3}{2}, -\frac{1}{2}\right\rangle = ?$

(v) $\left\langle 3, 1, \frac{3}{2}, \frac{3}{2}\right|J_z\left|3, 1, \frac{3}{2}, \frac{1}{2}\right\rangle = ?$

(d) What are

(i) $\left\langle 2, 1, \frac{3}{2}, \frac{3}{2}\right|p_z\left|2, 1, \frac{3}{2}, \frac{1}{2}\right\rangle = ?$

(ii) $\left\langle 1, 0, \frac{1}{2}, \frac{1}{2}\right|p_i p_j\left|1, 0, \frac{1}{2}, \frac{1}{2}\right\rangle = ?$

4. *Quantum Mechanics* (Spring 2007)

When π^+ mesons are scattered off a proton target, it is observed that the cross section becomes very large when the π^+ kinetic energy is near $E_0 = (150 \text{ MeV})$. At that energy it is also observed that the differential cross section, as a function of the scattering angle θ, is proportional to $\cos^2\theta$.

Near $E = E_0$ the total cross section has the resonance form

$$\sigma_{\text{tot}}(E) = \frac{c}{(E - E_0)^2 + \Gamma^2/4}$$

In terms of E_0, Γ, and the mass m_π of the π^+, what is the largest possible value of the constant c?

Note: In this problem ignore spin and relativistic effects, and assume the target proton is infinitely heavy.

5. *Quantum Mechanics* (Spring 2007)

Consider a particle of mass m in a one-dimensional potential of the form $V(x) = b\,|x|$, where b is a constant with $b > 0$. Apply the variational method using the trial wavefunction

$$\psi(x) = e^{-ax^2}$$

(a) Calculate $\langle\psi|\psi\rangle$ and $\langle\psi|H|\psi\rangle$ for the trial wavefunction.

(b) Find the extrema of the function

$$E(a) = \frac{\langle\psi|H|\psi\rangle}{\langle\psi|\psi\rangle}$$

(c) Calculate the ground state energy E_0 using the variational method.

(d) Write down a trial wavefunction which would be appropriate to calculate the energy of the first excited state. Give an argument for your choice.

6. *Statistical Mechanics and Thermodynamics* (Spring 2007)

Consider N noninteracting distinguishable particles of mass m in a three-dimensional harmonic oscillator with Hamiltonian

$$H = \sum_{i=1}^{N} \frac{\mathbf{p}_i{}^2}{2m} + \frac{m\omega^2}{2}\mathbf{r}_i{}^2$$

The particles are in contact with a heat bath of temperature T and are in thermal equilibrium.

(a) Calculate the partition function $Z(T, N)$ of the system.

(b) Calculate the internal energy $E(T, N)$ of the system.

(c) Calculate the heat capacity $c(T, N)$ of the system.

(d) Simplify the expression for the heat capacity for low and high temperature (with respect to $\hbar\omega/k_{\mathrm{B}}$).

7. *Statistical Mechanics and Thermodynamics* (Spring 2007)

(a) For the entropy $S(T,V)$ and the energy $E(T,V)$ derive from the first law of thermodynamics the following relation:

$$\left(\frac{\partial E}{\partial V}\right)_T = T\left(\frac{\partial p}{\partial T}\right)_V - p$$

(b) The equation of state of a thermodynamic system is given by

$$p = \alpha\,\epsilon(T)$$

where α is a constant and

$$\epsilon(T) = E(T,V)/V$$

Calculate the temperature dependence of the internal energy $E(T,V)$.

(c) Calculate the entropy $S(T,V)$ for a system with the equation of state given above.

(d) Calculate the Helmholtz free energy F and Gibbs free energy G for the case of a photon gas ($\alpha = 1/3$).

8. *Statistical Mechanics and Thermodynamics* (Spring 2007)

Consider a large number of photons, in thermal equilibrium at temperature T, inside a volume V. The photons are in vacuum and can have two possible directions of polarization.

(a) Let $u(\omega, T)\,\mathrm{d}\omega$ be the mean energy per unit volume of photons in the frequency range between ω and $\omega + \mathrm{d}\omega$. Find $u(\omega, T)$.

(b) What is the temperature dependence of the total energy density u_0?

Note: The constants of proportionality will involve a definite integral which you need not solve explicitly – find temperature dependence only.

