1. *Quantum Mechanics* (Fall 2005)

Let $H$ be the Hamiltonian for the hydrogen atom, including spin. $\hbar\mathbf{L} = \mathbf{r}\times\mathbf{p}$ and $\hbar\mathbf{s}$ are the orbital and spin angular momentum, respectively, and $\mathbf{J} = \mathbf{L}+\mathbf{s}$. Conventionally, the states are labeled $|n,l,j,m\rangle$ and they are eigenstates of $H$, $\mathbf{L}^2$, $\mathbf{J}^2$, and $J_z$.

In parts (a) and (d) you may state the answer to lowest nonvanishing order — ignore spin-orbit and relativistic effects.

(a) If the electron is in the state $|n,l,j,m\rangle$, what values will be measured for these four observables in terms of $\hbar$, $c$, the fine-structure constant $\alpha$, and the electron mass $m$?

(b) What are the restrictions on the possible values of $n$, $l$, $j$, and $m$?

(c) Let $J_\pm = J_x \pm iJ_y$. What are

(i) $\left\langle 3,1,\frac{3}{2},\frac{3}{2}\middle|J_+\middle|3,1,\frac{3}{2},-\frac{1}{2}\right\rangle = ?$

(ii) $\left\langle 3,1,\frac{3}{2},\frac{3}{2}\middle|J_+\middle|3,1,\frac{3}{2},\frac{1}{2}\right\rangle = ?$

(iii) $\left\langle 2,1,\frac{1}{2},-\frac{1}{2}\middle|\mathbf{L}^2\middle|2,1,\frac{1}{2},-\frac{1}{2}\right\rangle = ?$

(iv) $\left\langle 3,2,\frac{3}{2},-\frac{1}{2}\middle|\mathbf{J}^2\middle|3,2,\frac{3}{2},-\frac{1}{2}\right\rangle = ?$

(v) $\left\langle 3,1,\frac{3}{2},\frac{3}{2}\middle|J_z\middle|3,1,\frac{3}{2},\frac{1}{2}\right\rangle = ?$

(d) What are

(i) $\left\langle 2,1,\frac{3}{2},\frac{3}{2}\middle|p_z\middle|2,1,\frac{3}{2},\frac{1}{2}\right\rangle = ?$

(ii) $\left\langle 1,0,\frac{1}{2},\frac{1}{2}\middle|p_ip_j\middle|1,0,\frac{1}{2},\frac{1}{2}\right\rangle = ?$

2. *Quantum Mechanics* (Fall 2005)

Consider the one-dimensional harmonic oscillator. The Hamiltonian is

$$H_0 = \frac{p^2}{2m} + \frac{m\omega^2 x^2}{2}$$

(a) Define the operator

$$U = e^{ipb/\hbar}$$

for some real number $b$. Here $p$ is the momentum operator. What is the ground state wave function (up to the normalization constant) for the Hamiltonian

$$H = UH_0U^\dagger \quad ?$$

*Note*: If you know the answer, it is enough just to write it down. The derivation is allowed, but not required for full credit.

(b) Suppose a term $\alpha x^3$ is added to the Hamiltonian $H_0$. Calculate the change in the energy of each level, through second order in $\alpha$. Please write your answer as a constant independent of the level number $n$, times a polynomial or ratio of polynomials in $n$.

3. *Quantum Mechanics* (Fall 2005)

An electron in the $n = 3$, $l = 0$, $m = 0$ state of hydrogen decays by a sequence of electric dipole transitions to the ground state.

(a) What decay routes are possible? Specify them by listing the sequence of states $|nlm_l\rangle$ in each possible route.

(b) If you had a large number of atoms in this state $|300\rangle$, what fraction of them would decay via each route? Give an explicit justification for your answer from the expression for the matrix element of the relevant operator.

*Hint*: You may want to use some of the following:

$$Y_0^0 = \frac{1}{\sqrt{4\pi}} \qquad Y_1^0 = \sqrt{\frac{3}{4\pi}}\cos\theta \qquad Y_1^{\pm1} = \mp\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{\pm i\phi}$$

4. *Quantum Mechanics* (Fall 2005)

The Hamiltonian for a system consisting of three distinguishable spin half particles is

$$H = A(\mathbf{s}_1 \cdot \mathbf{s}_2 + \mathbf{s}_2 \cdot \mathbf{s}_3 + \mathbf{s}_3 \cdot \mathbf{s}_1)$$

where $\mathbf{s}_i$ is the spin of the $i^{\text{th}}$ particle, and all the components of the spin of one particle commute with all the components of the spins of the others. What are the eigenvalues of $H$, and what are the degeneracies of each energy level?

5. *Quantum Mechanics* (Fall 2005)

In this problem, neglect spin and relativistic effects, and use the Born approximation.

(a) Suppose an electron scatters off a spherically symmetric potential $V(r)$. Write down (or compute if you don't remember) the formula for the Born approximation to the scattering amplitude $f(\theta,\phi)$, in the form of a one-dimensional radial integral:

$$f(\theta,\phi) = \int_0^\infty (\text{some function of } r) \times V(r)\,\mathrm{d}r$$

(b) Now suppose that the electron scatters elastically off a spherically symmetric charge distribution, with charge density $\rho(r)$ centered at the origin. (This is not a local potential, but the answer to part (a) may still be useful.)

Calculate, in the Born approximation (that is, to first order in the potential), the scattering amplitude $f(\theta,\phi)$ and write it as

$$f(\theta,\phi) = f_R(q^2)F(q^2)$$

where $\mathbf{q}$ is the momentum transferred between the incident and the scattered electron, and $f_R(q^2)$ is the Rutherford amplitude for scattering off a point charge:

$$f_R(q^2) = \frac{2mZ\alpha}{\hbar^2 q^2}$$

Here $\alpha$ is the fine-structure constant. The function $F(q^2)$ is called the "form factor". Write an explicit formula for $F(q^2)$ in terms of $\rho(r)$.

(c) Now specialize to an electron scattering elastically off a uniformly charged sphere, centered at the origin, with radius $R$ and total charge $Ze$. What is $F(q^2)$ as a function of $q$ and $R$?

