

WRITTEN COMPREHENSIVE EXAMINATION FOR THE MASTER'S
DEGREE AND QUALIFYING EXAMINATION FOR THE PH.D. DEGREE
DEPARTMENT OF PHYSICS

Tuesday, March 26, and Wednesday, March 27, 2002

PART I – TUESDAY, MARCH 26

Important – please read carefully.

The exam (8 hours) is in two parts:

PART 1 Quantum Mechanics, Thermodynamics, Statistical Mechanics

March 26 7 Problems – DO ALL PROBLEMS.

9:00–1:00 This part will be collected at the end of four hours.
Each problem counts for 20 points; the total is 140.

PART 2 Electromagnetic Theory, Thermodynamics, Statistical Mechanics

March 27 7 Problems – DO ALL PROBLEMS.

9:00–1:00 This part will be collected at the end of four hours.
Each problem counts for 20 points; the total is 140.

Instructions

- 1) This is a closed book exam and calculators are not be used.
- 2) Work each problem on a separate sheet of paper. **Use one side only.**
- 3) Print **your name and problem number** on EACH AND EVERY page. (Note:
Pages without names may not be counted.)
- 4) Return the problem page as the first page of your answers.
- 5) If a part of any question seems ambiguous to you, state clearly what your interpretations and answer the question accordingly.

1. Quantum Mechanics.

The Hamiltonian of a one-dimensional harmonic oscillator in dimensionless units ($m = \hbar = \omega = 1$) is

$$H = a^\dagger a + \frac{1}{2}$$

where $a = \frac{1}{\sqrt{2}}(x + ip)$, $a^\dagger = \frac{1}{\sqrt{2}}(x - ip)$. One of the unnormalized eigenfunctions of this Hamiltonian is given by the expression

$$\psi(x) = (2x^3 - 3x) e^{-x^2/2}$$

- (a) What is the energy eigenvalue which corresponds to this wavefunction?
- (b) Find the two other (unnormalized) energy eigenfunctions which are closest in energy to this wavefunction.

2. Quantum Mechanics.

A system of two particles each with spin $\frac{1}{2}$ is described by the Hamiltonian

$$H = A(S_{1z} + S_{2z}) + B\mathbf{S}_1 \cdot \mathbf{S}_2$$

where S_1 and S_2 are the two spins, S_{1z} and S_{2z} are their z-components, and A and B are constants. Find all the energy levels of the Hamiltonian.

3. Quantum Mechanics.

- (a) Two Hermitian operators anticommute. Is it possible for them to have simultaneous eigenkets?
- (b) Do position operators at unequal times commute in general in the Heisenberg representation? Give a simple example illustrating your answer.
- (c) Explain how you would interpret the energy-time uncertainty relation. Illustrate your answer with a state that is a superposition of two energy eigenstates.

4. Quantum Mechanics.

Consider a mass m particle in one dimension moving in the potential

$$V(x) = V_0 \left| \frac{x}{x_0} \right| \quad (1)$$

where V_0 and x_0 are constants.

Estimate the ground state energy of the particle.

Your score on this problem will be

$$\text{Your Score} = 20 \times e^{-\left(\frac{E-E_0}{E_0}\right)^2} \quad (2)$$

20 is the maximum score, E is your estimate, E_0 is the exact ground state energy, and E and E_0 are evaluated at $V_0 = \frac{\hbar^2}{mx_0^2} = 1 \text{ eV}$.

5. Quantum Mechanics.

Charge on a circle: A small bead with charge e and mass m is confined to move on a circular ring in the $x - y$ plane with radius r . A weak, uniform electric field of intensity E_0 pointing in the positive x direction is turned on.

- (a) What are the eigenfunctions and energy eigenvalues for $E_0 = 0$? What are the degeneracies?
- (b) For $E_0 \neq 0$, show that the electric field operator has vanishing matrix elements between degenerate eigenstates of the unperturbed Hamiltonian.
- (c) Find an approximation for the energy levels which includes the first non-trivial term containing E_0 .

6. Statistical Mechanics and Thermodynamics

Calculate the collision frequency for the collisions between the molecules of a gas and a fixed sphere of diameter D . The molecules have an average diameter d . The gas has a temperature T .

7. Statistical Mechanics and Thermodynamics

This is an essay question. Answer two of the following three questions.

- (a) You are asked about the second law of thermodynamics, and you give one of the formulations, that there is no process the sole effect of which is the conversion of heat into work. The inquirer then points out that a steam engine converts heat into work. Explain how this is not a violation of the second law of thermodynamics. Your explanation should include an analysis of the steam engine, and a discussion of heat engines in general.
- (b) You read an article in a physics journal in which a group of researchers announce that they have cooled a system to absolute zero. Discuss why one ought to be skeptical of this claim. Invoke the appropriate laws of thermodynamics.
- (c) Explain, using the laws of thermodynamics, why a substance cannot have a negative heat capacity.

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PART II - WEDNESDAY, MARCH 27

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PART 2 Electromagnetic Theory, Thermodynamics, Statistical Mechanics

March 27 7 Problems – DO ALL PROBLEMS.

**9:00–1:00 This part will be collected at the end of four hours.
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- 8) If a part of any question seems ambiguous to you, state clearly what your interpretations and answer the question accordingly.

8. Electricity and Magnetism

Consider the AC current density in a conductor obeying Ohm's law, $\mathbf{J} = \sigma \mathbf{E}$, where σ is the constant real conductivity. Suppose that we have a conductor which is a 2D metal sheet of thickness d located in the $x - z$ plane and that current flows in the x -direction. If the AC current density is given by the expression $J_x(y, t) = \operatorname{Re}(\mathcal{J}_x(y) e^{i\omega t})$,

- (a) Find $\mathcal{J}_x(y)$.
- (b) Plot $|\mathcal{J}_x(y)|$ vs. y .
- (c) Find the phase shift of the current density between the center and the edge for $\omega\mu_0\sigma d \gg 1$.

9. Electricity and Magnetism

Lorentz transformation in one spatial dimension.

- (a) Assuming that x and t transform according to the Lorentz transformation law, show that the combination $(ds)^2 = (dx)^2 - c^2(dt)^2$ is the same in all inertial frames.
- (b) Show that the elapsed time $d\tau$ between two events occurring at the same location in the laboratory is related to the elapsed time dt' in a frame M moving along the $+x$ axis with speed v according to

$$d\tau = dt' \sqrt{1 - \frac{1}{c^2} \left(\frac{dx'}{dt'} \right)^2} = dt'/\gamma.$$

Here $dx'/dt' = -v$ is the velocity of the fixed laboratory point as seen in the frame M and $\gamma = 1/\sqrt{1 - \beta^2}$, where $\beta = v/c$.

- (c) Show that the spatial separation ds between two events occurring simultaneously ($dt = 0$) in the laboratory is related to the spatial separation dx' in the moving frame M by

$$ds = dx' \sqrt{1 - c^2 \left(\frac{dt'}{dx'} \right)^2} = A dx'$$

Here, dt' is the time in the moving frame which elapses between the two events. Determine the proportionality constant A .

- (d) The ratio dx'/dt' is a function of the relative speed v between the two frames. Is it the same function in parts (b) and (c)? Explain.

10. Electricity and Magnetism

An infinitely long solenoid of circular cross-section of radius a carries a current I along helical windings of n turns per axial length. The current is closed via a straight conductor of radius $b < a$ centered along the axis.

- (a) Find the magnetic field inside and outside the solenoid.
- (b) Calculate the self-inductance per axial length due to fields produced by the solenoidal and axial currents.
- (c) Find the "inner" inductance due to magnetic fields inside the conductors.

11. Electricity and Magnetism

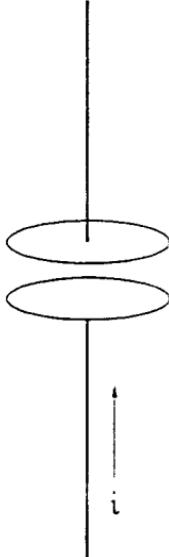
Forces on conductors

- (a) A grounded spherical conductor of radius a is placed at a distance $r \gg a$ far from a point charge q . Determine the direction and functional form of any force experienced by the conductor due to the presence of the point charge. (You don't need to calculate any proportionality constant.)
- (b) What is the direction and functional form of any force experienced by the conductor in part (a) if the conductor is isolated and uncharged and has an unknown shape but the same small volume? If your answer is different from that in part(a), explain the difference in physical terms. (Hint: Use a multipole expansion of the force acting on an arbitrary charge distribution in an external electric field.)

12. Electricity and Magnetism

Consider an infinitely long, straight wire along the \hat{z} -axis with a capacitor at $z = 0$, as shown in the picture below. The wire carries current i . Suppose that the plates of the capacitor are circular of radius r_c and are separated by a distance d ; there is vacuum between them.

- What is the capacitance of the capacitor? Assume that $r_c \gg d$ so that you can ignore fringing fields.
- What is the magnetic field at $z = 0$ at a distance $r \gg r_c$ from the capacitor.
- What is the magnetic field a distance r from the wire for $|z| \gg d$?



13. Statistical Mechanics and Thermodynamics

Consider an ideal monoatomic gas in which each atom has two internal energy states, one an energy Δ above the other. There are N atoms in a volume V at temperature T .

Find the a) chemical potential, b) free energy, c) entropy, d) pressure and e) heat capacity at constant pressure.

14. Statistical Mechanics and Thermodynamics

- (a) What is the free energy (as a function of temperature, T , volume, V , and particle number, N) of a ideal gas obeying Maxwell-Boltzmann statistics?
- (b) Assume that the ideal gas is made up of hydrogen atoms. Now the free energy must include a contribution reflecting the different possible electronic excited states of the hydrogen atoms. Show that this contribution diverges. What cuts off this divergence in a real gas?

1) (a) apply $a^+ a^- \Psi(x) \Rightarrow n = 3$

(b) $\Psi_2 \propto a^+ \Psi(x) = (2x^2 - 1) e^{-x^2/2}$

$\Psi_4 \propto a^+ \Psi(x) = (4x^4 - 12x^2 + 3) e^{-x^2/2}$

2) $H = \bar{A}S_z + \bar{B}\left(\frac{S^2}{2} - \frac{3}{4}\bar{\tau}^2\right)$ singlet: $\frac{1}{\sqrt{2}}(1\uparrow\downarrow - 1\downarrow\uparrow)$ $\Rightarrow E_{00} = -\frac{3B}{4}\hbar^2$
triplet: $E_{11} = \hbar A + \frac{B\hbar^2}{4}$, $E_{10} = \frac{B\hbar^2}{4}$, $E_{1,-1} = -\hbar A + \frac{B\hbar^2}{4}$

3) (a) no, unless one of the eigenstates is zero

(b) No! consider free particle: $x_i(t) = x_i(0) + \frac{p_i(0)}{m}t$ $[x_i, x_j] = -i\hbar \delta_{ij}$

(c) Δt = the time it takes to change subatomically; sharper the energy is \Rightarrow large time scale is

4) $E_{\min} = \frac{3}{2} \left(\frac{\hbar^2}{2m\lambda_0^2} \frac{V_0}{\pi} \right)^{1/3} = 1.02 \text{ eV}$

5) (c) $SE_n = -\frac{ne^2 r^4 |E|^2}{2\hbar^2} \quad n=0 \quad = \frac{ne^2 r^4 |E|^2}{2\hbar^2} \quad \frac{1}{4n^2-1} \quad n>0$

perturbation: $V(\phi) = e|E|r \cos\phi$

6) $v = (d+D)^2 n \sqrt{\frac{\pi k T}{2m}}$

7) (b) $dz = \frac{1}{c} dt' \quad (c) \quad ds = \frac{1}{c} dx' = dx' \sqrt{1 - c^2 \left(\frac{dt'}{dx'} \right)^2} \Rightarrow \frac{dx'}{dt'} = -\frac{c^2}{c^2}$