9. *Statistical Mechanics and Thermodynamics* (Spring 2007)

DNA is a double-stranded molecule which at sufficiently high temperature melts into two single strands. Consider the following simplified model of this process:

You have two half-ladders (see the figure below); each half-ladder represents one DNA strand. In the double-stranded molecule, each rung ("base") on one strand is bound to the opposite rung on the other strand by an energy ϵ_b ("base pairing"). For each base, there is 1 bound state and $\exp(S_b/k)$ states with the bond broken (i.e., S_b is the entropy gained because of one broken bond). Each base can also flip out of the plane of the page ("unstacking"), but there is an energy cost ϵ_S for that and also an entropy gain σ (i.e., there is 1 state with the base "stacked" and $\exp(\sigma/k)$ states with the base "unstacked").

To summarize:

- unpairing carries an energy cost ϵ_p and an entropy gain s per base or a total change in free energy of $\epsilon_p - Ts$
- unstacking carries an energy cost ϵ_s and an entropy gain σ per base or a total change in free energy of $\epsilon_s - T\sigma$

For a molecule with N base pairs:

(a) What is the partition function if "unstacking" is neglected?

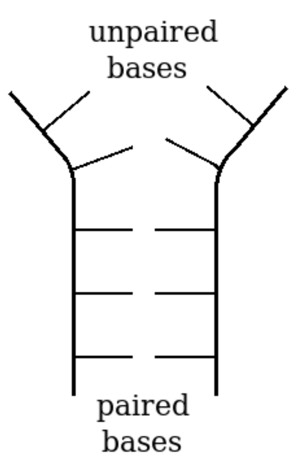
(b) For the partition function in part a), what is the average number of unpaired bases at temperature T in terms of the appropriate partial derivatives?

(c) What is the partition function if "unstacking" can occur? and what is the new expression for the average number of unpaired bases? You may leave your result in terms of the appropriate derivatives of the partition function.

Make the assumptions that:

(1) unpairing can proceed only serially from one end of the molecule (like a zipper; neglect states which are partially unzipped from both ends)

(2) a base can be unstacked only if it is unpaired.



10. *Electricity and Magnetism* (Spring 2007)

Consider the following scattering problem. A particle of charge q and mass m is constrained to move along the z-axis and is bounded by a harmonic potential centered at the origin, $x_1 = (0,0,0)$ with a natrual frequency ω_0. A plane e.m. wave propagating along x: $E = E_0 \exp[i(kx - \omega t)]$; $k = 2\pi\lambda$ is incident on this system. The incident wave is linearly polarized along z.

(a) Calculate the scattering cross-section $\mathrm{d}\sigma/\mathrm{d}\Omega$ in the spherical coordinate angle θ with respect to the z-axis. Note that this is not the scattering angle.

(b) Now add a second, identical charge, constrained in a similar fashion to move along the z-direction and harmonically bound with a natural frequency ω_0 centered at $x_2 = (0, \lambda/2, 0)$. Calculate the scattering cross-section $\mathrm{d}\sigma/\mathrm{d}\Omega$ due to both charges.

For this problem you might find the following formulas for the dipole fields useful where $\mathbf{p}$ is the amplitude of the electric dipole:

$$\begin{aligned} \mathbf{B} &= k^2 \frac{e^{ikr}}{r} (\hat{\mathbf{n}} \times \mathbf{p})\, e^{-i\omega t} \\ \mathbf{E} &= \mathbf{B} \times \hat{\mathbf{n}} \end{aligned}$$

11. *Electricity and Magnetism* (Spring 2007)

Consider a long electron beam with a flat top radial profile with a radius r_0 (density is constant in r for $r < a$), a length $L \gg a$, and a velocity v_b.

(a) What is the total force on an electron at the edge of the beam ($r = r_0$)?

(b) Suppose this beam enters a plasma with a density n_0 (an initially neutral collection of electrons and ions). Assume that the density of the beam is much larger than that of the plasma such that all of the plasma electrons are pushed outward. They will form a sheath that is distributed symmetrically in the azimuthal direction around the ions that have not moved. Assume that v_b is very close to the speed of light so that γ_b is very large and the force you calculated in part a) can be assumed to be zero. Show that for very large γ_b, each beam electron will approximately oscillate in the x and y directions with a frequency $\omega_\mathrm{p}/(2\gamma_\mathrm{b})^{1/2}$, where $\omega_\mathrm{p} = 4\pi e^2 n_0/m$ is the plasma frequency.