*Hint*: You might want the definite integral $\int_0^\infty e^{-\mu r}\sin(qr)\,\mathrm{d}r = \frac{q}{q^2+\mu^2}$ and the indefinite integrals

$$\int x\sin x\,\mathrm{d}x = \sin x - x\cos x \qquad \text{and} \qquad \int x\cos x\,\mathrm{d}x = \cos x + x\sin x$$

*Note*: The scattering amplitude is defined so that its square is the differential cross section: $|f|^2 = \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}$

6. *Statistical Mechanics and Thermodynamics* (Fall 2005)

"Cold" stars (that is, stars that have exhausted their nuclear fuel) are stabilized against gravitational collapse by the degeneracy pressure of the electrons, or, at higher densities, neutrons. To model this effect, consider a spherical star of mass $M$, mass density $\rho$, radius $R$, and volume $V$, consisting of neutrons of mass $m_n$.

(a) Calculate the gravitational potential of the star, that is, the gravitational potential of a uniform massive sphere of radius $R$.

*Answer*:

$$V_G = -\frac{(2\pi)^2\rho^2 G}{15} R^5$$

where $G$ is Newton's gravitational constant.

(b) Obtain the corresponding gravitational pressure $P_G$.

(c) View the neutrons as a cold, ideal neutron gas. Compute the energy and the degeneracy pressure of the fermion gas at $T = 0$ assuming

(i) the gas is nonrelativistic ($p \ll m$).

(ii) the gas is ultrarelativistic ($p \gg m$).

Is there an equilibrium radius for the star in

(i) the nonrelativistic case?

(ii) the ultrarelativistic case?

If there is no equilibrium radius, what is the critical particle number $N = N_e$ above which gravitational collapse is unavoidable?

7. *Statistical Mechanics and Thermodynamics* (Fall 2005)

In a temperature range near some absolute temperature $T$, the tension force $F$ of a stretched plastic rod is related to its length $L$ by the expression

$$F = aT^2(L - L_0)$$

where $a$ and $L_0$ are positive constants, $L_0$ being the unstretched length of the rod. When $L = L_0$, the heat capacity $C_L$ of the rod (measured at constant length) is given by the relation $C_L = bT$, where $b$ is a constant.

(a) Write down the fundamental thermodynamic relation for this system, expressing $\mathrm{d}S$ in terms of $\mathrm{d}L$ and $\mathrm{d}E$.

(b) The entropy $S(T, L)$ of the rod is a function of $T$ and $L$. Compute $\left(\frac{\partial S}{\partial L}\right)_T$.

(c) Knowing $S(T_0, L_0)$, find $S(T, L)$ at any *other* temperature $T$ and length $L$. (It is most convenient to calculate first the change of entropy with temperature at the length $L_0$ where the heat capacity is known.)

(d) If you start at $T = T_i$ and $L = L_i$ and stretch the thermally insulated rod quasi-statically until it attains the length $L_f$, what is the final temperature $T_f$?

(e) Calculate the heat capacity $C_L(L, T)$ of the rod when its length is $L$ instead of $L_0$.

(f) Calculate $S(T, L)$ by writing $S(T, L) - S(T_0, L_0) = [S(T, L) - S(T_0, L)] + [S(T_0, L) - S(T_0, L_0)]$ and using the result of part (e) to compute the first term in square brackets. Show that the final answer agrees with the one found in part (c).

8. *Electricity and Magnetism* (Fall 2005)

An anisotropic medium has a tensor conductivity given by

$$\overleftrightarrow{\sigma} = \begin{pmatrix} \sigma_\perp & 0 & 0 \\ 0 & \sigma_\perp & 0 \\ 0 & 0 & \sigma_\parallel \end{pmatrix}$$

where $\sigma_\parallel$ and $\sigma_\perp$ are real and independent of frequency. The symbol $\perp$ refers to the $(\hat{\mathbf{x}}, \hat{\mathbf{y}})$ direction and the symbol $\parallel$ to the $\hat{\mathbf{z}}$ direction in a Cartesian coordinate system.

(a) Find the dispersion relation $k = k(\omega)$ for an electromagnetic wave with O-mode (ordinary mode) polarization with the $k$ vector along $\hat{\mathbf{x}}$.

(b) Write an expression for the damping decrement $k_I = \mathrm{Im}\ k$ in the limit of high frequency.

(c) If the amplitude of the electric field is $E_0$ at $x = 0$, find the time-avaraged power per unit volume delivered to this medium at the location $x > 0$. (No need to write down $k_I$ explicitly.)

9. *Electricity and Magnetism* (Fall 2005)

Two small pieces of uncharged, continuous, polarizable matter (for example, glass) are placed in a region in which there is an externally generated, uniform field $E_0$. The two small pieces of matter have volumes $V_1$ and $V_2$ and electrical susceptibilities $\chi_1$ and $\chi_2$, respectively. If they are separated by a distance $d$, such that $d^3 \gg V_1$ and $d^3 \gg V_2$, find the energy associated with the interaction between the two pieces (that is, the part of the energy that depends on $d$.)

10. *Electricity and Magnetism* (Fall 2005)

A pulsar emits bursts of radio waves, which are observed from Earth at two different frequencies, say $\omega_1$ and $\omega_2$. An astronomer notes that the arrival time of the bursts is delayed at the lower frequency: the pulse at $\omega_1$ arrives after the pulse at $\omega_2$. The delay, $\tau$, is due to dispersion in the interstellar medium. Assuming this medium consists of ionized hydrogen, estimate the distance $s$ of the pulsar from the earth, as follows:

(a) Show that the electron plasma frequency for the dilute plasma — consisting of (heavy) ions and free electrons — is

$$\omega_p = \left(\frac{4\pi N e^2}{m_e}\right)^{1/2}$$

in e.s.u. Here $N$ is the number of electrons per unit volume.

(b) Show that the index of refraction of the plasma is

$$n = \sqrt{\epsilon} = \left(1 - \frac{{\omega_p}^2}{\omega^2}\right)^{1/2}$$

*Hint*: Write the equation of motion for a free electron in an oscillating ($e^{-i\omega t}$) electric field and find the plasma's polarizability $\chi$. Then $\epsilon = 1 + 4\pi\chi$.