(d) $\boxed{\frac{dx'}{dt'} = \left(\frac{dx'}{dt'} \right)_{\text{const } x}} \quad \boxed{\frac{dx'}{dt'} = -\frac{c^2}{c^2} = \left(\frac{dx'}{dt'} \right)_{\text{const } t} \neq \left(\frac{dx'}{dt'} \right)_{\text{const } x}}$

8) $\vec{F} = \frac{q\vec{r}'}{|F-r'|^3} \Rightarrow \left(-\frac{q^2 a}{r} \left(-\frac{\vec{r}}{r^2} \right) \right) \propto \frac{\vec{r}}{r^3} \quad \vec{F} \xrightarrow{q'} q$

(b) $\vec{F}_{\text{dipole}} \propto \frac{1}{r^5}$ answer different b/c the total induced charge is $Q=0$ here
for part (a), $Q \propto \frac{1}{r}$

13) see answer to Fall 2002 # 13

Spring 2002.

$$a = \frac{1}{\sqrt{2}}(x + ip) \quad a^\dagger = \frac{1}{\sqrt{2}}(x - ip)$$

$$\psi_n(x) = (2x^3 - 3x) e^{-x^2/2}$$

$$a^\dagger a |\psi_n\rangle = n |\psi_n\rangle$$

$$E_n = \hbar\omega(n + \frac{1}{2})$$

$$[x, p] = i$$

$$a^\dagger a^\dagger a a = \frac{1}{2}$$

$$a^\dagger a = \frac{1}{\sqrt{2}}(x - ip) \frac{1}{\sqrt{2}}(x + ip) = \frac{1}{2}(x^2 + p^2 - 1) = \frac{1}{2}(x^2 - \frac{\hbar^2 \partial^2}{m x^2} - 1) \quad \text{but } \hbar = 1$$

$$\frac{\partial}{\partial x} (2x^3 - 3x) e^{-x^2/2} = -x e^{-x^2/2} (2x^3 - 3x) + e^{-x^2/2} (6x^2 - 3)$$

$$\begin{aligned} \frac{\partial^2}{\partial x^2} (2x^3 - 3x) e^{-x^2/2} &= -e^{-x^2/2} (2x^3 - 3x) + x^2 e^{-x^2/2} (2x^3 - 3x) \\ &\quad - x e^{-x^2/2} (6x^2 - 3) - x e^{-x^2/2} (6x^2 - 3) + e^{-x^2/2} (12x) \\ &= x^2 e^{-x^2/2} (2x^3 - 3x) - 2x e^{-x^2/2} (6x^2 - 3) + e^{-x^2/2} (12x) - e^{-x^2/2} (2x^3 - 3x) \end{aligned}$$

$$\frac{1}{2} e^{-x^2/2} [2x^5 - 3x^3 - 2x^3 + 3x - 2x^5 + 3x^3 + 12x^3 - 6x - 12x + 2x^3 - 3x] =$$

$$x \frac{-x^2}{2} [12x^3 - 18x] = [6x^3 - 9x] e^{-x^2/2} = n [2x^3 - 3x] e^{-x^2/2}$$

$$n = 3$$

$$E_n = \hbar\omega(3 + \frac{1}{2})$$

b) Find next two closes + states

a^+ is the raising operator

$$\Psi_3 = a^+ \Psi(x) = \frac{1}{\sqrt{2}} (x - i\rho) \Psi(x) = \frac{1}{\sqrt{2}} \left(x - \frac{\partial}{\partial x} \right) \Psi(x)$$

$$\frac{\partial}{\partial x} \Psi(x) = e^{-x^2/2} (6x^2 - 3) - x e^{-x^2/2} (2x^3 - 3x)$$

$$\begin{aligned}\Psi_3(x) &= \frac{1}{\sqrt{2}} e^{-x^2/2} [2x^4 - 3x^2 - 6x^2 + 3 + 2x^4 - 3x^2] \\ &= \frac{1}{\sqrt{2}} e^{-x^2/2} [4x^4 - 12x^2 + 3]\end{aligned}$$

$$\begin{aligned}\Psi_4(x) &= a^+ \Psi(x) = \frac{1}{\sqrt{2}} (x + i\rho) \Psi(x) = \frac{1}{\sqrt{2}} \left(x + \frac{\partial}{\partial x} \right) \Psi(x) \\ &= \frac{1}{\sqrt{2}} e^{-x^2/2} [2x^4 - 3x^2 + 6x^2 - 3 - 2x^4 + 3x^2] \\ &= \frac{3}{\sqrt{2}} e^{-x^2/2} [2x^2 - 1]\end{aligned}$$

Spring 2002 #1 (p 1 of 3)

The Hamiltonian of a one-dimensional harmonic oscillator in dimensionless units ($m=\hbar=\omega=1$) is

$$H = a^\dagger a + \frac{1}{2}$$

where $a = \frac{1}{\sqrt{2}}(x+ip)$, $a^\dagger = \frac{1}{\sqrt{2}}(x-ip)$. One of the unnormalized eigenfunctions of the Hamiltonian is given by the expression

$$\Psi(x) = (2x^3 - 3x) e^{-x^2/2}$$

(a) What is the energy eigenvalue which corresponds to this wave function?

We know that $a^\dagger a |\Psi_n(x)\rangle = n |\Psi_n(x)\rangle$, where $a^\dagger a$ is the number operator

$$\text{where } \Psi(x) \propto \underset{\substack{\uparrow \\ \text{unnormalized}}}{\Psi_{n=?}(x)}$$

So, let's apply the lowering operator to $\Psi(x)$

$$a\Psi(x) = \frac{1}{\sqrt{2}}(x+ip)[(2x^3 - 3x)e^{-x^2/2}]$$

$$\text{where } p \rightarrow -i\frac{d}{dx} \text{ in 1-d}$$

That is,

$$\begin{aligned} a\Psi(x) &= \frac{1}{\sqrt{2}}(x + \frac{d}{dx})[(2x^3 - 3x)e^{-x^2/2}] \\ &= \frac{1}{\sqrt{2}} \left[(2x^4 - 3x^2)e^{-x^2/2} + (6x^2 - 3)e^{-x^2/2} - (2x^3 - 3x)x e^{-x^2/2} \right] \\ &= \frac{1}{\sqrt{2}} e^{-x^2/2} \left[\underbrace{2x^4 - 3x^2}_{\#} + \underbrace{6x^2 - 3}_{\#} - \underbrace{(2x^3 - 3x)x}_{\#} \underbrace{e^{-x^2/2}}_0 \right] \end{aligned}$$

$$\Rightarrow a\Psi(x) = \frac{3}{\sqrt{2}}(2x^2 - 1) e^{-x^2/2} \quad (1)$$

Spring 2002 #1 (p 2 of 3)

Now, let's apply a^+ to eq (1)

$$\begin{aligned} a^+ a \Psi(x) &= \frac{3}{\sqrt{2}} \frac{1}{\sqrt{2}} \left(x - \frac{d}{dx} \right) (2x^2 - 1) e^{-x^2/2} \\ &= \frac{3}{2} \left[(2x^3 - x) e^{-x^2/2} - 4x e^{-x^2/2} + (2x^2 - 1)x e^{-x^2/2} \right] \\ &= \frac{3}{2} e^{-x^2/2} [2x^3 - x - 4x + 2x^3 - x] \\ &= \frac{3}{2} e^{-x^2/2} (4x^3 - 6x) \end{aligned}$$

$$\Rightarrow a^+ a \Psi(x) = 3 e^{-x^2/2} (2x^3 \cdot 3x) = 3 \Psi(x)$$

Thus, $n = 3$

(b) Find the two other (unnormalized) energy eigenfunctions which are closest in energy to this wave function.

So, we want to find the $n=2$ and $n=4$ wave functions. We get $n=2$ state by applying the lowering operator. From eq (1), we know this is

$$n=2 \quad a \Psi(x) \propto (2x^2 - 1) e^{-x^2/2}$$

To get the $n=4$ stat, we need to apply the raising operator a^+ .

Spring 2002 #1 (p 3 of 3)

So, we have

$$\begin{aligned} a^+ \psi(x) &= \frac{1}{\sqrt{2}} \left(x - \frac{d}{dx} \right) (2x^3 - 3x) e^{-x^2/2} \\ &= \frac{e^{-x^2/2}}{\sqrt{2}} \left(2x^4 - 3x^2 - (6x^2 - 3) + (2x^4 - 3x^2) \right) \\ &= \frac{e^{-x^2/2}}{\sqrt{2}} (4x^4 - 12x^2 + 3) \end{aligned}$$

∴ $n=4$

$$a^+ \psi(x) \propto (4x^4 - 12x^2 + 3) e^{-x^2/2}$$

Spring 2002 #2

$$H = A(S_{1z} + S_{2z}) + B S_1 \cdot S_2 \quad \text{Spin } \frac{1}{2}$$

$$S = S_1 + S_2 \quad S^2 = S_1^2 + S_2^2 + 2 S_1 \cdot S_2 \quad S_1 \cdot S_2 = \frac{(S^2 - S_1^2 - S_2^2)}{2}$$

$$H = A(S_{1z} + S_{2z}) + B \frac{(S^2 - S_1^2 - S_2^2)}{2}$$

$$S^2 |S_m\rangle = S(S+1)$$

$$S_z |S_m\rangle = I |S_m\rangle$$

$$= A(S_{1z} + S_{2z}) + B \left(\frac{S^2}{2} - \frac{3}{4} \right)$$

$$\frac{1}{2}(k_2+1)$$

$$\frac{1}{2}\left(\frac{3}{2}\right) = \frac{3}{4}$$

$$S=1 \quad \left\{ \begin{array}{c} |11\rangle \\ Y_{10}[|1\uparrow\rangle + |1\downarrow\rangle] \\ |1\downarrow\downarrow\rangle \end{array} \right\}_{m=0}^{m=1} \quad \left\{ \begin{array}{c} m=1 \\ m=0 \\ m=-1 \end{array} \right.$$

$$\frac{1}{2} \otimes \frac{1}{2}$$

$$S=1 \text{ or } 0$$

$$S=0 \quad \frac{1}{\sqrt{2}}[|1\uparrow\rangle - |1\downarrow\rangle] \quad m=0$$

$$H = A S_2 + B \left(\frac{S^2}{2} - \frac{3}{4} \right) \quad \text{since } S_{1z} + S_{2z} = S_2$$

$$H\Psi_{11} = A + \frac{1}{4}B \quad H\Psi_{10} = \frac{1}{4}B \quad H\Psi_{1-1} = -A + \frac{1}{4}B$$

↑

Ψ_{sm}

$$H\Psi_{00} = -\frac{3}{4}B$$

Energies $-\frac{3}{4}B, \frac{1}{4}B, \frac{1}{4}B \pm A$

Spring 2002 #2 (p 1 of 2)

A system of two particles each with spin $1/2$ is described by the Hamiltonian

$$H = A(S_{1z} + S_{2z}) + B S_1 \cdot S_2$$

Find all the energy levels of the Hamiltonian.

$$S = S_1 + S_2 \Rightarrow S^2 = S_1^2 + S_2^2 + 2 S_1 \cdot S_2 \Rightarrow S_1 \cdot S_2 = \frac{1}{2}(S^2 - S_1^2 - S_2^2)$$

Making the substitution for $S_1 \cdot S_2$ in our Hamiltonian, we get

$$H = AS_2 + \frac{B}{2}(S^2 - S_1^2 - S_2^2)$$

$$\text{where } S_2 = S_{1z} + S_{2z}$$

The possible values of S are $|S_1 - S_2| \leq S \leq |S_1 + S_2| \Rightarrow 0 \leq S \leq 1$.

So, S can be either 0 or 1.