(c) The electrons will then radiate. What is the power radiated by a single electron that starts at the edge of the beam? Estimate the length of plasma it would take for the beam to radiate away its energy in terms of the plasma density, its initial radius, and the beam energy, i.e., γ_b. (Recall that the relativistic Larmor formula is $P = \frac{2}{3}\frac{e^2}{c}\gamma^6[\dot{\boldsymbol{\beta}}^2 - (\boldsymbol{\beta} \times \dot{\boldsymbol{\beta}})^2]$.)

(d) Estimate the length of plasma it would take for the beam to radiate away its energy.

12. *Electricity and Magnetism* (Spring 2007)

Consider a plane electromagnetic wave with a frequency, ω_0, propagating in vacuum in the $\hat{\mathbf{z}}$ direction with its electric field polarized in the $\hat{\mathbf{x}}$ direction.

(a) Write down the form for the electric field, magnetic field, vector potential, and scalar potential for this wave (should be easy).

(b) Suppose the wave is a pulse with a well-defined beginning and end, that is, the vector potential has an "envelope" that is a function of $z - ct$ (and the other fields have an appropriate form). Does such a wave also satisfy Maxwell's equations?

(c) Consider an electron starting at rest that sits in the path of the electromagnetic wave. What are the equations of motion and the conservation of energy equation? (Write out the equations for $\mathrm{d}\mathbf{P}/\mathrm{d}t$ and $\mathrm{d}\gamma/\mathrm{d}t$.)

(d) Use the equations from part c) to show that the canonical momentum in the $\hat{\mathbf{x}}$ direction and $\gamma - \frac{P_z}{mc}$ are constants of the motion.

Hint: For the second constant use a linear combination of the $\mathrm{d}\gamma/\mathrm{d}t$ and $\mathrm{d}P_z/\mathrm{d}t$ equations.

(e) If the electron starts at rest, then what is maximum kinetic energy as it oscillates in the wave?

(f) What is its kinetic energy after the wave completely passes by it?

13. *Electricity and Magnetism* (Spring 2007)

Consider a long cylider of radius r_0 that is sliced in half along z. Let one half be maintained at a potential ϕ_0 and the other half at $-\phi_0$.

(a) What is Poisson's equation in cylindrical coordinates?

(b) What is the potential for $r < r_0$?

(c) What is the potential for $r > r_0$?

14. *Electricity and Magnetism* (Spring 2007)

A wire consists of a long straight conductor of circular cross section with radius a and has a current I. Free electron charge carriers roam the otherwise rigid lattice of ions the wire is made out of. Assume Ohmic conduction so that the current density $\mathbf{J}$ is a constant inside the conductor, and assume that the free charge carriers are electrons with drift velocity v_{d}. The conductor is at rest in the S frame.

(a) Find the magnetic field inside the wire as a function of radius r. Find the force on the conduction electrons due to this magnetic field. Will the conduction electrons move in straight lines under the influence of this force alone? If not, what happens?

(b) In a steady state, is there an electric field perpendicular to the axis of the wire? If so, find its magnitude and direction and the charge density implied by that field.

(c) Consider the same problem as viewed from a frame S' moving to the right with velocity $v = v_{\mathrm{d}}$. Will there be a magnetic force on the conduction electrons? Find the perpendicular electric field and charge density in the wire in this frame.

(d) Let ρ_0 be the volume charge density in the rest frame of the charges and $\rho = \rho_0\, f(v)$ the charge density when viewed from the frame moving with velocity v. Find $f(v)$ from the requirement that a) and b) be consistent. Is charge conserved when changing the reference frame? Give your reasoning.

3. *Quantum Mechanics* (Spring 2007)

Let H be the Hamiltonian for the hydrogen atom, including spin. $\hbar\mathbf{L} = \mathbf{r} \times \mathbf{p}$ and $\hbar\mathbf{s}$ are the orbital and spin angular momentum, respectively, and $\mathbf{J} = \mathbf{L} + \mathbf{s}$. Conventionally, the states are labelled $|n, l, j, m_j\rangle$ and they are eigenstates of H, $\mathbf{L}^2$, $\mathbf{J}^2$, and J_z.

In parts (a) and (d) you may ignore spin-orbit and relativistic effects.

(a) If the electron is in the state $|n, l, j, m_j\rangle$, what values will be measured for these four observables in terms of $\hbar$, c, the fine-structure constant α, and the electron mass m?

(b) What are the restrictions on the possible values of n, l, j, and m_j?