(c) From the relation above find the group velocity of the light, and use this result to find the distance to the pulsar. (You may assume the frequencies are large compared to $\omega_p$.)

11. *Electricity and Magnetism* (Fall 2005)

A thin copper circular ring (conductivity $\sigma$, *mass* density $\rho_m$) is suspended so it can rotate freely about one diameter. There is a uniform magnetic field $\mathbf{B}$ perpendicular to the axis of rotation. The initial rotation frequency is $\omega_0$. Calculate the time it takes for the frequency to decrease to $1/e$ of its original value, assuming the energy all goes into Joule heating. (Assume the requested time $\tau$ is large compared to the rotation period.)

12. *Electricity and Magnetism* (Fall 2005)

An infinitely long cylinder of radius $a$ exhibits a permanent magnetization with its magnetization vector given by

$$\begin{aligned} \mathbf{M}(\mathbf{r}) &= \alpha r^2 \hat{\mathbf{z}} \quad && r \le a \\ \mathbf{M}(\mathbf{r}) &= 0 && r > a \end{aligned}$$

where $\alpha$ is a constant, $r$ is the (cylindrical) radial coordinate and $\hat{\mathbf{z}}$ is a unit vector along the axis of the cylinder.

(a) Find the magnetic vector field $\mathbf{B}$ for $r < a$ and for $r > a$.

*Hint*: In cylindrical coordinates

$$(\boldsymbol{\nabla} \times \mathbf{F})_\phi = \tfrac{\partial F_r}{\partial z} - \tfrac{\partial F_z}{\partial r}$$

(b) Determine the value of

$$\oint \mathbf{A} \cdot \mathrm{d}\mathbf{l}$$

along a circular path of radius $b > a$, encircling (in the $\hat{\boldsymbol{\phi}}$ direction) and concentric with the magnetized cylinder. $\mathbf{A}$ is the magnetic vector potential.

(c) Find the force per unit volume experienced by the material at a location $r < a$.

(d) What will happen to the cylinder if $\alpha$ is suddenly increased to a very large value?

13. *Statistical Mechanics and Thermodynamics* (Fall 2005)

The hydrogen molecule comes in two forms, in which the spin degrees of freedom of the two protons are in a spin triplet state (the "ortho" case) or in a spin singlet state (the "para" case) respectively. The rotational energies of a hydrogen molecule are given by

$$E(L) = \frac{\hbar^2}{2I} L(L+1)$$

with $I$ the moment of inertia and $L$ the orbital angular momentum quantum number.

(a) In the ortho case, only odd $L$ values are allowed and in the para case only even values. Why?

(b) Assuming that Boltzmann statistics are valid, find an expression for the specific heat of an ideal gas of hydrogen molecules for both the low temperature and the high temperature limits.

(c) Suppose protons were bosons instead of fermions. What would the low-temperature specific heat be then?

14. *Statistical Mechanics and Thermodynamics* (Fall 2005)

Consider a classical system of $N$ nonrelativistic charged particles in the presence of a constant external magnetic field $\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}$ at temperature $T$.

(a) Write down the partition function for the system.

(b) Compute the induced magnetization of the system along the direction of $\mathbf{B}$. From this you can answer the question whether paramagnetism occurs in classical physics.

1. *Quantum Mechanics* (Fall 2005)

Let $H$ be the Hamiltonian for the hydrogen atom, including spin. $\hbar\mathbf{L} = \mathbf{r}\times\mathbf{p}$ and $\hbar\mathbf{s}$ are the orbital and spin angular momentum, respectively, and $\mathbf{J} = \mathbf{L}+\mathbf{s}$. Conventionally, the states are labeled $|n,l,j,m\rangle$ and they are eigenstates of $H$, $\mathbf{L}^2$, $\mathbf{J}^2$, and $J_z$.

In parts (a) and (d) you may state the answer to lowest nonvanishing order — ignore spin-orbit and relativistic effects.

(a) If the electron is in the state $|n,l,j,m\rangle$, what values will be measured for these four observables in terms of $\hbar$, $c$, the fine-structure constant $\alpha$, and the electron mass $m$?

(b) What are the restrictions on the possible values of $n$, $l$, $j$, and $m$?

(c) Let $J_\pm = J_x \pm iJ_y$. What are

(i) $\left\langle 3,1,\frac{3}{2},\frac{3}{2}\middle|J_+\middle|3,1,\frac{3}{2},-\frac{1}{2}\right\rangle = ?$ $\quad J_+ \to T^1_1 \quad \Delta m \neq q = 1 \Rightarrow \boxed{0}$

(ii) $\left\langle 3,1,\frac{3}{2},\frac{3}{2}\middle|J_+\middle|3,1,\frac{3}{2},\frac{1}{2}\right\rangle = ?$ $\quad \sqrt{\frac{3}{2}(\frac{3}{2}+1)-\frac{1}{2}(\frac{1}{2}+1)} = \sqrt{\frac{15}{4}-\frac{3}{4}} = \boxed{\sqrt{3}}$

(iii) $\left\langle 2,1,\frac{1}{2},-\frac{1}{2}\middle|\mathbf{L}^2\middle|2,1,\frac{1}{2},-\frac{1}{2}\right\rangle = ?$ $\quad L^2 \to T^0_0 \quad l(l+1) = 1(1+1) = \boxed{2}$

(iv) $\left\langle 3,2,\frac{3}{2},-\frac{1}{2}\middle|\mathbf{J}^2\middle|3,2,\frac{3}{2},-\frac{1}{2}\right\rangle = ?$ $\quad J^2 \to T^0_0 \quad j(j+1) = \frac{3}{2}(\frac{3}{2}+1) = \boxed{\frac{15}{4}}$

(v) $\left\langle 3,1,\frac{3}{2},\frac{3}{2}\middle|J_z\middle|3,1,\frac{3}{2},\frac{1}{2}\right\rangle = ?$ $\quad J_z \to T^0_1 \quad \Delta m \neq q = 0 \Rightarrow \boxed{0}$