Now, since $S_1 = \frac{1}{2}$ and $S_2 = \frac{1}{2}$ ($\hbar = 1$), the Hamiltonian is

$$H = AS_2 + B\left(\frac{S^2}{2} - \frac{3}{4}\right)$$

where we used $S_i^2 = S_i(S_i + 1)$. Also note $S_z |m_z\rangle = m_z |m_z\rangle$

$$\underline{S=0}$$

We have the singlet state $\frac{|1\downarrow\rangle - |1\uparrow\rangle}{\sqrt{2}}$

$$\Rightarrow \frac{1}{\sqrt{2}} \left(\frac{AS_2|1\downarrow\rangle - AS_2|1\uparrow\rangle}{\sqrt{2}} + B\left(\frac{S^2}{2} - \frac{3}{4}\right)|1\downarrow\rangle - B\left(\frac{S^2}{2} - \frac{3}{4}\right)|1\uparrow\rangle \right) \Bigg|_{S=0}$$

Note $S_z |1\downarrow\rangle = (+1 - 0)|1\downarrow\rangle = 0$ and same for $S_z |1\uparrow\rangle$

$$\Rightarrow$$

$$E_{00} = \left(\frac{\langle 1\downarrow | - \langle 1\uparrow |}{\sqrt{2}} \right) H \left(\frac{|1\downarrow\rangle - |1\uparrow\rangle}{\sqrt{2}} \right) = \frac{1}{2} \left(-\frac{3B}{4} - \frac{3B}{4} \right)$$

Spring 2002 #2 (p 2 of 2)

$$\Rightarrow \boxed{E_{00} = -\frac{3B}{4}}$$

$$S=1$$

triplet states

$$|1\uparrow\uparrow\rangle \Rightarrow E_{11} = \left[A\left(\frac{1}{2} + \frac{1}{2}\right) + B\left(\frac{3}{2} - \frac{3}{4}\right) \right]$$

$$\Rightarrow \boxed{E_{11} = A + \frac{B}{4}}$$

$$\underbrace{|1\downarrow\downarrow\rangle + |1\uparrow\uparrow\rangle}_{\sqrt{2}} \Rightarrow E_{10} = \frac{1}{2} \left[A \cdot 0 + B\left(\frac{3}{2} - \frac{3}{4}\right) \right]$$

$$\Rightarrow \boxed{E_{10} = \frac{B}{4}}$$

$$|1\downarrow\downarrow\rangle \Rightarrow E_{1,-1} = \left[A\left(-\frac{1}{2} - \frac{1}{2}\right) + B\left(\frac{1}{4}\right) \right]$$

$$\Rightarrow \boxed{E_{1,-1} = -A + \frac{B}{4}}$$

Spring 2002 #3

a) If $AB + BA = 0$, it is not possible for them to have simultaneous eigenkets.

$$A|\psi\rangle = a|\psi\rangle \quad B|\psi\rangle = b|\psi\rangle$$

then $AB|\psi\rangle = Ab|\psi\rangle = ab|\psi\rangle$ eigenvalues commute

$$BA|\psi\rangle = Ba|\psi\rangle = ab|\psi\rangle$$

$$\Rightarrow (BA + AB)\psi\rangle = 2ab|\psi\rangle = 0 \Rightarrow a \text{ or } b \text{ must be zero.}$$

b) No. For example, a free particle

$$\begin{aligned} x(t) &= x(0) + v(0)t \\ &= x(0) + \frac{p(0)}{m}t \end{aligned}$$

$$[x(t), x(\omega)] \neq 0$$

c) Δt is the time it takes for the system to change substantially.

$$\Psi(x, t) = a\psi_1(x)e^{-iE_1 t/\hbar} + b\psi_2(x)e^{-iE_2 t/\hbar}$$

$$|\Psi(x, t)|^2 = a^2(\psi_1(x))^2 + b^2(\psi_2(x))^2 + 2ab\psi_1\psi_2 \cos\left(\frac{E_2 - E_1}{\hbar}t\right)$$

$$\Rightarrow \text{Period is } \frac{2\pi\hbar}{(E_2 - E_1)} = T \Rightarrow \Delta T \Delta E = 2\pi\hbar \geq \hbar \gamma_2 \quad \text{Griffith pg 11-1}$$

(a) Two Hermitian operators anticommute. Is it possible for them to have simultaneous eigenstates?

(see Sakurai # 1.16)

The only way this is possible is if one of the eigenstates is zero.

That is, for two operators to anticommute

$$AB + BA = 0$$

consider $A|\psi\rangle = a|\psi\rangle$ and $B|\psi\rangle = b|\psi\rangle$

$$\Rightarrow AB|\psi\rangle = ab|\psi\rangle \quad \& \quad BA|\psi\rangle = ab|\psi\rangle$$

$$\Rightarrow \underbrace{(AB + BA)}_{=0} |\psi\rangle = \underbrace{2ab}_{\text{only equals zero if } a \text{ or } b \text{ is zero}} |\psi\rangle$$

only equals zero if a or b is zero
or, if $|\psi\rangle$ is zero.

(b) Do position operators at unequal times commute in general in the Heisenberg representation? give an simple example illustrating your answer.

No, consider a free particle

$$x_i(t) = x_i(0) + v_i(0)t = x_i(0) + \frac{p_i}{m}t$$

$$\Rightarrow [x_i(t), x_j(0)] = [x_i(0), \cancel{x_j(0)}] + \frac{t}{m} [p_i, x_j] = -\frac{i\hbar t}{m} \neq 0$$

(c) Explain how you would interpret the energy-time uncertainty relation.

Δt is the time system takes to change subatomically,

sharper the energy is \Rightarrow larger time scale is

$$\Delta E \Delta t \geq \frac{\hbar}{2}$$

Spring 2002 #4

4. Quantum Mechanics.

Consider a mass m particle in one dimension moving in the potential

$$V(x) = V_0 \left| \frac{x}{x_0} \right| \quad (1)$$

where V_0 and x_0 are constants.

Estimate the ground state energy of the particle.

Your score on this problem will be

$$\text{Your Score} = 20 \times e^{-\left(\frac{E-E_0}{E_0}\right)^2} \quad (2)$$

20 is the maximum score, E is your estimate, E_0 is the exact ground state energy, and E and E_0 are evaluated at $V_0 = \frac{E_0^2}{m x_0^2} = 1 \text{ eV}$.

Let's pick trial wave function $\psi = e^{-\alpha x^2/2}$

$$H = -\frac{\partial^2}{2m \partial x^2} + V_0 \left| x \right| \quad V_0, x_0 \text{ are constants} \quad x_0 > 0$$

$$E(x) = \int_{-\infty}^{\infty} e^{-\alpha x^2/2} \left(-\frac{1}{2m} \frac{\partial^2}{\partial x^2} + \frac{V_0}{x_0} \left| x \right| \right) e^{-\alpha x^2/2} dx$$

$$= \int_{-\infty}^{\infty} e^{-\alpha x^2/2} \left(-\frac{1}{2m} \alpha^2 x^2 e^{-\alpha x^2/2} + \frac{V_0}{x_0} \left| x \right| e^{-\alpha x^2/2} \right) dx$$

$$= \int_{-\infty}^{\infty} \frac{e^{-\alpha x^2}}{2m} \alpha^2 x^2 dx + \int_{-\infty}^{\infty} \frac{V_0}{x_0} e^{-\alpha x^2} \left| x \right| dx + \int_{-\infty}^{\infty} \frac{V_0}{x_0} e^{-\alpha x^2} x dx$$

$$\int_{-\infty}^{\infty} x^2 e^{-\alpha x^2} dx = \frac{1}{2\alpha} \left(\frac{\pi}{\alpha} \right)^{1/2}$$

$$\int_{-\infty}^{\infty} e^{-\alpha x^2} x dx = \frac{1}{2\alpha}$$

$$U = -x \quad \frac{V_0}{x_0} \int_{-\infty}^{\infty} e^{-\alpha U^2} U (-dU) \Rightarrow \frac{V_0}{x_0} \int_0^{\infty} e^{-\alpha U^2} U dU \text{ set } \dots$$

$$= -\frac{\alpha^2}{2m} \int_{-\infty}^0 x^2 e^{-\alpha x^2} dx + \pm \frac{V_0}{X_0} \int_{-\infty}^0 e^{-\alpha x^2} x dx + \frac{V_0}{X_0} \int_{-\infty}^0 e^{-\alpha x^2} x dx$$

\curvearrowleft

\curvearrowright

$$= -\frac{\alpha^2}{2m} \cdot \frac{1}{2} \left(\frac{\pi}{\alpha}\right)^{1/2} + \frac{V_0}{X_0} \frac{1}{2\alpha}$$

$$\frac{V_0}{X_0} \cdot \frac{1}{2\alpha}$$

$$= -\frac{\alpha}{2m} \left(\frac{\pi}{\alpha}\right)^{1/2} + \frac{V_0}{X_0 \alpha} = -\frac{1}{2m} (\alpha \pi)^{1/2} + \frac{V_0}{X_0 \alpha}$$

$$\frac{\partial E}{\partial \alpha} = 0 = -\frac{1}{2m} \left(\frac{\pi}{\alpha}\right)^{1/2} - \frac{\alpha}{2m} \left(\frac{1}{2}\right) \left(\frac{\pi}{\alpha}\right)^{-1/2} (\pi) \alpha^{-2} - \frac{V_0}{X_0 \alpha^2}$$

$$= \frac{1}{\alpha} \frac{\pi}{4m} \left(\frac{\pi}{\alpha}\right)^{1/2} - \frac{V_0}{X_0 \alpha^2} - \frac{1}{2m} \left(\frac{\pi}{\alpha}\right)^{1/2}$$

$$0 = \frac{1}{4m\alpha} (\pi \alpha)^{1/2} - \frac{V_0}{X_0 \alpha^2} - \frac{1}{2m} \left(\frac{\pi}{\alpha}\right)^{1/2}$$

$$= \frac{\alpha}{4m} (\pi \alpha)^{1/2} - \frac{V_0}{X_0} - \frac{\alpha}{2m} (\alpha \pi)^{1/2}$$

$$\frac{\alpha}{4m} (\pi \alpha)^{1/2} = -\frac{V_0}{X_0} \quad \frac{\alpha^3 \pi}{(4m)^2} = \left(\frac{V_0}{X_0}\right)^2 \quad \left(\frac{V_0}{X_0}\right)^2 \frac{(4m)^2}{\pi} = \alpha^3$$

$$\alpha = \left(\frac{V_0}{X_0}\right)^{1/3} \frac{(4m)^{2/3}}{\pi^{1/3}}$$

$$E_0 \approx -\frac{1}{2m} \left(\frac{V_0}{X_0}\right)^{1/6} \frac{(4m)^{1/3}}{\pi^{1/6}} \pi^{1/2} + \frac{V_0}{X_0} \frac{\pi^{1/3}}{(4m)^{2/3}} \left(\frac{X_0}{V_0}\right)^{1/3}$$

$$= -\frac{1}{2m} \left(\frac{v_0}{x_0} \right)^{1/6} \pi^{1/3} (4m)^{1/3} + \left(\frac{v_0^2}{x_0^{5/3}} \right)^{1/3} \frac{\pi^{1/3}}{(4m)^{1/3}}$$

$$E_{m,n} = \pi^{1/3} (4m)^{1/3} \left(\frac{v_0^{2/3}}{4m x_0^{5/3}} - \frac{v_0^{1/6}}{2m x_0^{1/6}} \right)$$

$$= \frac{\pi^{1/3} (4m)^{1/3}}{m v_0^{1/2}} \left(\frac{v_0^{2/3} x_0^{1/3}}{4} - \frac{v_0^{1/6} x_0^{1/6}}{2} \right)$$

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QM S'02 #4

$$V(x) = V_0 \frac{|x|}{|x_0|}, V_0 \text{ & } x_0 \text{ are constants}$$

Estimate the ground state energy.