(c) Let $J_\pm = J_x \pm iJ_y$. What are

(i) $\langle 3, 1, \frac{3}{2}, \frac{3}{2}|J_+|3, 1, \frac{3}{2}, -\frac{1}{2}\rangle = ?$ $\quad J_+ \to T^1_1 \quad \Delta m = 2 \neq 1 \quad = \emptyset$

(ii) $\langle 3, 1, \frac{3}{2}, \frac{3}{2}|J_+|3, 1, \frac{3}{2}, \frac{1}{2}\rangle = ?$ $\quad \sqrt{\frac{3}{2}(\frac{3}{2}+1) - \frac{1}{2}(\frac{1}{2}+1)} = \sqrt{15/4 - 3/4} = \sqrt{3}$

(iii) $\langle 2, 1, \frac{1}{2}, -\frac{1}{2}|\mathbf{L}^2|2, 1, \frac{1}{2}, -\frac{1}{2}\rangle = ?$ $\quad \vec{L}^2 \to T^0_0 \quad 1(1+1) = 2$

(iv) $\langle 3, 2, \frac{3}{2}, -\frac{1}{2}|\mathbf{J}^2|3, 2, \frac{3}{2}, -\frac{1}{2}\rangle = ?$ $\quad \vec{J}^2 \to T^0_0 \quad \frac{3}{2}(\frac{3}{2}+1) = 15/4$

(v) $\langle 3, 1, \frac{3}{2}, \frac{3}{2}|J_z|3, 1, \frac{3}{2}, \frac{1}{2}\rangle = ?$ $\quad J_z \to T^0_1 \quad \Delta m = 1 \neq 0 \quad = \emptyset$

(d) What are

(i) $\langle 2, 1, \frac{3}{2}, \frac{3}{2}|p_z|2, 1, \frac{3}{2}, \frac{1}{2}\rangle = ?$ $\quad p_z \to T^0_1 \quad \Delta m = 1 \neq 0 \quad = \emptyset$

(ii) $\langle 1, 0, \frac{1}{2}, \frac{1}{2}|p_i p_j|1, 0, \frac{1}{2}, \frac{1}{2}\rangle = ?$

a) $H: \; -\frac{1}{2}\alpha^2 mc^2 \frac{1}{n^2} \qquad \vec{L}^2: \; l(l+1) \qquad \vec{J}^2: \; j(j+1) \qquad J_z: \; m_j$

b) $n \in \mathbb{Z}^+ = \{1, 2, 3, \ldots\} \qquad l \in \{0, 1, 2, \ldots, n-1\} \qquad s = \frac{1}{2}$

$j \in \{|l-s|, |l-s|+1, \ldots, l+s-1, l+s\} \qquad m_j \in \{-j, -j+1, \ldots, j-1, j\}$

c) Wigner-Eckart selection rules: $\langle \alpha', j', m'|T^q_k|\alpha, j, m\rangle = 0$

unless $\Delta m = q$

and $|\Delta j| \leq k \leq \Sigma j$

d) ii) $p_i p_j = \underbrace{\frac{1}{3}\vec{p}^2\delta_{ij}}_{T^0_0} + \frac{1}{2}\cancel{(p_i p_j - p_j p_i)} + \underbrace{\left[\frac{1}{2}(p_i p_j + p_j p_i) - \frac{1}{3}\vec{p}^2\delta_{ij}\right]}_{\sum_q T^q_2}$

$[p_i, p_j] = 0 \qquad k = 2 \nleq \Sigma j = 1$

$\langle p_i p_j\rangle = \frac{1}{3}\langle p^2\rangle\delta_{ij} = \frac{1}{3}2m\langle T\rangle\delta_{ij} = -\frac{2m}{3}\langle E\rangle\delta_{ij}$ by virial theorem

$= +\frac{1}{3}\alpha^2 m^2 c^2 \frac{1}{n^2}\delta_{ij}\Big|_{n=1} = \frac{1}{3}\alpha^2 m^2 c^2 \delta_{ij}$

6. *Statistical Mechanics and Thermodynamics* (Spring 2007)

Consider N noninteracting distinguishable particles of mass m in a three-dimensional harmonic oscillator with Hamiltonian

$$H = \sum_{i=1}^{N} \frac{\mathbf{p}_i^2}{2m} + \frac{m\omega^2}{2}\mathbf{r}_i^2$$

The particles are in contact with a heat bath of temperature T and are in thermal equilibrium.