(d) What are

(i) $\left\langle 2,1,\frac{3}{2},\frac{3}{2}\middle|p_z\middle|2,1,\frac{3}{2},\frac{1}{2}\right\rangle = ?$ $\quad p_z \to T^0_1 \quad \Delta m \neq q = 0 \Rightarrow \boxed{0}$

(ii) $\left\langle 1,0,\frac{1}{2},\frac{1}{2}\middle|p_ip_j\middle|1,0,\frac{1}{2},\frac{1}{2}\right\rangle = ?$

a) $H$: $E_n = -\frac{1}{2}\alpha^2 mc^2\frac{1}{n^2}$ $\quad L^2 = l(l+1)$ $\quad J^2 = j(j+1)$ $\quad J_z = m = m_j$

b) $n \in \mathbb{Z}^+ = \{1,2,3,\ldots\}$ $\quad l \in \{0,1,2,\ldots,n-1\}$ $\quad s = \frac{1}{2}$

$j \in \{|l-s|, |l-s|+1, |l-s|+2, \ldots, l+s-1, l+s\}$

$m \in \{-j,-j+1,\ldots,j-1,j\}$

c) Wigner-Eckart selection rules: $\langle\alpha' j' m'|T^q_k|\alpha j m\rangle = 0$

unless $m' = m+q$ and $|j-k| \leq j' \leq j+k$

d) ii) $p_ip_j = \underbrace{\frac{1}{3}\vec{p}^2\delta_{ij}}_{T^0_0} + \frac{1}{2}(\underbrace{\cancel{p_ip_j - p_jp_i}}_{[p_i,p_j]=0}) + \underbrace{\left[\frac{1}{2}(p_ip_j+p_jp_i) - \frac{1}{3}\vec{p}^2\delta_{ij}\right]}_{\sum_q T^q_2}$ $\quad |\frac{1}{2}-2| = \frac{3}{2} \not\leq \frac{1}{2} = j' \Rightarrow \emptyset$

$\Rightarrow \langle p_ip_j\rangle = \frac{1}{3}\langle\vec{p}^2\rangle\delta_{ij} = \frac{1}{3}2m\langle T\rangle\delta_{ij} \underset{\uparrow}{=} -\frac{2m}{3}\langle E\rangle\delta_{ij} = +\frac{1}{3}\alpha^2m^2c^2\frac{1}{n^2}\delta_{ij}$

Virial theorem

2. *Quantum Mechanics* (Fall 2005)

Consider the one-dimensional harmonic oscillator. The Hamiltonian is

$$H_0 = \frac{p^2}{2m} + \frac{m\omega^2 x^2}{2}$$

(a) Define the operator

$$U = e^{ipb/\hbar}$$

for some real number $b$. Here $p$ is the momentum operator. What is the ground state wave function (up to the normalization constant) for the Hamiltonian

$$H = UH_0U^\dagger \quad ?$$

*Note*: If you know the answer, it is enough just to write it down. The derivation is allowed, but not required for full credit.

(b) Suppose a term $\alpha x^3$ is added to the Hamiltonian $H_0$. Calculate the change in the energy of each level, through second order in $\alpha$. Please write your answer as a constant independent of the level number $n$, times a polynomial or ratio of polynomials in $n$.

a) $U(b)$ is the translation operator, so given that the ground state of $H_0$ is $\langle x|0\rangle = \frac{1}{(2\pi)^{1/4}\sqrt{x_0}} e^{-x^2/4x_0^2}$ where $x_0 = \frac{\hbar}{2m\omega}$ the ground state of $H$ is

$$\langle x|0\rangle' = \langle x-b|0\rangle = \frac{1}{(2\pi)^{1/4}\sqrt{x_0}} e^{-(x-b)^2/4x_0^2}$$

$\uparrow \pm ?$

b) $H = H_0 + H'$ $\quad H' = \alpha x^3 = \alpha x_0^3 (a^\dagger + a)^3 = \alpha x_0^3 (a^\dagger + a)(a^{\dagger 2} + a^\dagger a + a a^\dagger + a^2)$

$$= \alpha x_0^3 (\underset{+3}{a^{\dagger 3}} + \underset{+1}{a^{\dagger 2} a} + \underset{+1}{a^\dagger a a^\dagger} + \underset{-1}{a^\dagger a^2} + \underset{+1}{a a^{\dagger 2}} + \underset{-1}{a a^\dagger a} + \underset{-1}{a^2 a^\dagger} + \underset{-3}{a^3})$$

Using perturbation theory

$\Delta E_n^{(1)} = \langle n|H'|n\rangle = 0$ $\quad$ since $H'$ only "connects" states where $\Delta n \in \{\pm 1, \pm 3\}$ but here $\Delta n = n - n = 0$.

$$\Delta E_n^{(2)} = -\sum_{m\neq n} \frac{|\langle m|H'|n\rangle|^2}{E_m^0 - E_n^0} = -\frac{\alpha^2 x_0^6}{\hbar\omega}\left[\frac{|\langle n-3|a^3|n\rangle|^2}{n-3-n} + \frac{|\langle n-1|(a^\dagger a^2 + a a^\dagger a + a^2 a^\dagger)|n\rangle|^2}{n-1-n}\right.$$

$$\left. + \frac{|\langle n+1|(a^{\dagger 2}a + a^\dagger a a^\dagger + a a^{\dagger 2})|n\rangle|^2}{n+1-n} + \frac{|\langle n+3|a^{\dagger 3}|n\rangle|^2}{n+3-n}\right]$$

$$= -\frac{\alpha^2}{\hbar\omega}\left(\frac{\hbar}{2m\omega}\right)^3\left[-\frac{1}{3}n(n-1)(n-2) - \{\sqrt{n}(n-1) + \sqrt{n}\,n + \sqrt{n}(n+1)\}^2 + \{\sqrt{n+1}\,n + \sqrt{n+1}(n+1) + \sqrt{n+1}(n+2)\}^2 + \frac{1}{3}(n+1)(n+2)(n+3)\right]$$

$$= -\frac{\alpha^2}{\hbar\omega}\left(\frac{\hbar}{2m\omega}\right)^3\left[-\frac{1}{3}n(n-1)(n-2) - 9n^3 + (n+1)(3n+3)^2 + \frac{1}{3}(n+1)(n+2)(n+3)\right]$$

4. *Quantum Mechanics* (Fall 2005)

The Hamiltonian for a system consisting of three distinguishable spin half particles is

$$H = A(\mathbf{s}_1 \cdot \mathbf{s}_2 + \mathbf{s}_2 \cdot \mathbf{s}_3 + \mathbf{s}_3 \cdot \mathbf{s}_1)$$

where $\mathbf{s}_i$ is the spin of the $i^{\text{th}}$ particle, and all the components of the spin of one particle commute with all the components of the spins of the others. What are the eigenvalues of $H$, and what are the degeneracies of each energy level?