$$-(\frac{E-E_0}{E_0})^2$$

Your score = $20 \times c$

$$E_0 \text{ is the exact ground state; } V_0 = \frac{\pi^2}{2m x_0^2} = 1 \text{ eV}$$

Via WKB

$$x' \Rightarrow E = V_0 \left| \frac{x'}{x_0} \right| \Rightarrow |x'| = \frac{|x_0|}{V_0} E$$

$$\int_0^{x'} p dx = (\pi - V_0) \pi \hbar$$

$$\int_0^{x'} \sqrt{2m(E-V)} dx = \int_0^{x'} \sqrt{2m \left(E - \frac{V_0}{|x_0|} x \right)} dx = -\frac{|x_0|}{2m V_0} \int_0^0 \sqrt{u} du = \frac{|x_0|}{2m V_0} \int_0^{2mE} \sqrt{u} du$$

$$\begin{aligned} u &= 2m \left(E - \frac{V_0}{|x_0|} x \right) \\ du &= -\frac{2m V_0}{|x_0|} dx \end{aligned} \quad \left| \begin{aligned} &= \frac{|x_0|}{2m V_0} \frac{u}{3} \Big|_0^{2mE} = \frac{|x_0|}{3m V_0} (2mE)^{3/2} \end{aligned} \right.$$

$$\text{So } \frac{|x_0|}{3m V_0} (2mE)^{3/2} = (\pi - V_0) \pi \hbar$$

$$\text{For ground state } n=1 \Rightarrow (2mE)^{3/2} = \frac{3m V_0}{|x_0|} \frac{\pi \hbar}{2}$$

$$\begin{aligned} \Rightarrow E &= \frac{1}{2m} \left(\frac{3m V_0}{|x_0|} \frac{\pi \hbar}{2} \right)^{2/3} = \frac{1}{2m} \left(\frac{3m \pi \hbar}{2|x_0|} \frac{\pi^2}{8|x_0|^2} \right)^{2/3} = \frac{1}{2m} \left(\frac{3\pi}{2} \frac{\pi^2}{1|x_0|^3} \right)^{2/3} = \frac{\pi^2}{2m|x_0|^2} \left(\frac{3\pi}{2} \right)^{2/3} \\ &= \frac{V_0}{2} \left(\frac{3\pi}{2} \right)^{2/3} \end{aligned}$$

According to Merzbacher (p. 138-139) $E_0 = 0.8086 V_0$ (exactly)

so our estimate gave us:

$$E_0 = 1.41 V_0 \therefore \text{My score} = \underline{9.56}$$

Via variational method:

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$$\psi(x) = \left(\frac{2a}{\pi}\right)^{1/4} e^{-ax^2}$$

$$\langle H \rangle = \left(\frac{2a}{\pi}\right)^{1/2} \int_{-\infty}^{\infty} e^{-ax^2} \left(-\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + \frac{V_0}{1x01} |x| \right) e^{-ax^2} dx$$

$$= \left(\frac{2a}{\pi}\right)^{1/2} \left[\underbrace{\int_{-\infty}^{\infty} e^{-ax^2} \frac{-\hbar^2}{2m} \frac{d^2}{dx^2} e^{-ax^2} dx}_{I} + \underbrace{\frac{V_0}{1x01} \int_{-\infty}^{\infty} |x| e^{-2ax^2} dx}_{II} \right]$$

I

II

$$I: -\frac{\hbar^2}{2m} \int_{-\infty}^{\infty} e^{-ax^2} \frac{d^2}{dx^2} e^{-ax^2} dx = -\frac{\hbar^2}{m} \int_0^{\infty} e^{-ax^2} \frac{d^2}{dx^2} e^{-ax^2} dx$$

$$\text{now } \frac{d^2}{dx^2} e^{-ax^2} = \frac{d}{dx} \frac{d}{dx} e^{-ax^2} = \frac{d}{dx} (-da x e^{-ax^2}) = 4a^2 x^2 e^{-ax^2} - da e^{-ax^2}$$

$$\text{so } -\frac{\hbar^2}{m} \int_0^{\infty} e^{-ax^2} (4a^2 x^2 - da) e^{-ax^2} dx = -\frac{\hbar^2}{m} \int_0^{\infty} (4a^2 x^2 - da) e^{-2ax^2} dx = -\frac{\hbar^2}{m} \left[\underbrace{\int_0^{\infty} 4a^2 x^2 e^{-2ax^2} dx}_A - \underbrace{\int_0^{\infty} da e^{-2ax^2} dx}_B \right]$$

$$A: \int_0^{\infty} x^2 e^{-2ax^2} dx = \frac{1}{2} \frac{\Gamma((2+1)/2)}{(da)^{(2+1)/2}} = \frac{1}{2} \frac{\sqrt{\pi}}{(da)^{3/2}} = \frac{\sqrt{\pi}}{4(a)^{3/2}}$$

$$B: \int_0^{\infty} e^{-2ax^2} dx = \frac{1}{2} \sqrt{\frac{\pi}{2a}}$$

\sqrt{a}

\sqrt{a}

$$\begin{aligned} \text{So integral I: } & -\frac{\hbar^2}{m} \left[\frac{4a^2 \frac{\sqrt{\pi}}{2}}{(da)^{3/2}} - \frac{da \frac{1}{2} \sqrt{\frac{\pi}{2a}}}{(da)^{3/2}} \right] = -\frac{\hbar^2}{m} \left[\frac{\sqrt{\pi}}{(da)^{3/2}} \frac{a^2}{a^{1/2}} - \frac{\sqrt{\pi}}{(da)^{1/2}} \frac{a^{1/2}}{a^2} \right] \\ & = -\frac{\hbar^2 \sqrt{\pi a}}{m} \left[\frac{1}{(da)^{3/2}} - \frac{1}{(da)^{1/2}} \right] = -\frac{\hbar^2 \sqrt{\pi a}}{m} \left[\frac{(-2)}{(da)^{3/2}} \right] = \frac{-\hbar^2 \sqrt{\pi a}}{m (da)^{3/2}} \end{aligned}$$

$$\text{II: } \frac{V_0}{1x01} \int_{-\infty}^{\infty} |x| e^{-2ax^2} dx = \frac{2V_0}{1x01} \int_0^{\infty} |x| e^{-2ax^2} dx = \frac{V_0}{1x01} \frac{1}{2a} - 1 = \frac{1}{2a} \frac{V_0}{1x01}$$

QM S'02 #4

So

$$\langle H \rangle = \left(\frac{2a}{\pi}\right)^{1/2} \left[\frac{\frac{\pi^2 \sqrt{\pi a}}{m(a)^{3/2}} + \frac{1}{2a} \frac{v_0}{kx_0}}{1} \right] = \frac{\pi^2 a}{2m} + \frac{v_0}{kx_0} \frac{1}{\sqrt{2\pi a}}$$

Need to minimize this with respect to "a"!

$$\frac{d}{da} \langle H \rangle = 0 \Rightarrow \frac{d}{da} \left(\frac{\pi^2 a}{2m} + \frac{v_0}{kx_0} \frac{1}{\sqrt{2\pi a}} \right) = 0$$

$$\frac{\pi^2}{2m} - \frac{1}{2} \frac{2\pi}{(2\pi a)^{3/2}} \frac{v_0}{kx_0} = 0$$

$$\Rightarrow \frac{\pi^2}{2m} = \frac{v_0 \pi}{kx_0} \frac{1}{(2\pi a)^{3/2}} \Rightarrow (2\pi a)^{3/2} = \frac{2m v_0 \pi}{\pi^2 k x_0}$$

$$a = \frac{1}{2\pi} \left(\frac{2m v_0 \pi}{\pi^2 k x_0} \right)^{2/3} = \frac{1}{2\pi} \left(\frac{2m \pi}{k x_0} \frac{\pi^2}{2\pi a^2} \right)^{2/3} = \frac{1}{2\pi} \left(\frac{2\pi}{k x_0^2} \right)^{2/3}$$

$$= \frac{1}{2\pi} \frac{1}{x_0^2} \left(\frac{2\pi}{k} \right)^{2/3} = \frac{1}{(2\pi)^{1/3}} \frac{1}{x_0^2}$$

So

$$\langle H \rangle = \frac{\pi^2}{2m} \frac{1}{(2\pi)^{1/3}} \frac{1}{x_0^2} + \frac{v_0}{kx_0} \frac{1}{\sqrt{\frac{2\pi}{(2\pi)^{1/3} x_0^2}}} \left\{ \sqrt{\frac{(2\pi)^{2/3}}{x_0^2}} \right\}$$

$$= \underbrace{\frac{\pi^2}{2m x_0^2} \frac{1}{(2\pi)^{1/3}}}_{v_0} + \frac{v_0}{kx_0} \frac{kx_0}{(2\pi)^{1/3}} = v_0 \left(\frac{1}{(2\pi)^{1/3}} + \frac{1}{(2\pi)^{1/3}} \right)$$

$$= \frac{3}{2} \frac{v_0}{(2\pi)^{1/3}}$$

This gives us 0.813 v₀ or 19.9 points

via uncertainty relation:

$$H = \frac{p^2}{2m} + \frac{V_0}{x_0} |x| \Rightarrow \langle H \rangle = \frac{\langle p^2 \rangle}{2m} + \frac{V_0}{x_0} \langle |x| \rangle$$

now $\langle p^2 \rangle = \langle p \rangle^2$ as we are dealing with an eigenstate (stationary state)

$$\langle |x| \rangle = \langle x \rangle$$

$$\text{also } \langle p \rangle \langle x \rangle = \frac{1}{2} \Rightarrow \langle x \rangle = \frac{1}{2\langle p \rangle}$$

$$\text{so } \langle H \rangle = E = \frac{\langle p \rangle^2}{2m} + \frac{V_0}{x_0} \frac{1}{2\langle p \rangle} \Rightarrow \frac{dE}{dp} = \frac{\langle p \rangle}{m} - \frac{V_0}{2x_0 \langle p \rangle^2} = 0$$

$$\Rightarrow \frac{\langle p \rangle}{m} = \frac{V_0 t}{2x_0 \langle p \rangle^2} \Rightarrow \langle p \rangle^3 = \frac{V_0 t m}{2x_0} \Rightarrow \langle p \rangle = \left(\frac{V_0 t m}{2x_0} \right)^{1/3}$$

$$\text{plugging this back into } E: E = \frac{1}{2m} \left(\frac{V_0 t m}{2x_0} \right)^{2/3} + \frac{V_0 t}{2x_0} \left(\frac{2x_0}{V_0 t m} \right)^{1/3}$$

$$E = \frac{1}{2} \left(\frac{V_0^2 t^2 m^2}{m^2 4 x_0^3} \right)^{1/3} + \left(\frac{2x_0 V_0^{2/3} t^{2/3}}{2 x_0^{2/3} t^{2/3} m} \right)^{1/3} = \frac{1}{2} \left(\frac{V_0^2 t^2}{4 m x_0^3} \right)^{1/3} + \left(\frac{V_0^2 t^2}{4 m x_0^3} \right)^{1/3}$$

$$= \frac{3}{2} \left(\frac{t^2}{2 m x_0^2} \cdot V_0^2 \right)^{1/3} = \frac{3}{2} \left(\frac{1}{4} V_0^3 \right)^{1/3} = \frac{3}{2} \left(\frac{1}{4} \right)^{1/3} V_0 = 0.945 V_0 = 0.945 \text{ eV}$$

$$\text{as } V_0 = \frac{t^2}{m x_0^2}$$

My score = 19.44

Spring 2002 #4 (p 1 of 2)

Consider a mass m particle in one dimension moving in the potential

$$V(x) = V_0 \left| \frac{x}{x_0} \right|$$

where V_0 and x_0 are constants. Estimate the ground state of the particle.