(a) Calculate the partition function $Z(T, N)$ of the system.

(b) Calculate the internal energy $E(T, N)$ of the system.

(c) Calculate the heat capacity $c(T, N)$ of the system.

(d) Simplify the expression for the heat capacity for low and high temperature (with respect to $\hbar\omega/k_B$).

Since part (d) asks for a low-temperature limit, we suspect that we'll have to solve for these quantities quantum-mechanically. Furthermore, we know that by the equipartition theorem (which is valid in the classical, high-temperature, high-particle-number (?) limit), since there are six quadratic terms in the single-particle Hamiltonian, $\langle\epsilon\rangle = 6\,\frac{1}{2}kT = 3kT$, so $E = N\langle\epsilon\rangle = 3NkT$, and as $dE = đQ$ $(đW = 0)$, $C = \frac{đQ}{dT} = \frac{dE}{dT} = 3Nk$, which can't be simplified. Thus we must solve quantum-mechanically and expect $C = 3Nk$ in the high-temperature limit.

a) N 3-D QSHOs $\Rightarrow$ 3N 1-D QSHOs

$\Rightarrow$ act as if there are 3N particles w/ energy $\epsilon = \hbar\omega(n+\frac{1}{2})$

non-interacting, nondistinguishable $\Rightarrow$ $Z = z^{3N}$

$$z = \sum_i e^{-\beta\epsilon_i} = \sum_{n=0}^{\infty} e^{-\beta\hbar\omega(n+\frac{1}{2})} = e^{-\alpha/2}\sum_n e^{-\alpha n} \quad \text{letting } \alpha \equiv \beta\hbar\omega = \frac{\hbar\omega}{kT}$$

$$= e^{-\alpha/2}\left(\frac{1}{1-e^{-\alpha}}\right) = \frac{1}{e^{\alpha/2}-e^{-\alpha/2}} = \frac{1}{2\sinh\alpha/2} = \left[2\sinh\alpha/2\right]^{-1}$$

$$\Rightarrow \quad Z = \left[2\sinh\alpha/2\right]^{-3N} = \left[2\sinh\frac{\hbar\omega}{2kT}\right]^{-3N}$$

b)
$$E = -\frac{\partial}{\partial\beta}\ln Z = -\hbar\omega\frac{\partial}{\partial\alpha}\ln Z = -\hbar\omega\,\partial_\alpha\left[-3N\ln\left(2\sinh\alpha/2\right)\right]$$

$$= 3N\hbar\omega\,\partial_\alpha\left[\ln 2 + \ln\left(\sinh\alpha/2\right)\right] = 3N\hbar\omega\frac{\frac{1}{2}\cosh\alpha/2}{|\sinh\alpha/2|} = \frac{3}{2}N\hbar\omega\frac{\cosh\alpha/2}{\sinh\alpha/2}$$

6. Statistical Mechanics and Thermodynamics (Spring 2007)

b) (continued)

$$= \frac{3}{2} N\hbar\omega \coth \alpha/2 \quad \Rightarrow \quad E = \frac{3}{2} N\hbar\omega \coth \frac{\hbar\omega}{2kT}$$

c) $$C = \frac{dE}{dT} = \frac{d\alpha}{dT}\frac{d}{d\alpha} E = \left(-\frac{\hbar\omega}{k}\frac{1}{T^2}\right) \partial_\alpha \left(\frac{3}{2} N\hbar\omega \coth \alpha/2\right)$$

$$= -\frac{\hbar\omega}{kT^2}\left[\frac{3}{2} N\hbar\omega \frac{\frac{1}{2}\sinh^2 \alpha/2 - \frac{1}{2}\cosh^2 \alpha/2}{\sinh^2 \alpha/2}\right] = -\frac{\hbar\omega}{kT^2}\left[\frac{3}{4} N\hbar\omega \frac{-1}{\sinh^2 \alpha/2}\right]$$

$$= \frac{3}{4} Nk \left(\frac{\hbar\omega}{kT}\right)^2 \operatorname{csch}^2 \alpha/2 = \frac{3}{4} Nk \left(\frac{\hbar\omega}{kT}\right)^2 \operatorname{csch}^2 \frac{\hbar\omega}{2kT}$$

d) Low T $\Rightarrow$ High α

$$\Rightarrow \operatorname{csch}^2 \alpha/2 = \frac{1}{\sinh^2 \alpha/2} = \left(\frac{2}{e^{\alpha/2} - e^{-\alpha/2}}\right)^2 \rightarrow \frac{4}{e^{\alpha/2} - 0}$$

$$= 4 e^{-\alpha/2}$$

$$\Rightarrow C \rightarrow \frac{3}{4} Nk \left(\frac{\hbar\omega}{kT}\right)^2 4 e^{-\alpha/2} = 3Nk \left(\frac{\hbar\omega}{kT}\right)^2 e^{-\hbar\omega/2kT}$$