It is easiest (I think) to analyze this Hamiltonian in terms of squares of spin operators. Since the operators commute with each other, we have $\vec{S}^2 = (\vec{S}_1 + \vec{S}_2 + \vec{S}_3)^2 = \vec{S}_1^{\,2} + \vec{S}_2^{\,2} + \vec{S}_3^{\,2} + 2(\vec{S}_1 \cdot \vec{S}_2 + \vec{S}_2 \cdot \vec{S}_3 + \vec{S}_3 \cdot \vec{S}_1)$

so $H = \frac{A}{2}(\vec{S}^2 - \vec{S}_1^{\,2} - \vec{S}_2^{\,2} - \vec{S}_3^{\,2}) = \frac{A\hbar^2}{2}\left(s(s+1) - 3s_i(s_i+1)\right)$ where $s_i = \frac{1}{2}$

$= \frac{A\hbar^2}{2}\left(s(s+1) - \frac{9}{4}\right)$

Define $\vec{S}_{12} \equiv \vec{S}_1 + \vec{S}_2$ $\quad m_{12} \equiv m_1 + m_2$

$\vec{S} = \vec{S}_{12} + \vec{S}_3$ $\quad m = m_{12} + m_3$

| $s_i$ | $m_i$ |
|---|---|
| $\frac{1}{2}$ | $-\frac{1}{2}, \frac{1}{2}$ |

$|s_1 - s_2| \le s_{12} \le s_1 + s_2$

$0 \le s_{12} \le 1$

$\Rightarrow s_{12} \in \{0, 1\}$

| $s_{12}$ | $m_{12}$ |
|---|---|
| 0 | 0 |
| 1 | $-1, 0, 1$ |

$|s_{12} - s_3| \le s \le s_{12} + s_3$

1. $(s_{12} = 0)$ $\quad \frac{1}{2} \le s \le \frac{1}{2}$

$\Rightarrow s \in \{\frac{1}{2}\}$

2. $(s_{12} = 1)$ $\quad \frac{1}{2} \le s \le \frac{3}{2}$

$\Rightarrow s \in \{\frac{1}{2}, \frac{3}{2}\}$

| | $s$ | $m$ |
|---|---|---|
| 1. | $\frac{1}{2}$ | $-\frac{1}{2}, \frac{1}{2}$ |
| 2. | $\frac{1}{2}$ | $-\frac{1}{2}, \frac{1}{2}$ |
| | $\frac{3}{2}$ | $-\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}$ |

So the possible values of $s$ are $\frac{1}{2}$ (degeneracy 4) and $\frac{3}{2}$ (deg. 4).

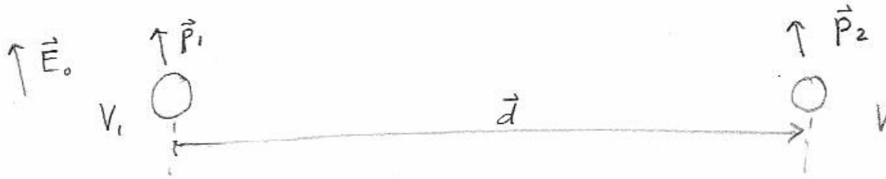
$\therefore$ The eigenvalues of $H$ are

$$E_{s=\frac{1}{2}} = \frac{A\hbar^2}{2}\left(\frac{1}{2}\left(\frac{1}{2}+1\right) - \frac{9}{4}\right) = \frac{A\hbar^2}{2}\left(\frac{3}{4} - \frac{9}{4}\right) = -\frac{3}{4}A\hbar^2 \quad \text{(degeneracy 4)}$$

$$E_{s=\frac{3}{2}} = \frac{A\hbar^2}{2}\left(\frac{3}{2}\left(\frac{3}{2}+1\right) - \frac{9}{4}\right) = \frac{A\hbar^2}{2}\left(\frac{15}{4} - \frac{9}{4}\right) = \frac{3}{4}A\hbar^2 \quad \text{(degeneracy 4)}$$

9. *Electricity and Magnetism* (Fall 2005)

Two small pieces of uncharged, continuous, polarizable matter (for example, glass) are placed in a region in which there is an externally generated, uniform field $E_0$. The two small pieces of matter have volumes $V_1$ and $V_2$ and electrical susceptibilities $\chi_1$ and $\chi_2$, respectively. If they are separated by a distance $d$, such that $d^3 \gg V_1$ and $d^3 \gg V_2$, find the energy associated with the interaction between the two pieces (that is, the part of the energy that depends on $d$.)




$$\vec{p}_1 = V_1 \vec{P}_1 = V_1 \varepsilon_0 \chi_1 \vec{E}_0 \qquad \vec{p}_2 = V_2 \vec{P}_2 = V_2 \varepsilon_0 \chi_2 \vec{E}_0$$

$$W_{12} = K_e \frac{\vec{p}_1 \cdot \vec{p}_2 - 3(\vec{p}_1 \cdot \hat{d})(\vec{p}_2 \cdot \hat{d})}{d^3} = K_e \frac{p_1 p_2 - 3 p_1 p_2 (\hat{E}_0 \cdot \hat{d})^2}{d^3}$$

$$= K_e \frac{p_1 p_2}{d^3}\left[1 - 3(\hat{E}_0 \cdot \hat{d})^2\right] = \frac{1}{4\pi\varepsilon_0} \frac{\varepsilon_0^2 E_0^2 V_1 V_2 \chi_1 \chi_2}{d^3}\left[1 - 3(\hat{E}_0 \cdot \hat{d})^2\right]$$

$$= \frac{\varepsilon_0 E_0^2 V_1 V_2 \chi_1 \chi_2}{4\pi d^3}\left[1 - 3(\hat{E}_0 \cdot \hat{d})^2\right]$$