E_0 is the exact ground state energy. $V_0 = \frac{\hbar^2}{mx_0^2} = 1 \text{ eV}$.

Use variational method. ... for other approaches see Bertrand's solution. Choose trial function

$$\psi(x) = \left(\frac{2a}{\pi} \right)^{1/4} e^{-ax^2}$$

The Hamiltonian is given by

$$H = \frac{p^2}{2m} + V(x) = -\frac{1}{2m} \frac{d^2}{dx^2} + V_0 \left| \frac{x}{x_0} \right|$$

Now find $\langle H \rangle$.

$$\langle H \rangle = \frac{1}{2m} \langle p^2 \rangle + \frac{V_0}{|x_0|} \langle |x| \rangle$$

$$\frac{1}{2m} \langle p^2 \rangle = -\frac{1}{2m} \sqrt{\frac{2a}{\pi}} \int_{-\infty}^{\infty} e^{-ax^2} \frac{d^2}{dx^2} e^{-ax^2} dx = -\frac{1}{2m} \sqrt{\frac{2a}{\pi}} \int_{-\infty}^{\infty} e^{-ax^2} \frac{d}{dx} (-2ax) e^{-ax^2} dx$$

$$= -\frac{1}{2m} \sqrt{\frac{2a}{\pi}} \int_{-\infty}^{\infty} (-2a) e^{2ax^2} dx - \frac{1}{2m} \sqrt{\frac{2a}{\pi}} \int_{-\infty}^{\infty} 4a^2 x^2 e^{-2ax^2} dx$$

$$\Rightarrow \frac{1}{2m} \langle p^2 \rangle = \frac{a}{m} \sqrt{\frac{2a}{\pi}} \sqrt{\frac{\pi}{2a}} - \frac{2a^2}{m} \sqrt{\frac{2a}{\pi}} \frac{\sqrt{\pi}}{2(2a)^{3/2}} = \frac{a}{m} - \frac{a^2}{m} \frac{1}{2a} = \boxed{\frac{a}{2m}} \quad (1)$$

$$\frac{V_0}{|x_0|} \langle |x| \rangle = \frac{V_0}{|x_0|} \int_{-\infty}^{\infty} e^{-ax^2} \underbrace{|x| e^{-ax^2}}_{\text{even function}} dx = \frac{2V_0}{|x_0|} \int_0^{\infty} x e^{-2ax^2} dx$$

$$\text{let } u = x^2 \Rightarrow du = 2x dx$$

Spring 2002 #4 (p 2 of 2)

$$\Rightarrow \frac{V_0}{|x_0|} \langle |x| \rangle = \frac{2V_0}{|x_0|} \int_0^\infty \frac{1}{2} e^{-2au} du = \frac{V_0}{|x_0|} \left(-\frac{1}{2a} \right) \left[e^{-2au} \right]_0^\infty = \frac{V_0}{2a|x_0|}$$

Thus, combining eq's (1) and (2) we get

$$\langle H \rangle = \frac{q}{2m} + \frac{V_0}{2a|x_0|}$$

now find the extremum.

$$0 = \frac{\partial \langle H \rangle}{\partial a} = \frac{1}{2m} - \frac{V_0}{2a^2|x_0|} \xrightarrow{\text{solve for } a} \frac{V_0}{2|x_0|} = \frac{1}{2m} a^2$$

$$\therefore a = \sqrt{\frac{mV_0}{|x_0|}}$$

then

$$E = \langle H(a) \rangle = \frac{1}{2m} \sqrt{\frac{mV_0}{|x_0|}} + \frac{V_0}{2|x_0|} \sqrt{\frac{|x_0|}{mV_0}} = \frac{1}{2} \sqrt{\frac{V_0}{m|x_0|}} + \frac{1}{2} \sqrt{\frac{V_0}{m|x_0|}}$$

$$= \sqrt{\frac{V_0}{m|x_0|}} = \boxed{\sqrt{V_0^2}} = \boxed{V_0}$$

$$\text{so, my score is } 20 \times e^{-\left(\frac{V_0 - V_0}{V_0}\right)^2} = 20$$

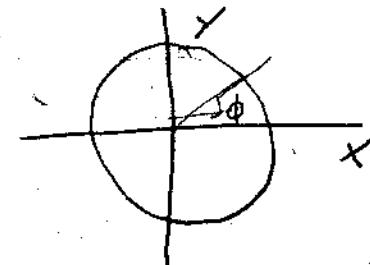
Spring 2002 #5.

$$H = \frac{1}{2}m(r^2 + r^2\dot{\theta}^2 + r^2 \sin^2\theta \dot{\phi}^2) + V(r)$$

$V(r) = \dot{r} = \dot{\theta} = 0$ since it is stuck on a ring.

$$= \frac{1}{2}mr^2 \sin^2\theta \dot{\phi}^2 \quad \text{But } \dot{\theta} = \dot{\phi}_2 \\ \text{and } r=R$$

$$= \frac{1}{2}mr^2 \dot{\phi}^2$$



$$H = \frac{p^2}{2m} = \frac{1}{2}mR^2\dot{\phi}^2 = \frac{L_2^2}{2mR^2} \quad L_2 \rightarrow -i\frac{\partial}{\partial\phi}$$

$$L = m\dot{\phi}R$$

$$\frac{1}{2mR^2} \left(-i\frac{\partial}{\partial\phi} \right)^2 \Psi(\phi) = E\Psi(\phi)$$

⇒ degeneracy
2 per state.

Solutions have the form

$$\Psi(\phi) = \frac{1}{\sqrt{2\pi}} e^{in\phi}, \text{ and } \frac{1}{\sqrt{2\pi}} e^{-in\phi} \text{ is valid}$$

$$\frac{1}{2mR^2} \left(-i\frac{\partial}{\partial\phi} \right)^2 \frac{e^{in\phi}}{\sqrt{2\pi}} = E_n \frac{e^{in\phi}}{\sqrt{2\pi}}$$

$$E_n = \frac{n^2}{2mR^2}$$

$$n = 1, 2, 3, \dots$$

Can think of $\Psi(\phi) = \frac{1}{\sqrt{2\pi}} e^{in\phi}$

$$\Psi(\phi) = \frac{1}{\sqrt{2\pi}} e^{in\phi}$$

b) $H' = \vec{eE} \cdot \vec{r} = eE_x R \cos\phi = eE_x x$

$$\langle \Psi_n | eE_x | \Psi_n \rangle = 0 \text{ since } x \text{ is odd parity}$$

and the bra and kets can't be the same parity for odd operators $\Rightarrow 0$

c) 2nd order shift

$$\Delta E^{(2)} = \sum_{m \neq n} \frac{|\langle \Psi_m^0 | H' | \Psi_n^0 \rangle|^2}{E_n^0 - E_m^0}$$

$$\langle \Psi_m^0 | H' | \Psi_n^0 \rangle = \frac{1}{2\pi} \int_0^{2\pi} e^{im\phi} e^{E_0 r \cos \phi} e^{in\phi} d\phi$$

$$\cos \phi = \frac{e^{i\phi} + e^{-i\phi}}{2}$$

$$\langle \Psi_m^0 | H' | \Psi_n^0 \rangle = \frac{e E_0 r}{4\pi} \int_0^\pi e^{i\phi(-m+n+1)} + e^{i\phi(-m+n-1)} d\phi$$

$$= \frac{e E_0 r}{4\pi} (2\pi) \quad \text{for } m=n+1 \quad , \quad 0 \text{ otherwise}$$

$$= \frac{e E_0 r}{2}$$

$$\Delta E^{(2)} = \frac{|\langle \Psi_{n+1}^0 | H' | \Psi_n^0 \rangle|^2}{E_n^0 - E_{n+1}^0} + \frac{|\langle \Psi_{n-1}^0 | H' | \Psi_n^0 \rangle|^2}{E_n^0 - E_{n-1}^0}$$

$$E = \frac{n^2}{2mr^2}$$

$$= \frac{e^2 E_0^2 r^2}{4 \frac{1}{(2mr^2)} [n^2 - (n+1)^2]} + \frac{e^2 E_0^2 r^2}{4 \frac{1}{(2mr^2)} [n^2 - (n-1)^2]}$$

$$= \frac{e^2 E_0^2 mr^4}{2 [n^2 - n^2 - 2n - 1]} + \frac{e^2 E_0^2 mr^4}{2 [n^2 - n^2 + 2n - 1]}$$

$$= \frac{e^2 E_0^2 mr^4}{2} \left[\frac{1}{(-2n-1)} + \frac{1}{(2n-1)} \right] = \frac{e^2 E_0^2 mr^4}{2} \left[\frac{(2n-1)}{(-2n-1)(2n-1)} + \frac{(-2n-1)}{(2n-1)(2n-1)} \right]$$

$$= \frac{e^2 E_0^2 mr^4}{2} \left[\frac{2}{4n^2 - 1} \right] = \frac{e^2 E_0^2 mr^4}{(4n^2 - 1)}$$

Spring 2002 #5 (p 1 of 3)

charge on a circle; A small bead with charge e and mass m is confined to move on a circular ring in the $x-y$ plane with radius r . A weak, uniform electric field of intensity E_0 pointing in the positive x direction is turned on.

(a) What are the eigenfunctions and energy eigenvalues for $E_0 = 0$? What are the degeneracies? (See Fall 1996 #11)

When $E_0 = 0$, we just have

$$H = \frac{P_r^2}{2m} = \frac{P_r^2}{2m} + \frac{P_\theta^2}{2mr^2} + \frac{P_\phi^2}{2mr^2 \sin^2 \theta}$$

Since the charge is constrained to a circular ring in the $x-y$ plane, we know that

$$P_r = 0 = P_\theta$$

and $\theta = \frac{\pi}{2}$. So, we have

$$H = \frac{P_\phi^2}{2mr^2} = \frac{L_z^2}{2mr^2}$$

With $\hbar = 1$, the Schrödinger eq. is

$$-\frac{1}{2mr^2} \frac{d^2}{d\phi^2} \psi = E \psi \Rightarrow \frac{d^2}{d\phi^2} \psi + n^2 \psi = 0, n^2 = 2mr^2 E$$

The solutions to this ΔE are

$$\psi(x) \propto e^{\pm in\phi}$$

Since these functions repeat every 2π , n must be an integer

So, the energy levels are

$$\boxed{E_n = \frac{n^2}{2mr^2}} \quad , n = 0, \pm 1, \pm 2, \pm 3$$

There is 2-fold degeneracy for every n except $n=0$. This comes from the energy being the same no matter if going clockwise or counter clockwise

Spring 2002 #5 (p 2 of 3)

(b) For $E_0 \neq 0$, show that the electric field operator has vanishing matrix elements between degenerates of the unperturbed Hamiltonian.

The perturbation is given by

$$H' = -q\bar{E}x = e\bar{E}x$$

The diagonal elements vanish.

The reason is that x is odd and diagonal elements have the same function so they have the same parity. So, we have

$$E_0^{(1)} = e\bar{E} \langle \psi(x) | x | \psi(x) \rangle = 0$$

↑ ↑ ↑
 odd same parity

(c) Find an approximation for the energy levels which includes the first non-trivial term containing E_0 .