High T $\Rightarrow$ Low α

$$\Rightarrow \operatorname{csch}^2 \alpha/2 = \frac{1}{\sinh^2 \alpha/2} \rightarrow \frac{1}{(\alpha/2)^2} = \frac{4}{\alpha^2} = 4\left(\frac{kT}{\hbar\omega}\right)^2$$

$$\Rightarrow C \rightarrow \frac{3}{4} Nk \cancel{\left(\frac{\hbar\omega}{kT}\right)^2} 4 \cancel{\left(\frac{kT}{\hbar\omega}\right)^2} = 3Nk$$ ✓

6. *Statistical Mechanics and Thermodynamics* (Spring 2007)

Consider N noninteracting distinguishable particles of mass m in a three-dimensional harmonic oscillator with Hamiltonian

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(d) Simplify the expression for the heat capacity for low and high temperature (with respect to $\hbar\omega/k_B$).

Since part (d) asks for a low-temperature limit, we suspect that we'll have to solve for these quantities quantum-mechanically. Furthermore, we know that by the equipartition theorem (which is valid in the classical, high temperature, high particle number (?) limit), since there are six quadratic terms in the Hamiltonian, $\langle \epsilon \rangle = 6 \frac{1}{2} kT$ so $E = N\langle\epsilon\rangle = 3NkT$, and as $dE = đQ$ ($đW = 0$), $C = \frac{đQ}{dT} = \frac{dE}{dT} = 3Nk$, which can't be simplified. Thus we must solve quantum-mechanically and expect $C = 3Nk$ in the high-temperature limit.

a) $Z = z^N \qquad z = \sum_i e^{-\beta\epsilon_i} \qquad \epsilon = \hbar\omega(n_x + n_y + n_z + \frac{3}{2}) = \hbar\omega(N + \frac{3}{2})$

$N = n_x + n_y + n_z$; The energies are degenerate since several combinations of values for n_x, n_y, and n_z yield the same value of N. How many combinations? Given a particular value for n_z, $n_x + n_y = N - n_z$, and n_x may range from zero to $N - n_z$. $\Rightarrow$ $(N - n_z + 1)$ comb.s for n_x and n_y. And n_z may range from zero to N, so the number of comb.s, and thus the multiplicity of the energies, is

$$m(N) = \sum_{n_z=0}^{N} (N - n_z + 1) \overset{*}{=} \sum_{n=1}^{N+1} n = \frac{1}{2}(N+1)(N+2) \qquad * \left(\text{reversing the summation}\right)$$

$$\Rightarrow \quad z = \sum_{N=0}^{\infty} m(N)\, e^{-\beta\epsilon(N)} = \sum_N \frac{1}{2}(N+1)(N+2)\, e^{-\beta\hbar\omega(N+3/2)} \qquad \alpha \equiv \beta\hbar\omega$$

$$= \frac{1}{2} e^{-\beta\hbar\omega 3/2} \sum_N (N^2 + 3N + 2)\, e^{-\beta\hbar\omega N} = \frac{1}{2} e^{-\alpha 3/2} \sum_N (N^2 + 3N + 2)\, e^{-\alpha N}$$

$$= \frac{1}{2} e^{-\alpha 3/2} (\partial_\alpha^2 - 3\partial_\alpha + 2) \sum_N e^{-\alpha N} = \frac{1}{2} e^{-\alpha 3/2} (\partial_\alpha^2 - 3\partial_\alpha + 2)(1 - e^{-\alpha})^{-1}$$

$$= \frac{1}{2} e^{-\alpha 3/2} \left[\partial_\alpha \{ -(1-e^{-\alpha})^{-2}(e^{-\alpha}) \} + 3(1-e^{-\alpha})^{-2}(e^{-\alpha}) + 2(1-e^{-\alpha})^{-1} \right]$$