10. *Electricity and Magnetism* (Fall 2005)

A pulsar emits bursts of radio waves, which are observed from Earth at two different frequencies, say $\omega_1$ and $\omega_2$. An astronomer notes that the arrival time of the bursts is delayed at the lower frequency: the pulse at $\omega_1$ arrives after the pulse at $\omega_2$. The delay, $\tau$, is due to dispersion in the interstellar medium. Assuming this medium consists of ionized hydrogen, estimate the distance $s$ of the pulsar from the earth, as follows:

(a) Show that the electron plasma frequency for the dilute plasma — consisting of (heavy) ions and free electrons — is

$$\omega_p = \left(\frac{4\pi N e^2}{m_e}\right)^{1/2}$$

in e.s.u. Here $N$ is the number of electrons per unit volume.

(b) Show that the index of refraction of the plasma is

$$n = \sqrt{\epsilon} = \left(1 - \frac{{\omega_p}^2}{\omega^2}\right)^{1/2}$$

*Hint*: Write the equation of motion for a free electron in an oscillating ($e^{-i\omega t}$) electric field and find the plasma's polarizability $\chi$. Then $\epsilon = 1 + 4\pi\chi$.

(c) From the relation above find the group velocity of the light, and use this result to find the distance to the pulsar. (You may assume the frequencies are large compared to $\omega_p$.)

a) The plasma frequency is the natural frequency* of charge density oscillations in a conductor (e.g. plasma, metal). If we assume that $\rho$ is oscillating with no externally applied field and ignore damping and any restoring force, we may easily derive $\omega_p$.

$\rho(\vec x, t) = \rho_0(\vec x)\, e^{-i\omega_p t}$ where $\rho_0(\vec x)$ varies positively and negatively due to the displacement of electrons against a constant proton charge density, but over large scales averages to zero. ($\rho_0(\vec x)$ must vary in sign for charge to be conserved.)

$\vec\nabla \cdot \vec E = \frac{\rho}{\varepsilon_0} \Rightarrow \vec E(\vec x, t) = \vec E_0(\vec x)\, e^{-i\omega_p t}$

Take $\vec d$ to be the displacement an electron near position $\vec x$:

$m_e \ddot{\vec d} = \vec F_{net} = -e\vec E(\vec x, t) = -e\vec E_0(\vec x)\, e^{-i\omega_p t}$

The velocity of that electron is thus $\vec v = \dot{\vec d} = \overset{\vec 0}{\cancel{\vec v_0}} + \frac{e}{i\omega_p m_e}\vec E_0(\vec x)\, e^{-i\omega_p t}$

The current is due to the motion of the electrons, since the protons are essentially stationary, and the electrons have an average number density of $n_e = N$ (so an average charge density $\bar\rho_e = -n_e e$).

10. Electricity and Magnetism (Fall 2005)

a) (continued)

$$\Rightarrow \quad \underset{\sim}{\vec{J}}(\vec{x}) \approx \bar{\rho}_e \underset{\sim}{\vec{U}}(\vec{x}) = (-n_e e)\left[\frac{e}{i\omega_p m_e} \vec{E}_0(\vec{x})\, e^{-i\omega_p t}\right]$$

Finally, we use continuity:

$$\vec{\nabla}\cdot\vec{J} = -\frac{\partial\rho}{\partial t} \quad \Rightarrow \quad -\frac{n_e e^2}{i\omega_p m_e} \underbrace{\vec{\nabla}\cdot\vec{E}_0(\vec{x})}_{\rho_0(\vec{x})/\varepsilon_0} e^{-i\omega_p t} = +i\omega_p \rho_0(\vec{x}) e^{-i\omega_p t}$$

$$\Rightarrow \quad i\frac{n_e e^2}{\varepsilon_0 \omega_p m_e} = i\omega_p \quad \Rightarrow \quad \omega_p = \sqrt{\frac{n_e e^2}{\varepsilon_0 m_e}}$$

In e.s.u. $\varepsilon_0 = \frac{1}{4\pi}$, and using $n_e = N$, $\quad \omega_p = \sqrt{\frac{4\pi N e^2}{m_e}}$ ✓

b) Now we include damping, restoring, and driving forces:

$$\underset{\sim}{\vec{P}} = \varepsilon_0 \chi \underset{\sim}{\vec{E}} \approx (-n_e e)\underset{\sim}{\vec{d}} \qquad \underset{\sim}{\vec{E}}(\vec{x},t) = \vec{E}_0(\vec{x})\, e^{-i\omega t} \qquad \left(\underset{\sim}{\vec{d}} = -\frac{\varepsilon_0 \chi}{n_e e}\underset{\sim}{\vec{E}}\right)$$

$$m_e \ddot{\underset{\sim}{\vec{d}}} = \underset{\sim}{\vec{F}}_{net} = -e\,\underset{\sim}{\vec{E}}(\vec{x},t) - k_d \dot{\underset{\sim}{\vec{d}}} - k_r \underset{\sim}{\vec{d}}$$

$$\Rightarrow \quad m_e\left[\ddot{\underset{\sim}{\vec{d}}} + \gamma_d \dot{\underset{\sim}{\vec{d}}} + \omega_p^2 \underset{\sim}{\vec{d}}\right] = -e\,\underset{\sim}{\vec{E}} \qquad \text{and let } \underset{\sim}{\vec{d}} = \underset{\sim}{d_0}\, e^{-i\omega t}$$

$$\left(\gamma_d \equiv \frac{k_d}{m_e} \qquad \omega_p^2 = \frac{k_r}{m_e}\right)$$

$$\Rightarrow \quad \left[-\omega^2 - i\omega\gamma_d + \omega_p^2\right]\underset{\sim}{\vec{d}} = -\frac{e}{m_e}\underset{\sim}{\vec{E}}$$

$$\Rightarrow \quad -\frac{\varepsilon_0 \chi}{n_e e} = -\frac{e}{m_e\left[-\omega^2 - i\omega\gamma_d + \omega_p^2\right]} \quad \Rightarrow \quad \chi = \frac{\overbrace{\left(\frac{n_e e^2}{\varepsilon_0 m_e}\right)}^{\omega_p^2}}{\left[-\omega^2 - i\omega\gamma_d + \omega_p^2\right]}$$