From part (b) we know that 1st order perturbation vanishes. So, try 2nd order.

$$E_0^{(2)} = \sum_{m \neq n} \frac{|\langle \psi_m | H' | \psi_n \rangle|^2}{E_n^{(0)} - E_m^{(0)}}$$

$$\text{where } E_n^{(0)} - E_m^{(0)} = \frac{1}{2Mr^2} (n^2 - m^2)$$

↑
charge expression from mass to avoid confusion with index.

and

$$\langle \psi_m | H' | \psi_n \rangle = e\bar{E} \int_{-\infty}^{\infty} e^{-im\phi} x e^{in\phi} d\phi = e\bar{E} r \int_{-\infty}^{\infty} e^{-im\phi} \cos\phi e^{in\phi} d\phi$$

Spring 2002 #5 (p 3 of 3)

since $\cos \phi = \frac{1}{2} (e^{-i\phi} + e^{i\phi})$, we have

$$\begin{aligned}\langle \psi_m | H' | \psi_n \rangle &= \frac{eE_r}{2} \int_{-\infty}^{\infty} e^{-i\phi(m-n)} (e^{-i\phi} + e^{i\phi}) d\phi \\ &= \frac{eE_r}{2} \int_{-\infty}^{\infty} \left[e^{-i\phi(m-n-1)} + e^{-i\phi(m-n+1)} \right] d\phi \\ &= \frac{eE_r}{2} [\delta_{m-n-1,0} + \delta_{m-n+1,0}]\end{aligned}$$

So, only two values of m survive

$$m = n \pm 1$$

$$\text{for } m=n+1, \quad \langle \psi_m | H' | \psi_n \rangle = \frac{1}{2} e E_r$$

$$m=n-1, \quad \langle \psi_m | H' | \psi_n \rangle = \frac{1}{2} e E_r$$

So,

$$E_0^{(2)} = \sum_{m \neq n} \frac{|\langle \psi_m | H' | \psi_n \rangle|^2}{E_n^{(0)} - E_m^{(0)}} = \frac{\left(\frac{1}{2} e E_r\right)^2}{\frac{1}{2 M r^2}} \left[\frac{1}{n^2 - (n+1)^2} + \frac{1}{n^2 - (n-1)^2} \right]$$

$$\begin{aligned}&= \frac{1}{4} e^2 E^2 r^2 (2 M r^2) \left[\underbrace{\frac{1}{n^2 - n^2 - 2n - 1} + \frac{1}{n^2 - n^2 + 2n - 1}}_{= -\frac{1}{2n+1} + \frac{1}{2n-1}} \right] \\ &= -\frac{1}{2n+1} + \frac{1}{2n-1} = \frac{-2n+1 + 2n+1}{4n^2 - 1} = \frac{2}{4n^2 - 1}\end{aligned}$$

Thus,

$$\boxed{E_0^{(2)} = \frac{e^2 E^2 r^4 M}{4n^2 - 1}}$$

Spring 2002 #6

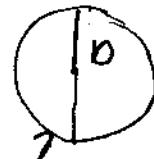
$$l = \tau \bar{v} = \frac{\bar{v}}{V} \frac{1}{\sigma_0} \text{ average speed of incident particle}$$

mean free path

$$\frac{1}{V} \sigma_0 \rightarrow \text{scattering cross section}$$

↑
relative average speed

$$\frac{1}{\tau} = \omega \leftarrow \text{frequency of collisions}$$



$$\sigma_0 = \pi \left(\frac{d+D}{2} \right)^2 \quad \text{Ref 12.2.1}$$

$$\bar{v} = \bar{v} \quad \text{since target is fixed}$$

$$V \bar{v} = \sqrt{\frac{8kT}{\pi m}} \quad \text{Ref 7.10.13}$$

$$\begin{aligned} \frac{1}{\tau} &= \frac{1}{n \sigma_0} & \omega &= \bar{v} n \sigma_0 = \pi \left(\frac{d+D}{2} \right)^2 n \sqrt{\frac{8kT}{\pi m}} \\ &&&= \left(\frac{d+D}{2} \right)^2 n \sqrt{\frac{\pi 8kT}{m}} \\ &&&= \frac{1}{4} (d+D)^2 n \sqrt{\frac{\pi 8kT}{m}} \\ &&&= (d+D)^2 n \sqrt{\frac{\pi kT}{2m}} \end{aligned}$$

Spring 2002 #6 (p 1 of 1)

Calculate the collision frequency for the collisions between the molecules of a gas and a fixed sphere of diameter D . The molecules have an average diameter d . The gas has a temperature T .

(See Spring 2001 #6)

$$\text{maxwell distribution function: } D(v) = \underbrace{\left(\frac{m}{2\pi kT}\right)^{3/2} 4\pi r^2}_{\text{Normalization condition}} e^{-\frac{mv^2}{2kT}}$$

\sim probability of a molecule having v^2

mean molecular speed

$$\rightarrow v_{\text{average}} = \sum_v v D(v) dv \Rightarrow \int_0^\infty \left(\frac{m}{2\pi kT}\right)^{3/2} 4\pi v^3 e^{-\frac{mv^2}{2kT}} dv$$

$$\text{let } x = v^2 \Rightarrow dx = 2v dv$$

So, we have

$$v_{\text{average}} = \left(\frac{m}{2\pi kT}\right)^{3/2} \int_0^\infty 2\pi \times e^{-mx/2kT} dv = 2\pi \left(\frac{m}{2\pi kT}\right)^{3/2} \int_0^\infty x e^{-\frac{mx}{2kT}} dx$$

$$= 2\pi \left(\frac{m}{2\pi kT}\right)^{3/2} \frac{1}{\left(\frac{m}{2kT}\right)^2} = 2\pi \left(\frac{m}{2\pi kT}\right)^{3/2} \left(\frac{2kT}{m}\right)^2$$

$$v_{\text{average}} = \frac{2\pi}{\pi^{3/2}} \left(\frac{2kT}{m}\right)^{1/2} = \sqrt{\frac{8kT}{m\pi}} \quad (1) \quad \leftarrow \text{equation 12.3.10 in Reif}$$

Now, we have

$$v = \frac{v_{\text{average}}}{l} = \frac{\sqrt{\frac{8kT}{m\pi}}}{\frac{4}{\pi(d+D)^2 n}} = \sqrt{\frac{8kT}{16m\pi}} \pi (d+D)^2 n$$

∴ $v = n(d+D)^2 \sqrt{\frac{kT\pi}{2m}}$

We start out with the wave equation for \vec{E} :

$$\nabla^2 \vec{E} - \frac{4\pi\sigma\mu}{c^2} \frac{\partial \vec{E}}{\partial t} = 0 \quad \text{but } \vec{J} = \sigma \vec{E}$$

so

$$\nabla^2 \vec{J} - \frac{4\pi\sigma\mu}{c^2} \frac{\partial \vec{J}}{\partial t} = 0$$

we can assume the solution $\vec{J} = J_x(y) e^{i\omega t}$, and as we are dealing with only one dimension (y):

$$\frac{d^2}{dy^2} J(y) - \frac{4\pi\sigma\mu}{c^2} i\omega J(y) = 0$$

now define $\tau^2 = i \frac{4\pi\sigma\mu\omega}{c^2}$, as $\sqrt{\tau^2} = \frac{\sqrt{2}}{2} (1+i)$

$$\tau = (1+i) \sqrt{\frac{4\pi\sigma\mu\omega}{c^2}} = \frac{1+i}{\delta} \quad \text{with } \delta = \frac{c}{\sqrt{4\pi\sigma\mu\omega}} = \frac{c}{B}$$

so we now have:

$$\frac{d^2}{dy^2} J(y) - \tau^2 J(y) = 0 \Rightarrow J(y) = A e^{-\tau y} + B e^{\tau y}$$

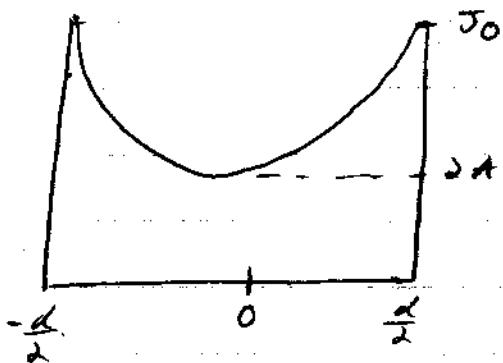
by planar symmetry this can be written as:

$$J(y) = A(e^{-\tau y} + e^{\tau y})$$

as $J(y = -\frac{L}{2}) = J_0 = A(e^{-\tau \frac{L}{2}} + e^{\tau \frac{L}{2}})$
 $J(y = +\frac{L}{2}) = J_0 = A(e^{-\tau \frac{L}{2}} + e^{\tau \frac{L}{2}})$

} they are the same

so the plot of $|J(z)|$ is:



Now for the phase difference:

$$J(0) = 2A; \quad J\left(z = \frac{d}{2}\right) = A(e^{-\frac{\pi d k}{2}} + e^{\frac{\pi d k}{2}}) = J\left(z = -\frac{d}{2}\right)$$

taking the ratio:

$$\frac{J\left(z = \pm \frac{d}{2}\right)}{J(0)} = \frac{e^{-\frac{\pi d k}{2}} + e^{\frac{\pi d k}{2}}}{2}$$

$$\text{but } \alpha = \frac{1+i}{\delta} = \frac{(1+i)}{c} \beta$$

$$\text{so } = \frac{1}{2} \left(e^{-\frac{(1+i)\beta d k}{2}} + e^{\frac{(1+i)\beta d k}{2}} \right)$$

$$\text{for } \beta d \gg 1 \quad \text{the above} = \frac{1}{2} e^{\frac{(1+i)\beta d k}{2}}$$

Spring 2002 #8 (p 10F3)

Consider the AC current density in a conductor obeying Ohm's law, $\vec{J} = \sigma \vec{E}$, where σ is the constant real conductivity. Suppose that we have a conductor which is a 2D metal sheet of thickness d located in the $x-z$ plane and that current flows in the x -direction. If the AC current density is given by the expression $J_x(y, t) = \text{Re} [J_x(y) e^{i\omega t}]$.

(a) Find $J_x(y)$.

See Marion and Heald 2nd edition section 5.7 "Current distribution in Conductors - The 'skin depth'"

In a good conductor, the conduction current dominates the displacement current. So, ignoring this term, we have a diffusion equation for \vec{E} .

$$\nabla^2 \vec{E} - \frac{4\pi\sigma\mu}{c^2} \frac{\partial \vec{E}}{\partial t} = 0$$

Substituting in Ohm's law $\vec{J} = \sigma \vec{E}$, we get

$$\nabla^2 \vec{J} - \frac{4\pi\sigma\mu}{c^2} \frac{\partial \vec{J}}{\partial t} = 0 \quad (1)$$

As given assume the solution (as given) is of the form

$$J_x(y, t) = J_x(y) e^{i\omega t}$$

so, eq(1) becomes

$$e^{i\omega t} \frac{d^2 J_x(y)}{dy^2} = \gamma J_x(y) (i\omega) e^{i\omega t}$$

$$\Rightarrow \frac{d^2 J_x(y)}{dy^2} - \gamma^2 J_x(y) = 0 \quad , \quad \gamma^2 = i \frac{4\pi\sigma\mu\omega}{c^2}$$

Spring 2002 #8 (p 2 of 3)

the solution is

$$J_x(y) = A e^{-\tau y} \quad \text{or} \quad J_x(y) = A(e^{-\tau y} + e^{\tau y})$$

where

$$\tau = \sqrt{i} \frac{\sqrt{4\pi\sigma\mu\omega}}{c} = \frac{\sqrt{2}}{2}(1+i) \frac{\sqrt{4\pi\sigma\mu\omega}}{c} = (1+i) \frac{\sqrt{2\pi\sigma\mu\omega}}{c}$$

recall the skin depth is defined as

$$\delta = \frac{c}{\sqrt{2\pi\sigma\mu\omega}}$$

so,

$$\boxed{\tau = \frac{(1+i)}{\delta}}$$

) Now, let J_0 be the value of the current on the surface. Then,

$$J_x\left(\frac{d}{2}\right) = J_0 = A \left(e^{-\tau \frac{d}{2}} + e^{\tau \frac{d}{2}}\right) = 2A \cosh\left(\frac{\tau d}{2}\right)$$

$$\Rightarrow A = \frac{J_0}{2 \cosh\left(\frac{\tau d}{2}\right)}$$

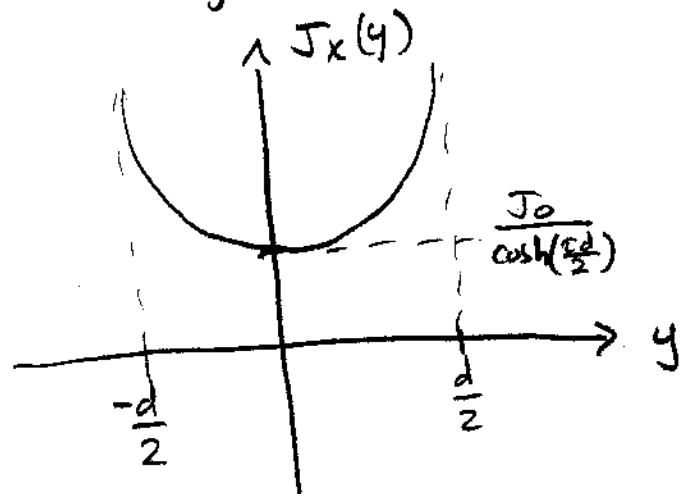
(note: $J_x(-\frac{d}{2}) = J_0$ yields same result). Thus,

$$J_x(y) = \frac{J_0}{2 \cosh\left(\frac{\tau d}{2}\right)} (e^{-\tau y} + e^{\tau y}) = \frac{J_0}{2 \cosh\left(\frac{\tau d}{2}\right)} 2 \cosh(\tau y)$$

$$\therefore \boxed{J_x(y) = J_0 \frac{\cosh(\tau y)}{\cosh\left(\frac{\tau d}{2}\right)}}$$

Spring 2002 #8 (p 3 of 3)

(b) Plot $|J_x(y)|$ vs. y .