6. Statistical Mechanics and Thermodynamics

a) (continued)

$$= \frac{1}{2} e^{-\alpha 3/2} \left[2(1-e^{-\alpha})^{-3}(e^{-\alpha})^2 + (1-e^{-\alpha})^{-2}(e^{-\alpha}) + 3(1-e^{-\alpha})^{-2}(e^{-\alpha}) + 2(1-e^{-\alpha})^{-1} \right]$$

$$= \frac{1}{2} e^{-\alpha 3/2} (1-e^{-\alpha})^{-1} \left[2e^{-2\alpha}(1-e^{-\alpha})^{-2} + 4e^{-\alpha}(1-e^{-\alpha})^{-1} + 2 \right]$$

$$= e^{-\alpha 3/2}(1-e^{-\alpha})^{-1}\left[e^{-2\alpha}(1-e^{-\alpha})^{-2} + 2e^{-\alpha}(1-e^{-\alpha})^{-1} + 1 \right]$$

$$= z(T) \qquad \text{where} \qquad \alpha = \beta \hbar\omega = \frac{\hbar\omega}{kT}$$

$$Z(T,N) = z^N$$

since $\beta = \frac{1}{\hbar\omega}\alpha$

b) $$E = -\frac{\partial}{\partial\beta} \ln Z = -\frac{\partial}{\partial\beta} N \ln z = -N\frac{1}{z}\frac{\partial z}{\partial\beta} = -N\hbar\omega \frac{1}{z}\frac{\partial z}{\partial\alpha}$$

$$= -N\hbar\omega \left[-\frac{3}{2} + \frac{e^{-\alpha\frac{3}{2}}}{z} \left\{ \begin{array}{l} -2e^{-2\alpha}(1-e^{-\alpha})^{-3} - 3e^{-2\alpha}(1-e^{-\alpha})^{-4}(e^{-\alpha}) \\ -2e^{-\alpha}(1-e^{-\alpha})^{-2} - 4e^{-\alpha}(1-e^{-\alpha})^{-3}(e^{-\alpha}) - (1-e^{-\alpha})^{-2}(e^{-\alpha}) \end{array} \right\} \right]$$

$$= N\hbar\omega \left[\frac{3}{2} + \frac{3e^{-3\alpha}(1-e^{-\alpha})^{-4} + 6e^{-2\alpha}(1-e^{-\alpha})^{-3} + 3e^{-\alpha}(1-e^{-\alpha})^{-2}}{e^{-2\alpha}(1-e^{-\alpha})^{-3} + 2e^{-\alpha}(1-e^{-\alpha})^{-2} + (1-e^{-\alpha})^{-1}} \right]$$

$$= E(T,N) \qquad \text{where} \qquad \alpha = \frac{\hbar\omega}{kT}$$

$\left\{ \begin{array}{l} \text{numerator} \equiv H \\ \text{denominator} \equiv L \end{array} \right.$

c) $$C = \frac{dE}{dT} = \frac{d\alpha}{dT}\frac{dE}{d\alpha} = -\frac{\hbar\omega}{k}\frac{1}{T^2}\frac{dE}{d\alpha} \qquad \text{where} \qquad E = N\hbar\omega\left(\frac{3}{2} + \frac{H}{L}\right)$$

$$= -\frac{\frac{N}{k}\left(\frac{\hbar\omega}{T}\right)^2}{L^2} \left[\begin{array}{l} L\{-9e^{-3\alpha}(1-e^{-\alpha})^{-4} - 12e^{-3\alpha}(1-e^{-\alpha})^{-5}(e^{-\alpha}) - 12e^{-2\alpha}(1-e^{-\alpha})^{-3} \\ \quad -18e^{-2\alpha}(1-e^{-\alpha})^{-4}(e^{-\alpha}) - 3e^{-\alpha}(1-e^{-\alpha})^{-2} - 6e^{-\alpha}(1-e^{-\alpha})^{-3}(e^{-\alpha})\} \\ -H\{\partial_\alpha L = -H\} \end{array} \right]$$

$$= \frac{N}{k}\left(\frac{\hbar\omega}{T}\right)^2 \left[\begin{array}{l} \frac{3}{L}\{4e^{-4\alpha}(1-e^{-\alpha})^{-5} + 9e^{-3\alpha}(1-e^{-\alpha})^{-4} + 6e^{-2\alpha}(1-e^{-\alpha})^{-3} + e^{-\alpha}(1-e^{-\alpha})^{-2}\} \\ - H^2/L^2 \end{array} \right]$$

d) High T $\Rightarrow$ low α $\Rightarrow$ $e^{-\alpha} \to 1$, $(1-e^{-\alpha}) \to \alpha$; $H \to 3\alpha^{-4}$, $L \to \alpha^{-3}$

$$\Rightarrow \quad C \to \frac{N}{k}\left(\frac{\hbar\omega}{T}\right)^2 \left[3\frac{4\alpha^{-5}}{\alpha^{-3}} - \left(3\frac{\alpha^{-4}}{\alpha^{-3}}\right)^2 \right] \quad \text{keeping leading terms only}$$