For $\omega \gg \gamma_d, \omega_p$ $\quad \chi = -\frac{\omega_p^2}{\omega^2}$ $\quad$ and $\quad \varepsilon = \varepsilon_0(1+\chi) = \varepsilon_0\left(1 - \frac{\omega_p^2}{\omega^2}\right)$

So $\quad n = \frac{c}{v} = \sqrt{\frac{\mu\varepsilon}{\mu_0\varepsilon_0}} \approx \sqrt{\frac{\varepsilon}{\varepsilon_0}} = \left(1 - \frac{\omega_p^2}{\omega^2}\right)^{1/2}$ ✓

c) $\quad v_g = \frac{d\omega}{dk}, \quad k^2 = \omega^2\mu\varepsilon \quad \Rightarrow \quad k = \omega\sqrt{\mu_0\varepsilon_0}\,(1 - \omega_p^2/\omega^2)^{1/2} = \frac{\omega}{c}(1 - \omega_p^2/\omega^2)^{1/2}$

$$= \frac{1}{c}(\omega^2 - \omega_p^2)^{1/2}$$

$$\Rightarrow \quad \omega = (c^2k^2 + \omega_p^2)^{1/2}$$

$$v_g = \frac{1}{\cancel{2}}(c^2k^2 + \omega_p^2)^{-1/2}\,\cancel{2}c^2k = \left((\omega^2 \cancel{-\omega_p^2}) \cancel{+\omega_p^2}\right)^{-1/2} c\,(\omega^2 - \omega_p^2)^{1/2}$$

$$= c\,(1 - \omega_p^2/\omega^2)^{1/2}$$

10. Electricity and Magnetism (Fall 2005)

c) (continued)

distance to pulsar:

$$D = v_{g1} t_1 = v_{g2} t_2 \qquad \tau = t_1 - t_2 \qquad t_1 > t_2 \qquad \omega_1 < \omega_2$$

$$v_{g1} < v_{g2}$$

$$D = v_{g1} t_1$$

$$D = v_{g2}(t_1 - \tau)$$

$$v_{g2} D = \cancel{v_{g1} v_{g2} t_1}$$

$$- v_{g1} D = - \cancel{v_{g1} v_{g2} t_1} + v_{g1} v_{g2} \tau$$

$$(v_{g2} - v_{g1}) D = v_{g1} v_{g2} \tau$$

$$D = \frac{v_{g1} v_{g2} \tau}{(v_{g2} - v_{g1})} = \frac{\tau}{\left(\frac{1}{v_{g1}} - \frac{1}{v_{g2}}\right)} = \frac{c\tau}{\left[(1 - \omega_p^2/\omega_1^2)^{-1/2} - (1 - \omega_p^2/\omega_2^2)^{-1/2}\right]}$$

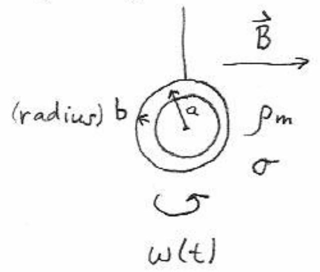
$$\approx \frac{c\tau}{\left[(1 + \frac{1}{2}\omega_p^2/\omega_1^2) - (1 + \omega_p^2/\omega_2^2)\right]} \qquad \text{for } \omega_1, \omega_2 \gg \omega_p$$

$$= \frac{2c\tau}{\omega_p^2\left(\frac{1}{\omega_1^2} - \frac{1}{\omega_2^2}\right)}$$

* This definition of the plasma frequency corresponds to Jackson's assertions (pg 313, 3rd Ed.).

11. *Electricity and Magnetism* (Fall 2005)

A thin copper circular ring (conductivity $\sigma$, *mass* density $\rho_m$) is suspended so it can rotate freely about one diameter. There is a uniform magnetic field $\mathbf{B}$ perpendicular to the axis of rotation. The initial rotation frequency is $\omega_0$. Calculate the time it takes for the frequency to decrease to $1/e$ of its original value, assuming the energy all goes into Joule heating. (Assume the requested time $\tau$ is large compared to the rotation period.)




$$\omega(t=0) = \omega_0 \qquad \omega(\tau) = \frac{1}{e}\omega_0 \qquad \tau = ?$$

$$E(t) = \frac{1}{2} I \omega^2(t) \qquad (1)$$

$$I = \int r^2 dm \approx a^2 M = a^2 \rho_m V = a^2 \rho_m (\pi b^2 2\pi a)$$
$$= 2\pi^2 \rho_m b^2 a^3$$

Joule heating (averaged over many cycles):

$$\frac{dE}{dt} = -\frac{\overline{V^2(t)}}{R} \qquad R = \rho \frac{L}{A} \approx \frac{1}{\sigma} \frac{2\pi a}{\pi b^2} = \frac{2a}{\sigma b^2}$$

$$V(t) = -\partial_t \Phi_B(t) = -\partial_t (\vec{B} \cdot \vec{A}) = -\partial_t (B \pi a^2 \sin[\omega(t) t])$$
$$= -B\pi a^2 \cos[\omega(t) t] \frac{d\omega}{dt}$$

$$\frac{dE}{dt} = -\frac{(B\pi a^2)^2}{R} \overline{\cos^2[\omega(t) t] \left(\frac{d\omega}{dt}\right)^2} = -\frac{(B\pi a^2)^2}{2R} \left(\frac{d\omega}{dt}\right)^2 \equiv -K \dot{\omega}^2$$

Using (1) above:

$$\frac{dE}{dt} = \frac{d}{dt}\left(\frac{1}{2} I \omega^2\right) = I \omega \dot{\omega} = -K \dot{\omega}^2$$

$$\Rightarrow -\frac{K}{I} \dot{\omega} = \omega \qquad \Rightarrow \qquad \omega = \omega_0 e^{-\frac{K}{I} t}$$

$$\Rightarrow \tau = \frac{I}{K} = \frac{(2\cancel{\pi^2} \rho_m \cancel{b^2} a^3)}{B^2 \cancel{\pi^2} a^4} \, 2\left(\frac{2a}{\sigma \cancel{b^2}}\right) = \frac{8\rho_m}{\sigma B^2}$$

14. *Statistical Mechanics and Thermodynamics* (Fall 2004)

Consider black body radiation at temperature $T$. What is the average energy per photon in units of $kT$?