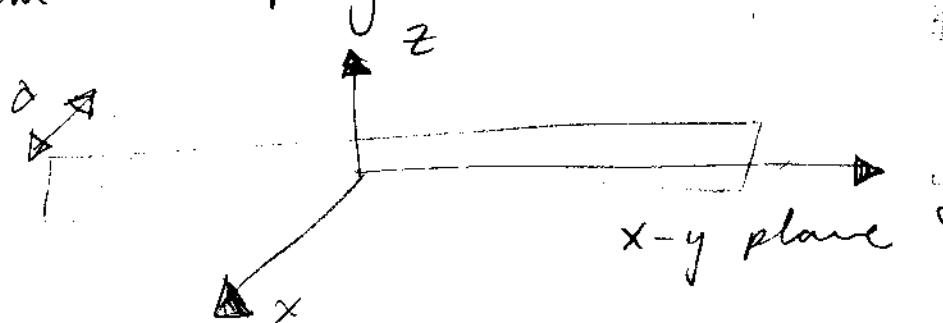
(c) Find the phase shift of the current density between the center and the edge for $\omega \mu_0 \sigma d \gg 1$

in this limit, $\gamma \rightarrow 0$. So $\cosh(\gamma y) \rightarrow 1$

So in this limit, there can't be a phase shift since the current density is constant.

Problem # 8 Spring 2002

$$\vec{J} = \sigma \vec{E}$$



$$J_x(y, t) = \operatorname{Re} (\tilde{J}_x(y) e^{i\omega t})$$

a) $\nabla \times \vec{B} = \frac{4\pi}{c} \vec{J} + \frac{1}{c} \frac{\partial \vec{E}}{\partial t}$

~~$$\nabla \times \frac{\partial \vec{B}}{\partial t} = \frac{4\pi}{c} \frac{\partial \vec{J}}{\partial t} + \frac{\partial^2 \vec{E}}{\partial t^2}$$~~

~~$$\nabla \times \vec{E} = -\frac{1}{c} \frac{\partial \vec{B}}{\partial t}$$~~

~~$$\nabla \cdot \vec{E} = 4\pi \rho$$~~

SI

$$\nabla \times \vec{B} = \mu_0 \vec{J} + \epsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t}$$

$$\nabla \times \frac{\partial \vec{B}}{\partial t} = \mu_0 \frac{\partial \vec{J}}{\partial t} + \epsilon_0 \mu_0 \frac{\partial^2 \vec{E}}{\partial t^2}$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$-\nabla \times (\nabla \times \vec{E}) = \mu_0 \sigma \frac{\partial \vec{E}}{\partial t} + \epsilon_0 \mu_0 \frac{\partial^2 \vec{E}}{\partial t^2}$$

$$-\nabla (\nabla \cdot \vec{E}) + \nabla^2 \vec{E} = \mu_0 \sigma \frac{\partial \vec{E}}{\partial t} + \epsilon_0 \mu_0 \frac{\partial^2 \vec{E}}{\partial t^2}$$

$$\vec{\nabla} \cdot \vec{E} = \frac{f}{\epsilon_0}$$

$$\frac{\partial^2}{\partial y^2} \vec{E} = \mu_0 \sigma \frac{\partial \vec{E}}{\partial t} + \epsilon_0 \mu_0 \frac{\partial^2 \vec{E}}{\partial t^2}$$

$$\frac{\partial^2}{\partial y^2} \mathcal{Y}_x(y) e^{i\omega t} = \mu_0 \sigma + \frac{\mathcal{Y}_x(y) e^{i\omega t}}{\partial t} + \epsilon_0 \mu_0 \frac{\partial^2 \mathcal{Y}_x(y) e^{i\omega t}}{\partial t^2}$$

$$\frac{\partial^2}{\partial y^2} \mathcal{Y}_x(y) = \mu_0 \sigma i\omega \mathcal{Y}_x(y) - \epsilon_0 \mu_0 \omega^2 \mathcal{Y}_x(y)$$

$$\frac{\partial^2}{\partial y^2} \mathcal{Y}_x(y) = \mu_0 (\sigma i\omega - \epsilon \omega^2) \mathcal{Y}_x(y)$$

$$K^2 = \mu_0 (\sigma \omega i - \epsilon \omega^2)$$

$$= \sqrt{\mu_0} (\sqrt{\sigma \omega}(1-i) + \sqrt{\epsilon} \omega)(\sqrt{\sigma \omega}(1-i) + \sqrt{\epsilon} \omega)$$

$$\zeta^2 = \alpha i + \beta \quad \zeta = \sqrt{\alpha i - \beta} = \sqrt{\alpha i} \pm \sqrt{\beta}$$

$$(K + K_i)(K - K_i) = \mu_0 (\sigma \omega i - \epsilon \omega^2)$$

$$\mathcal{Y}_x(y) = e^{\frac{1}{2}(\mu_0 \sigma \omega i - \mu_0 \epsilon \omega^2)y}$$

a) Assuming that x and t transform according to the Lorentz transformation law, show that the combination $(ds)^2 = (dx)^2 - c^2(dt)^2$ is the same in all inertial frames.

L.T. in 1-D: $x' = \gamma(x - vt)$ where the primed stands
 $t' = \gamma(t - \frac{v}{c^2}x)$ for the moving reference
frame (at velocity v)

In the rest frame:

$$(ds)^2 = (dx)^2 - c^2(dt)^2$$

As for the moving frame:

$$(ds')^2 = (dx')^2 - c^2(dt')^2$$

now $dx' := x'_A = \gamma(x_A - vt_A); x'_B = \gamma(x_B - vt_B)$
 $\Rightarrow dx' = \gamma x_A - v\gamma t_A - \gamma x_B + v\gamma t_B$
 $= \gamma \delta x - \gamma v \delta t = \gamma(\delta x - v \delta t)$

similarly dt' : $t'_A = \gamma(t_A - \frac{v}{c^2}x_A); t'_B = \gamma(t_B - \frac{v}{c^2}x_B)$
 $\delta t' = \gamma t_A - \frac{v}{c^2} \gamma x_A - \gamma t_B + \frac{v}{c^2} \gamma x_B$
 $= \gamma \delta t - \frac{v}{c^2} \gamma \delta x = \gamma(\delta t - \frac{v}{c^2} \delta x)$

then:

$$\begin{aligned} (ds')^2 &= [\gamma(\delta x - v \delta t)]^2 - c^2[\gamma(\delta t - \frac{v}{c^2} \delta x)]^2 \\ &= \gamma^2 [dx^2 + v^2 dt^2 - 2v \delta x \delta t - (\delta t^2 + \frac{v^2}{c^2} \delta x^2 - 2v \delta x \delta t)] \\ &= \gamma^2 [dx^2 + v^2 dt^2 - 2v \delta x \delta t - c^2 \delta t^2 - \frac{v^2}{c^2} \delta x^2 + 2v \delta x \delta t] \\ &= \gamma^2 [(1 - \frac{v^2}{c^2}) \delta x^2 + (v^2 c^2) \delta t^2] \\ &= \gamma^2 [\frac{1}{\gamma^2} \delta x^2 - c^2 \frac{1}{\gamma^2} \delta t^2] \\ &= (dx)^2 - c^2(dt)^2 = \underline{(ds)^2} \end{aligned}$$

b) Show that the elapsed time dt between two events occurring at the same location in the laboratory is related to the elapsed time dt' in a frame M moving along the $+x$ axis with speed v according to

$$dt = dt' \sqrt{1 - \frac{v}{c} \left(\frac{dx'}{dt'} \right)^2} = dt' / \gamma.$$

Here $dx/dt' = -v$ is the velocity of the fixed laboratory point as seen in the frame M and $\gamma = 1/\sqrt{1-v^2/c^2}$, where $\beta = v/c$.

Easy way: $dx' = \gamma(dx - v/c \cdot dt)$ from part a

$$\text{now } dx=0 \Rightarrow dx' = \gamma dt \text{ or } dt = \frac{dx'}{\gamma}$$

"Harder" way: $(ds)^2 = (dx)^2 - c^2(dt)^2 = (ds')^2 = (dx')^2 - c^2(dt')^2$

$$\text{so } (dx)^2 - c^2(dt)^2 = (dx')^2 - c^2(dt')^2$$

$$\text{but } dx=0 \Rightarrow -c^2(dt)^2 = (dx')^2 - c^2(dt')^2$$

$$(dt)^2 = (dt')^2 + \frac{1}{c^2} (dx')^2$$

$$= (dt')^2 \left(1 - \frac{1}{c^2} \left(\frac{dx'}{dt'} \right)^2 \right)$$

$$\text{so } dt = dt' \sqrt{1 - \frac{1}{c^2} \left(\frac{dx'}{dt'} \right)^2} = dt' / \gamma$$

c) Show that the spatial separation ds between two events occurring simultaneously ($dt=0$) in the laboratory is related to the spatial separation dx' in the moving frame M by

$$ds = dx' \sqrt{1 - c^2 \left(\frac{dx'}{dt'} \right)^2} = A dx'$$

Here, dt' is the time in the moving frame which elapses between the two events. Determine the proportionality constant A.