6. Statistical Mechanics and thermodynamics

d) (continued)

$$= \frac{N}{k}\left(\frac{\hbar\omega}{T}\right)^2\left[12\alpha^{-2} - 9\alpha^{-2}\right] = 3\frac{N}{k}\left(\frac{\hbar\omega}{T}\right)^2\left(\frac{kT}{\hbar\omega}\right)^2$$

$$= 3Nk \quad \checkmark$$

Low T $\Rightarrow$ high α $\Rightarrow$ $e^{-\alpha} \to 0, e^{-\alpha}$ $\quad (1-e^{-\alpha})^{-n} \to 1, 1+ne^{-\alpha}$

$H \to 3e^{-\alpha} \quad L \to 1$

$$\Rightarrow \quad C \to \frac{N}{k}\left(\frac{\hbar\omega}{T}\right)^2\left[3e^{-\alpha} - (3e^{-\alpha})^2\right]$$

$$\to 3Nk\left(\frac{\hbar\omega}{kT}\right)^2 e^{-(\hbar\omega/kT)}$$

8. *Statistical Mechanics and Thermodynamics* (Spring 2007)

Consider a large number of photons, in thermal equilibrium at temperature T, inside a volume V. The photons are in vacuum and can have two possible directions of polarization.

(a) Let $u(\omega, T)\, d\omega$ be the mean energy per unit volume of photons in the frequency range between ω and $\omega + d\omega$. Find $u(\omega, T)$.

(b) What is the temperature dependence of the total energy density u_0?

Note: The constants of proportionality will involve a definite integral which you need not solve explicitly – find temperature dependence only.

a) $u_0 = \frac{1}{V}\langle U\rangle = \frac{1}{V}\int_0^\infty \epsilon\, f(\epsilon)\,\rho(\epsilon)\, d\epsilon \qquad \epsilon = pc = \hbar\omega, \quad f(\epsilon) = \frac{1}{e^{\beta\epsilon}-1}$

$\rho = \frac{d\Omega}{d\epsilon} = 2\,\frac{\frac{1}{h^3}\, dV_{phase}}{d\epsilon}$ (where the 2 comes from polarization multiplicity)

$dV_{phase} = \left(\int d^3r\right)\left(\int p^2\, d\Omega_p\right) dp = V\, 4\pi p^2\, dp = V\, 4\pi \left(\frac{\hbar}{c}\right)^3 \omega^2\, d\omega$

since $p = \frac{\hbar}{c}\omega$

$\Rightarrow \rho = 2\cdot 4\pi V \left(\frac{1}{2\pi c}\right)^3 \omega^2 \frac{d\omega}{d\epsilon} = \frac{V}{\pi^2 c^3}\, \omega^2 \frac{d\omega}{d\epsilon}$

$$\Rightarrow u_0 = \frac{1}{V}\int_0^\infty \hbar\omega \left(\frac{1}{e^{\beta\epsilon}-1}\right)\frac{V}{\pi^2 c^3}\,\omega^2 \frac{d\omega}{d\epsilon}\, d\epsilon$$

$$= \frac{\hbar}{\pi^2 c^3}\int_0^\infty \frac{\omega^3}{e^{\beta\hbar\omega}-1}\, d\omega = \int_0^\infty u(\omega, T)\, d\omega$$

$$\Rightarrow u(\omega, T) = \frac{\hbar}{\pi^2 c^3}\,\frac{\omega^3}{e^{\hbar\omega/kT}-1} \qquad \text{(Planck's law)}$$

b) $u_0 \sim \int_0^\infty \frac{\omega^3}{e^{\beta\hbar\omega}-1}\, d\omega \sim \left(\frac{1}{\beta}\right)^4 \int_0^\infty \frac{x^3\, dx}{e^x - 1}$ where $x = \beta\hbar\omega$

so $\omega \sim \frac{1}{\beta}x$

$\sim T^4$ (~ Stefan-Boltzmann Law)