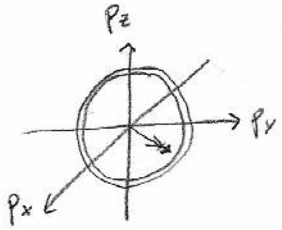
You may find the following formulae useful:

$$\int_0^\infty \frac{x^3 dx}{e^x - 1} = \frac{\pi^4}{15} \approx 6.5; \qquad \int_0^\infty \frac{x^2 dx}{e^x - 1} \approx 2.4$$

$$\epsilon = pc$$

$$\langle \epsilon \rangle = \sum_i \epsilon_i P(\epsilon_i) = \frac{\int \epsilon f(\epsilon) w(\epsilon) d\epsilon}{\int f(\epsilon) w(\epsilon) d\epsilon} \qquad \text{for photons} \quad f(\epsilon) = \frac{1}{e^{\beta\epsilon} - 1}$$

$$w(\epsilon) = \frac{d\Omega}{d\epsilon} = \frac{\frac{1}{h^3} dV_{phase}}{d\epsilon}$$




$$dV_{phase} = \int_{\mathbb{R}^3} d^3r \int_{S_{octant}} p^2 d\Omega_p \, dp = (V)(4\pi p^2) dp$$

$$= \frac{4\pi V}{c^3} \epsilon^2 d\epsilon \quad \Rightarrow \quad w(\epsilon) = \frac{4\pi V}{(hc)^3} \epsilon^2$$

$$\langle \epsilon \rangle = \frac{\frac{4\pi V}{(hc)^3} \int \frac{\epsilon^3 d\epsilon}{e^{-\beta\epsilon} - 1}}{\frac{4\pi V}{(hc)^3} \int \frac{\epsilon^2 d\epsilon}{e^{-\beta\epsilon} - 1}} = \frac{\left(\frac{1}{\beta}\right)^4 \int \frac{x^3 dx}{e^{-x} - 1}}{\left(\frac{1}{\beta}\right)^3 \int \frac{x^2 dx}{e^{-x} - 1}} \qquad \text{with } x \equiv \beta\epsilon, \quad \epsilon = \frac{x}{\beta}$$

$$\approx \left(\frac{6.5}{2.4}\right)\left(\frac{1}{\beta}\right) \approx 2.7 \, (kT)$$

14. *Statistical Mechanics and Thermodynamics* (Fall 2004)

Consider black body radiation at temperature $T$. What is the average energy per photon in units of $kT$?

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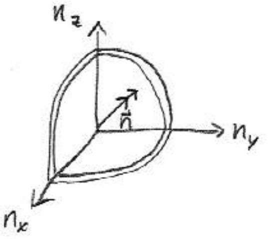
Consider a cubical cavity for simplicty. The photons oscillate in distinct modes:

$$\epsilon = pc = \hbar k c = \hbar c\sqrt{(n_x\pi/L)^2 + (n_y\pi/L)^2 + (n_z\pi/L)^2} = \frac{\pi\hbar c}{L}\sqrt{n_x^2+n_y^2+n_z^2} = \frac{\pi\hbar c}{L} n$$

$$\langle\epsilon\rangle = \sum_i \epsilon_i P(\epsilon_i) = \frac{\int_0^\infty \epsilon\, f(\epsilon)\,\omega(\epsilon)\,d\epsilon}{\int_0^\infty f(\epsilon)\,\omega(\epsilon)\,d\epsilon} \qquad f(\epsilon) = \frac{1}{e^{-\beta\epsilon}-1} \text{ for photons}$$




$$\#\text{ states} = \rho(n)\,dn = \frac{1}{8}4\pi n^2 dn = \omega(\epsilon)\,d\epsilon$$

$$n = \frac{L}{\pi\hbar c}\epsilon \qquad dn = \frac{L}{\pi\hbar c}d\epsilon$$

$$\Rightarrow \omega(\epsilon)\,d\epsilon = \frac{1}{8}4\pi\left(\frac{L}{\pi\hbar c}\right)^3\epsilon^2 d\epsilon = \frac{V}{2\pi^2}\frac{\epsilon^2}{(\hbar c)^3}d\epsilon$$

$$I_1 \equiv \int_0^\infty \epsilon\, f(\epsilon)\,\omega(\epsilon)\,d\epsilon = \frac{V}{2\pi^2(\hbar c)^3}\left[\int_0^\infty \frac{\epsilon^3\,d\epsilon}{e^{-\beta\epsilon}-1} = \frac{1}{\beta^4}\int_0^\infty \frac{x^3dx}{e^{-x}-1} \approx 6.5\,(kT)^4\right] \qquad x \equiv \beta\epsilon$$

$$I_2 \equiv \int_0^\infty f(\epsilon)\,\omega(\epsilon)\,d\epsilon = \frac{V}{2\pi^2(\hbar c)^3}\left[\int_0^\infty \frac{\epsilon^2 d\epsilon}{e^{-\beta\epsilon}-1} = \frac{1}{\beta^3}\int_0^\infty \frac{x^2dx}{e^{-x}-1} \approx 2.4\,(kT)^3\right]$$

$$\Rightarrow \langle\epsilon\rangle = \frac{I_1}{I_2} \approx \frac{6.5\,(kT)^4}{2.4\,(kT)^3} \approx 2.7\,(kT)$$