Easy way: $dx' = \gamma(dx - vdt) ; dt=0$

$$\Rightarrow dx' = \gamma dx \rightarrow ds = \gamma dx' \text{ so } A = 1/\gamma$$

"Harder" way: $(\Delta s)^2 = (\Delta s')^2$

$$(\Delta x)^2 - c^2(\Delta z)^2 = (\Delta x')^2 - c^2(\Delta z')^2$$

but $\Delta t = 0$

$$\rightarrow (\Delta x)^2 = (\Delta x')^2 - c^2(\Delta z')^2$$

$$= (\Delta x')^2 \left(1 - c^2 \left(\frac{\Delta z'}{\Delta x'} \right)^2 \right)$$

$$\text{so } \Delta x = \Delta x' \sqrt{1 - c^2 \left(\frac{\Delta z'}{\Delta x'} \right)^2}$$

now $\Delta z' = \gamma(\Delta x - \frac{v}{c} \Delta z)$; $\Delta x' = \gamma(\Delta x - v \Delta z)$

but $\Delta t = 0 \rightarrow \Delta z' = -\frac{v}{c} \Delta x$; $\Delta x' = \gamma \Delta x$

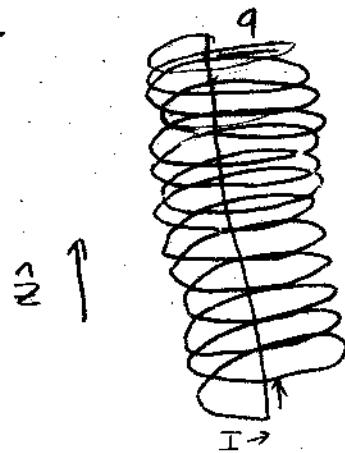
$$\text{so } \frac{\Delta z'}{\Delta x'} = -\frac{v}{c} \frac{\Delta x}{\Delta x} = -\frac{v}{c}$$

hence $\Delta x = \Delta x' \sqrt{1 - c^2 \left(\frac{-v}{c} \right)^2} = \Delta x' \sqrt{1 - \frac{v^2}{c^2}} = \Delta x' / \gamma$

d) The ratio $\Delta x'/\Delta t'$ is a function of the relative speed v between the two frames. Is it the same function in parts (b) and (c)? Explain.

In (b) it refers to the motion of a particle (same position in lab frame) while (c) just describes the space-time separation between two locations.

Spring 2002 #16



Infinite Solenoid radius a ,

carries current I $\frac{n \text{ turns}}{\text{axial length}}$

Closed via a straight wire
of radius $b < a$.

a) magnetic Field

Inside

$$B = B_{\text{wire}} + B_{\text{rel}}$$

$$\int B \cdot d\ell = \mu_0 I N$$

$$BL = \mu_0 NF$$

$$B = \frac{\mu_0 I}{2\pi r} \hat{\phi} + \mu_0 n I \hat{z}$$

$$B_{\text{rel}} = \mu_0 n I \hat{z}$$

$$\int B_{\text{wire}} \cdot d\ell = \mu_0 I$$

Outside

$$B_{\text{wire}} 2\pi r = \mu_0 I$$

$$B = \frac{\mu_0 I}{2\pi r} \hat{\phi}$$

$$B_{\text{wire}} = \frac{\mu_0 I}{2\pi r} \hat{\phi}$$

b) $\Phi = LI$ $\Phi = \int B \cdot da$

$\Phi = \pi r^2 B$ (central current) \Rightarrow no inductance,

since $d\vec{a} > da \hat{z}$ and $B \hat{\phi} \cdot d\vec{a} \hat{z} = 0$

$$\Phi = \mu_0 n \pi \sigma (a^2 - b^2)$$

$$\Phi_{\text{total}} = \frac{\mu_0 N^2 I \pi (a^2 - b^2)}{l} = L F$$

of loops
unit length

$$L = \frac{\mu_0 N^2 \pi (a^2 - b^2)}{l} = \mu_0 N n \pi a^2$$

$$\frac{L}{l} = \mu_0 n^2 \pi (a^2 - b^2)$$

(c) the wire has radius b



The magnetic field inside
wire is constant.

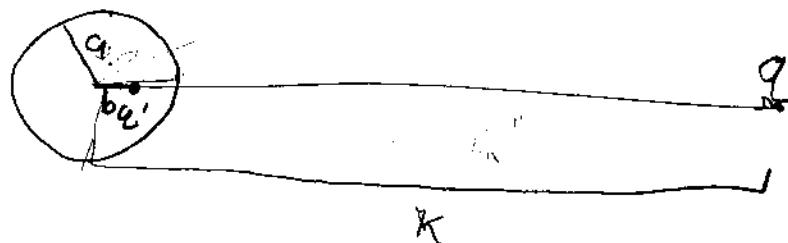
$$\Rightarrow B_{\text{wire}} = \frac{\mu_0 I}{2\pi b} \neq 0 \text{ so still no inductance.}$$

$$\Rightarrow \frac{L}{l} = \mu_0 n^2 \pi b^2$$

Spring 2002 #11

a)

$$r \rightarrow ?$$



$$\vec{F} = \frac{q\vec{r}}{(r-r)^2}$$

1/cubed for large distances Jackson page 60

$$q' = -\frac{a\epsilon}{k} \quad b = \frac{a^2}{k}$$

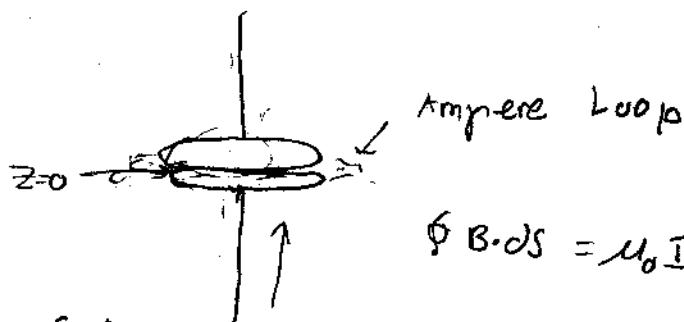
- Griffiths pg 125

$$F = \frac{\hat{r}q'}{(k-b)^2} = \frac{-q^2a}{k(k-b)^2} = \frac{-q^2a}{-k(k-a^2/k)^2} = \frac{-q^2a}{k^2(k^2-a^2)^2} = \frac{-q^2ak}{(k^2-a^2)^3}$$

$$k \gg a \Rightarrow F = \frac{-q^2ak}{k^4} \propto \frac{-q^2a}{k^3} \hat{r}$$

b) $F \propto \frac{1}{r^5}$ since we know the total induced charge is $Q=0$
which is different from part a since
there was an induced charge that
is $\propto \frac{1}{r}$

Spring 2002 #12



Amperian Loop

$$\oint \mathbf{B} \cdot d\mathbf{s} = \mu_0 I + \mu_0 \epsilon_0 \frac{d\Phi_E}{dt}$$

a) $C = \frac{\epsilon_0 A}{d}$

$$\Phi_E = \text{Flux} = \pi r_c^2 E$$

b)

$$\frac{d\Phi_E}{dt} = \pi r_c^2 \frac{dE}{dt} = \frac{\pi r_c^2}{d} \frac{dV}{dt} = \frac{\pi r_c^2}{\epsilon_0 d} \frac{dQ}{dt} = \frac{\pi r_c^2}{\epsilon_0 A d t} \cancel{dE}$$

From $V = - \int \epsilon_0 dE$

$$C = \frac{Q}{V} = \frac{\epsilon_0 A}{d} \text{ for parallel plates}$$

$$2\pi r B(r) = \mu_0 \epsilon_0 \frac{\pi r_c^2 dQ}{\epsilon_0 A dt}$$

$$B(r) = \frac{r_c^2 \mu_0}{2\pi r} \frac{dQ}{dt} = \frac{\mu_0 r_c^2}{2\pi r_c^2 r} i = \frac{\mu_0 i}{2\pi r} \cancel{\phi}$$

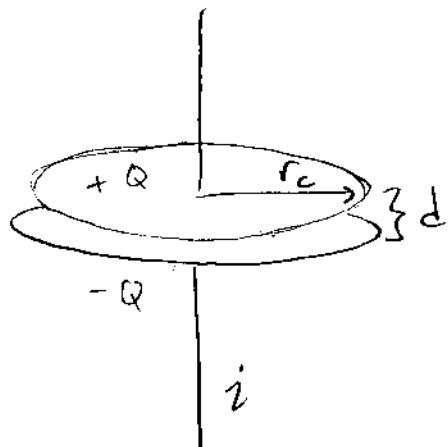
c) Now amperian loop far away from capacitor.

$$\oint \mathbf{B} \cdot d\mathbf{s} = \mu_0 I + \mu_0 \epsilon_0 \cancel{\frac{d\Phi_E}{dt}}$$

$$2\pi r B(r) = \mu_0 i$$

$$B(r) = \frac{\mu_0 i}{2\pi r} \cancel{\phi}$$

problem #12 Spring 2002



a) put charge $\pm Q$ on plates

$$\sigma_{\text{top}} = \frac{Q}{\pi r_c^2} \quad \sigma_{\text{bottom}} = -\frac{Q}{\pi r_c^2}$$

for one plate

$$\int \vec{E} \cdot d\vec{a} = \frac{1}{\epsilon_0} Q_{\text{enc}}$$

$$2|E|\pi r_c^2 = \frac{1}{\epsilon_0} \frac{Q}{\pi r_c^2} \pi r_c^2$$

$$E = \frac{\frac{1}{\epsilon_0} Q}{2\pi r_c^2} = \frac{\sigma}{2\epsilon_0} \hat{n}$$

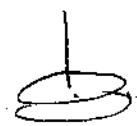
$$E_{\text{inside}} = \frac{\sigma}{\epsilon_0}$$

$$V = - \int \vec{E} \cdot d\vec{l} = - \frac{\sigma d}{\epsilon_0} = - \frac{Qd}{A\epsilon_0}$$

$$Q = CV$$

$$C = \frac{Q}{V} = - \frac{A\epsilon_0}{d} = \frac{\pi r_c^2 \epsilon_0}{d}$$

b)



$$\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{\text{enc}} + \mu_0 \epsilon_0 \int \left(\frac{d\vec{E}}{dt} \right) \cdot d\vec{a}$$

$$\frac{d\vec{E}}{dt} = \frac{1}{\epsilon_0} \left(\frac{Q}{\pi r_c^2 \epsilon_0} \right) = \frac{I}{\epsilon_0 \pi r_c^2}$$

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{\text{enc}} + \mu_0 \int_0^{l_0} \frac{I}{\pi r_c^2} \cdot d\vec{a}$$

$$\oint \vec{B} \cdot d\vec{l} = |\vec{B}| 2\pi r = \mu_0 I$$

$$\boxed{\vec{B} = \frac{\mu_0 I}{2\pi r} \hat{\phi}}$$

$$\frac{d\vec{E}}{dt} = \frac{I}{\epsilon_0 A} \Rightarrow \int \left(\frac{d\vec{E}}{dt} \right) \cdot d\vec{a} = \frac{I}{\epsilon_0}$$

$$\int \frac{I}{\epsilon_0 A} \cdot d\vec{a} = \frac{I}{\epsilon_0}$$

c) $\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{\text{enc}} + \mu_0 \epsilon_0 \int \left(\frac{d\vec{E}}{dt} \right) \cdot d\vec{a}$

$$|\vec{B}| 2\pi r = \mu_0 I$$

$$\boxed{\vec{B} = \frac{\mu_0 I}{2\pi r} \hat{\phi}}$$

Problem #13 Spring 2002

$$Z = \frac{1}{N!} \left(\frac{V}{(2\pi\hbar^2/MkT)^{3/2}} \right)^N \left(e^{-\beta E} + e^{-\beta(E+\Delta)} \right)^N$$

or

$$Z = \frac{1}{N!} \left(\frac{V}{(2\pi\hbar^2/MkT)^{3/2}} \right)^N \left(e^{+\beta\Delta/2} + e^{-\beta\Delta/2} \right)^N$$

or

$$Z = \frac{1}{N!} \left(\frac{V}{(2\pi\hbar^2/MkT)^{3/2}} \right)^N \left(1 + e^{-\beta\Delta} \right)^N$$

$$F = -kT \ln Z, \quad F = \vec{E} - TS$$

$$S = k(\ln Z + \beta \vec{E})$$

$$\mu = \left(\frac{\partial F}{\partial N} \right) = -kT \frac{\partial}{\partial N} \ln Z$$

$$\vec{E} = - \frac{\partial}{\partial \beta} \ln Z$$

$$P = \frac{1}{\beta} \frac{\partial}{\partial V} \ln Z$$

$$C_p = \left(\frac{\partial \vec{E}}{\partial T} \right)_p$$